Examples of Groups: Coxeter Groups

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Definition

A *cell complex* is a union of convex polytopes ($=$ “cells”) in some Euclidean space so that the intersection of any two is either empty or a common face of both.

Definition

A *simplex* is the convex hull of a finite set T of affinely independent points in some Euclidean space. Its *dimension* is $\text{Card}(T) - 1$. For S a finite set, Δ^S, the *simplex on* S, is the convex hull of the standard basis of $\mathbf{R}^S$ (where $\mathbf{R}^S := \{x : S \to \mathbf{R}\}$).

Example

A 1-*simplex* is an interval; a 2-*simplex* is a triangle; a 3-*simplex* is a tetrahedron.

Definition

An *abstract simplicial complex* consists of a set S (of *vertices*) and a poset $\mathcal{S}$ of finite subsets of S s.t.

- $\{s\} \in \mathcal{S}$, $\forall s \in S$.
- If $T \in \mathcal{S}$ and $U \subset T$, then $U \in \mathcal{S}$.

Definition

A (geometric) *simplicial complex* is a cell complex in which all cells are geometric simplices.

Suppose L is a geometric simplicial cx. Put

$$S := \mathrm{Vert}(L), \quad \text{and}$$
$$\mathcal{S}(L) := \{T \subset S \mid T \text{ is the vertex set of a simplex in } L\}.$$

Definition

A *geometric realization* of an abstract simplicial cx $\mathcal{S}$ is a geometric simplicial cx L s.t. $\mathcal{S} = \mathcal{S}(L)$.

Theorem

Every abstract simplicial cx $\mathcal{S}$ has a geometric realization.

Given a set S and a function $x : S \to \mathbf{R}$, $\text{Supp}(x) := \{s \in S \mid x_s \neq 0\}$. $\mathbf{R}^S$ denotes the Euclidean space of finitely supported functions $x : S \to \mathbf{R}$ and Δ^S is the simplex on S. We want to prove:

Theorem

Every abstract simplicial cx $\mathcal{S}$ has a geometric realization.

Proof.

Given $\mathcal{S}$, define a subcx $L \subset \Delta^S$ by

$$L := \{x \in \Delta^S \mid \text{Supp}(x) \in \mathcal{S}\} = \bigcup_{T \in \mathcal{S}} \Delta^T$$

Clearly, $\mathcal{S}(L) = \mathcal{S}$. □

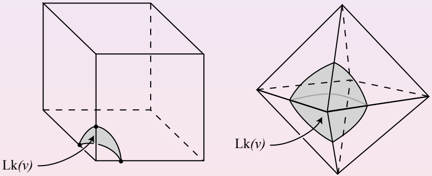
Definition

The *standard cube on a set S* is: $\square^S := [-1,1]^S \subset \mathbf{R}^S$. Its dimension is $\mathrm{Card}(S)$.

For each $T \subset S$, put

$$\square^T := [-1,1]^T \times \{0\}^{S-T} \subset \mathbf{R}^S.$$

- The link of a vertex v in a cell is the intersection of a small sphere about the vertex with the cell.
- For example, the link of a vertex in a cube is a (spherical) simplex.



Similarly, the link, Lk(v), of a vertex in a cubical cell complex is a simpicial cx.

Suppose v is a vertex in a cubical cell complex P.

- As an abstract simplicial complex $\mathrm{Lk}(v; P)$ is isomorphic to the poset of cells of P which properly contain v.
- A neighborhood of v in P is homeomorphic to the cone on $\mathrm{Lk}(v)$.

Recall $\square^T := [-1,1]^T \times \{0\}^{S-T} \subset \mathbf{R}^S$.
A face of $\square^S$ is *parallel* to $\square^T$ if it has the form $[-1,1]^T \times \{\varepsilon\}$ for some $\varepsilon \in \{\pm 1\}^{S-T}$.

The cubical complex P_L

Given a simplicial complex L with vertex set S, define a subcomplex P_L of $\square^S$ by

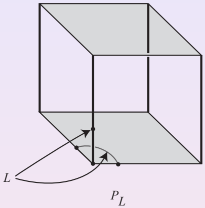
$$P_L := \bigcup_{T \in \mathcal{S}(L)} \text{all faces parallel to } \square^T$$

Main Property

For each vertex v, $\mathrm{Lk}(v, P_L) = L$

Example

Suppose S consists of 3 points and L is the union of an interval and a point. Then $\square^S$ is a 3-cube and P_L is indicated subcx.



More examples

- If $L = \Delta^{n-1}$, then $P_L = \square^n$
- If $L = \partial\Delta^{n-1}$, then $P_L = \partial\square^n = S^{n-1}$.
- If L is a set of n points, then P_L is the 1-skeleton of $\square^n$, eg, if $L = S^0$, then $P_L = \partial\square^2 = S^1$.

Joins

If T and U are disjoint sets, then $\Delta^T * \Delta^U = \Delta^{T \cup U}$. (Note: $\dim(\Delta^T * \Delta^U) = \mathrm{Card}(T \cup U) - 1 = \dim(\Delta^T) + \dim(\Delta^U) + 1$.) Similarly, if L_1 and L_2 are simplicial complexes, then $L_1 * L_2$ is defined by taking the joins of simplices in L_1 with those in L_2 (including the two empty simplices). We have: $\mathcal{S}(L_1 * L_2) = \mathcal{S}(L_1) \times \mathcal{S}(L_2)$

More

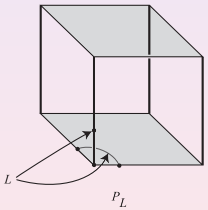
- $P_{(L_1 * L_2)} = P_{L_1} \times P_{L_2}$, eg, if $L = S^0 * S^0$, then $P_L = S^1 \times S^1 = T^2$, or if L is the n-fold join $S^0 * \cdots * S^0$ (the bdry of an n-dim octahedron), then $P_L = T^n$.
- If L is a k-gon, then P_L is the orientable surface of Euler characteristic $2^{k-2}(4-k)$

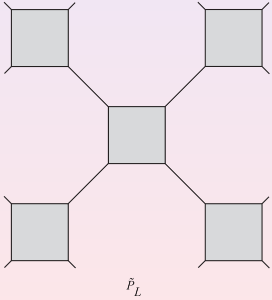
If L is a triangulation of S^{n-1}, then P_L is an n-mfld.

The $(\mathbb{Z}/2)^S$-action

- Let $\{r_s\}_{s\in S}$ be standard basis for $(\mathbb{Z}/2)^S$. Represent r_s as reflection on $\square^S$ across the hyperplane $x_s = 0$, ie, r_s changes the sign of the s^{th}-coordinate. This defines a $(\mathbb{Z}/2)^S$-action on $\square^S$.
- The subcomplex P_L is $(\mathbb{Z}/2)^S$-stable.
- $[0,1]^S$ is a (strict) fundamental domain for $(\mathbb{Z}/2)^S$-action on $\square^S \quad (= [-1,1]^S)$.
- $K := P_L \cap [0,1]^S$ is a (strict) fundamental domain for $(\mathbb{Z}/2)^S$-action on P_L.

The universal cover of P_L is denoted $\widetilde{P}_L$. The cubical structure on P_L lifts to one on $\widetilde{P}_L$.





- Let W_L be the gp of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{P}_L$. Let $\varphi : W_L \to (\mathbb{Z}/2)^S$ be the projection. We have the short exact sequence
- $$1 \to \pi_1(P_L) \to W_L \to (\mathbb{Z}/2)^S \to 1.$$
- $(\mathbb{Z}/2)^S$ acts simply transitively on $\mathrm{Vert}(P_L)$, so, W_L acts simply transitively on $\mathrm{Vert}(\widetilde{P}_L)$.
- Let $v \in K$ be the vertex $(1, \dots, 1)$. Choose a lift $\widetilde{K}$ of K in $\widetilde{P}_L$ (N.B. K is a cone) and let $\tilde{v}$ be the lift of v in $\widetilde{K}$. The 1-cells at v or $\tilde{v}$ correspond to elements of S. The reflection r_s flips the 1-cell at v labeled by s. Let $\tilde{s}$ be the unique lift of r_s which stabilizes the corresponding 1-cell at $\tilde{v}$. (Eventually, I will drop the ˜ from $\tilde{s}$.)

A presentation for W_L

- Since $\tilde{s}^2$ fixes $\tilde{v}$ and covers the identity on P_L, it follows that $\tilde{s}^2 = 1$.
- Since W_L is simply transitive on Vert$(\widetilde{P}_L)$, the 1-skeleton of $\widetilde{P}_L$ is the Cayley graph of $(W_L, \widetilde{S})$.
- Suppose $\{s, t\}$ is an edge of L. The corresponding 2-cell at $\tilde{v}$ has edges labeled successively by $\tilde{s}, \tilde{t}, \tilde{s}, \tilde{t}$. It follows that $(\tilde{s}\tilde{t})^2 = 1$.
- Since the 2-skeleton of $\widetilde{P}_L$ is simply connected, it is the Cayley 2-complex of a presentation. Therefore, W_L has a presentation with generating set $\widetilde{S} = \{\tilde{s}\}_{s \in S}$ and relations: $\tilde{s}^2 = 1$ and $(\tilde{s}\tilde{t})^2 = 1$, $\forall \{s, t\} \in$ Edge(L).
- W_L is a *right-angled Coxeter group*.

Definition

A simplicial cx L is a *flag complex* iff any finite set of vertices which are pairwise connected by edges spans a simplex of L.

Examples

- $\partial\Delta^n$ is not a flag cx for $n \geq 2$
- A k-gon (i.e. a triangulation of S^1) is a flag cx iff $k \geq 4$
- The barycentric subdivision of any cell complex is a flag cx. (This shows that the condition of being a flag cx does not restrict the topological type of L: it can be any polyhedron.)

Theorem

$\widetilde{P}_L$ is contractible iff L is a flag cx.

First Proof.

One shows that $\widetilde{H}_*(\widetilde{P}_L) = 0$ (ie, $\widetilde{P}_L$ is acyclic). $\widetilde{P}_L$ is tessellated by translates of K (each of which is a “chamber”). Order these by word length and glue on one at a time. Mayer-Vietoris sequence shows that at each stage result is acyclic. □

Second Proof (Gromov).

As a cubical cx, $\widetilde{P}_L$ has a piecewise Euclidean metric. Gromov: this is $\mathrm{CAT}(0) \iff$ it is simply connected and the link of each vertex is a flag cx. □

Here is a generalization of the construction of P_L which has received a good deal of recent interest.

- Let (X, A) be a pair of spaces and L a simplicial cx with $\text{Vert}(L) = S$. We define certain subspaces of the product $\prod_{s \in S} X$.
- For each $T \in \mathcal{S}(L)_{>\emptyset}$, let X^T be the set of $(x_s)_{s \in S}$ in the product defined by

$$\begin{cases} x_s \in X & \text{if } s \in T, \\ x_s \in A & \text{if } s \notin T. \end{cases}$$

- $$Z(L; X, A) := \bigcup_{T \in \mathcal{S}(L)_{>\emptyset}} X^T$$

Examples

- $(X, A) = ([-1, 1], \{\pm 1\})$. Then $Z(L; [-1, 1], \{\pm 1\}) = P_L$.
- $(X, A) = (S^1, \{1\})$. Then the fundamental gp of $Z(L; S^1, \{1\})$ is the right-angled Artin gp determined by the 1-skeleton of L. If L is a flag cx, then $Z(L; S^1, \{1\})$ is the standard $K(\pi, 1)$ for the Artin gp.
- $(X, A) = (D^2, S^1)$. Then $Z(L; D^2, S^1)$ is the *moment angle cx* of L. The group $(S^1)^S$ acts on $(Z(L; D^2, S^1)$. The quotient space is the same space K ($\subset [0, 1]^S$) considered earlier. If K is a n-dim convex polytope and L is the bdry cx of its dual, then $Z(L; D^2, S^1)$ is a smooth mfld, and if T is an appropriate subgp of codim n in $(S^1)^S$, then $Z(L; D^2, S^1)/T$ is a "toric variety".

L is a flag cx with vertex set S and W_L is associated right-angled Coxeter gp. S is its fundamental set of generators..

The basic construction

A *mirror structure* on a space X is a family of closed subspaces $\{X_s\}_{s\in S}$. For $x \in X$, put $S(x) = \{s \in S \mid x \in X_s\}$. Define

$$\mathcal{U}(W, X) := (W \times X)/\sim,$$

where $\sim$ is the equivalence relation: $(w, x) \sim (w', x') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$ (the subgp generated by $S(x)$). $\mathcal{U}(W, X)$ is formed by gluing together copies of X (the *chambers*). The gp W_L $(= W)$ acts on it.

Another construction of $\widetilde{P}_L$

Recall $K := P_L \cap [0,1]^S$. For each $s \in S$, K_s is the intersection of K with the hyperplane $x_s = 0$. This is a mirror structure on K.

Theorem

The natural maps $\mathcal{U}((\mathbb{Z}/2)^S, K) \to P_L$ and $\mathcal{U}(W_L, K) \to \widetilde{P}_L$ are homeomorphisms.

The basic idea

The topology of the simplicial cx L is reflected in properties of the gp W_L.

- Suppose π is a torsion-free gp. Its *cohomological dimension*, $\mathrm{cd}(\pi)$ is defined to be the maximum integer k st $H^k(\pi; M) \neq 0$ for some π-module M.
- Its *geometric dimension*, $\mathrm{gd}(\pi)$ is the smallest dimension of a $K(\pi, 1)$ complex. Obviously, $\mathrm{cd}(\pi) \leq \mathrm{gd}(\pi)$.
- Eilenberg-Ganea proved equality if $\mathrm{cd}(\pi) \geq 3$ and Stallings, Swan proved it for $\mathrm{cd}(\pi) = 1$.

The Eilenberg-Ganea Problem

Is there a gp π with $\mathrm{cd}(\pi) = 2$ and $\mathrm{gd}(\pi) = 3$?

Conjectured answer

Yes.

- Suppose L is a flag triangulation of an acyclic 2-complex with $\pi_1(L) \neq 1$. Put $\pi_L := \pi_1(P_L) = \mathrm{Ker}(\varphi : W_L \to (\mathbb{Z}/2)^S)$.
- π_L is torsion-free. It is a conjectured Eilenberg-Ganea counterexample.
- Put $\partial K := K - \overset{\circ}{K}$. In our case it is acyclic. It follows that $\mathcal{U}(W_L, \partial K)$ is acyclic (but not simply connected). Hence, $\mathrm{cd}(\pi_L) = 2$.
- $\dim \widetilde{P}_L = \dim L + 1 = 3$ and the only contractible complex which π_L seems to act on is $\widetilde{P}_L$ $(= \mathcal{U}(W_L, K))$.

Remark

Brady, Leary, Nucinkis proved these W_L are counterexamples to the version of the Eilenberg-Ganea Problem for groups with torsion.

Example (Different cd over $\mathbb{Z}$ than $\mathbb{Q}$)

- Suppose L is a flag triangulation of $\mathbf{R}P^2$. Then
- $H^3(\pi_L; \mathbb{Z}\pi_L) = H^3(\widetilde{P}_L; \mathbb{Z}) \cong H^2_c(\mathbf{R}P^2) = \mathbb{Z}/2$.
- $H^3(\widetilde{P}_L; \mathbb{Q}) = 0$ and $H^2(\widetilde{P}_L; \mathbb{Q})$ is a countably generated $\mathbb{Q}$ vector space. Hence,
- $\mathrm{cd}_{\mathbb{Z}}(\pi_L) = 3$ and $\mathrm{cd}_{\mathbb{Q}}(\pi_L) = 2$

Facts

- A closed m-mfld M^m, $m \geq 3$, with the same homology as S^m need not be homeomorphic to S^m, because it need not be simply connected. However,
- If $\pi_1(M^m) = 1$, then $M^m \cong S^m$ (Poincaré Conjecture).
- Similarly, a contractible open mfld Y^m, $m \geq 3$, is homeomorphic to $\mathbf{R}^m$ iff it is simply connected at ∞. (Stallings, Freedman, Perelman).
- Every such homology m-sphere M^m (simply connected or not) bounds a contractible $(m+1)$-mfld.

- Suppose L^{n-1} is a non simply connected homology $(n-1)$-sphere triangulated as a flag cx.
- Then $\widetilde{P}_L$ is a contractible n-dim homology mfld (the non manifold points are the vertices) and $\widetilde{P}_L$ is not simply connected at ∞. (Its fundamental gp at ∞ is the inverse limit of free products of an increasing number of copies of $\pi_1(L)$.)
- $\widetilde{P}_L$ can be modified to be a contractible n-mfld. Let C be a contractible n-mfld bounded by L $(= \partial K)$. Remove $\overset{\circ}{K}$ and replace it by $\overset{\circ}{C}$. Then $Y^n := \mathcal{U}(W_L, C)$ is a contractible n-mfld $\not\cong \mathbf{R}^n$ and $M^n := Y^n/\pi$ (where $\pi = \pi_1(P_L)$) is a closed aspherical mfld with universal cover Y^n.

Reflection Group Trick

- Given a group π which has a finite $K(\pi, 1)$ complex, this is a technique for constructing an aspherical mfld M which retracts back onto $K(\pi, 1)$. (*Aspherical* means its universal cover is contractible.) In a nutshell the trick goes as follows:
- Thicken $K(\pi, 1)$ to X, a compact mfld with bdry. (X is homotopy equivalent to $K(\pi, 1)$.)
- Put $L := \partial X$. Triangulate L as a flag cx and let W $(= W_L)$ be the corresponding right-angled Coxeter gp. As before modify P_L to a mfld by removing each copy of $\overset{\circ}{K}$ and replacing it by $\overset{\circ}{X}$, ie, form
- $M := \mathcal{U}((\mathbb{Z}/2)^S, X)$, the desired aspherical mfld.

Sample applications

Point: many interesting gps have finite $K(\pi, 1)$-complexes (even 2-dimensional ones).

- By choosing π a Baumslag-Solitar gp, we can get $\pi_1(M)$
  - to be non-residually finite, or
  - to have an infinitely divisible subgp ($\cong \mathbb{Z}[1/2]$).
- By choosing π to have unsolvable word problem (can do this with a 2-dim $K(\pi, 1)$), we get $\pi_1(M)$ with unsolvable word problem.

$M := \mathcal{U}((\mathbb{Z}/2)^S, X)$.

- As before, we can construct $\mathcal{U}(W_L, X)$. It is not contractible. But it is aspherical. (Pf: It is a union of copies of X glued together along contractible pieces.) Hence, M is aspherical (since it is covered by $\mathcal{U}(W_L, X)$).
- $\widetilde{M} =$ (univ cover of M). We can explicitly describe $\widetilde{M}$ as follows. Let $\widetilde{X} =$ (univ cover of X) and $\pi = \pi_1(X)$.
- $\widetilde{L}$ is the induced triangulation of $\partial\widetilde{X}$ and $\widetilde{S} = \text{Vert}(\widetilde{L})$. $\widetilde{W}$ $(= W_{\widetilde{L}})$ the corresponding right-angled Coxeter gp. Give $\widetilde{X}$ the induced mirror structure (indexed by $\widetilde{S}$). Then $\widetilde{M} = \mathcal{U}(\widetilde{W}, \widetilde{X})$.

The gp $\widetilde{W} \rtimes \pi$ acts on $\mathcal{U}(\widetilde{W}, \widetilde{X})$ with quotient space X and if Γ is the inverse image of the commutator subgp of W_L in $\widetilde{W}$, then $\Gamma \rtimes \pi$ acts freely with quotient space M.

Exercises

Exercise 1

Prove the following formula for the Euler characteristic of P_L:

$$\chi(P_L) = \sum_{T \in \mathcal{S}(L)} \left(-\frac{1}{2}\right)^{|T|}.$$

The Hopf Conjecture

If M^{2n} is a closed, aspherical mfld, then $(-1)^n \chi(M^{2n}) \geq 0$.

Exercises

The Charney-Davis Conjecture

If L is a flag triangulation of S^{2n-1}, then $(-1)^n\chi(P_L) \geq 0$, ie, if $\kappa(L)$ denotes the RHS of the formula in EXercise 1, then $(-1)^n\kappa(L) \geq 0$.

Exercise 2

Prove the Charney-Davis Conjecture for flag triangulations of S^1. Do any nontrivial example of the conjecture for S^3. For example, calculate $\kappa(L)$ for L the barycentric subdivision of $\partial\Delta^4$.

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