Bordification of hyperplane arrangement complements and curve complexes for spherical Artin groups

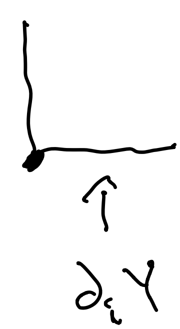
(with Jingying Huang)

1) Harvey compactification of moduli space of Riemann surface with punctures

2) Borel-Serre compactification of $\Gamma \backslash G/K = SL(n,\mathbb{Z}) \backslash SL(n,\mathbb{R}) / SO(n)$

In both cases we have mfld $\mathring{X}$ = interior of mfld with bdry

X = mfld with corners
$= \mathring{X} \cup \partial X$



On level of universal covers:

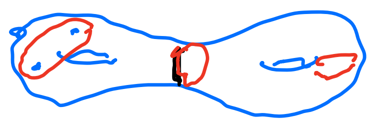
$$\begin{array}{ccc} \mathring{Y} & & Y \\ \downarrow & & \downarrow \\ X^o & & X \end{array}$$

Connection with curve cx $\mathcal{C}$

$$\partial Y = \bigcup \partial_i Y$$

covered by codim 1 "faces"
Nerve of covering is
indexed by simplicial cx $\mathcal{C}$
$\mathrm{Vert}\,\mathcal{C} = \{\partial_i Y\}$

In case 1) $\mathcal{C}$ = curve cx



In case 2) $\mathcal{C}$ is Tits bldg
for $GL(n, \mathbb{Q})$.

In both cases, $\partial Y \sim |\mathcal{C}| \sim \bigvee$ spheres

The Basic Example

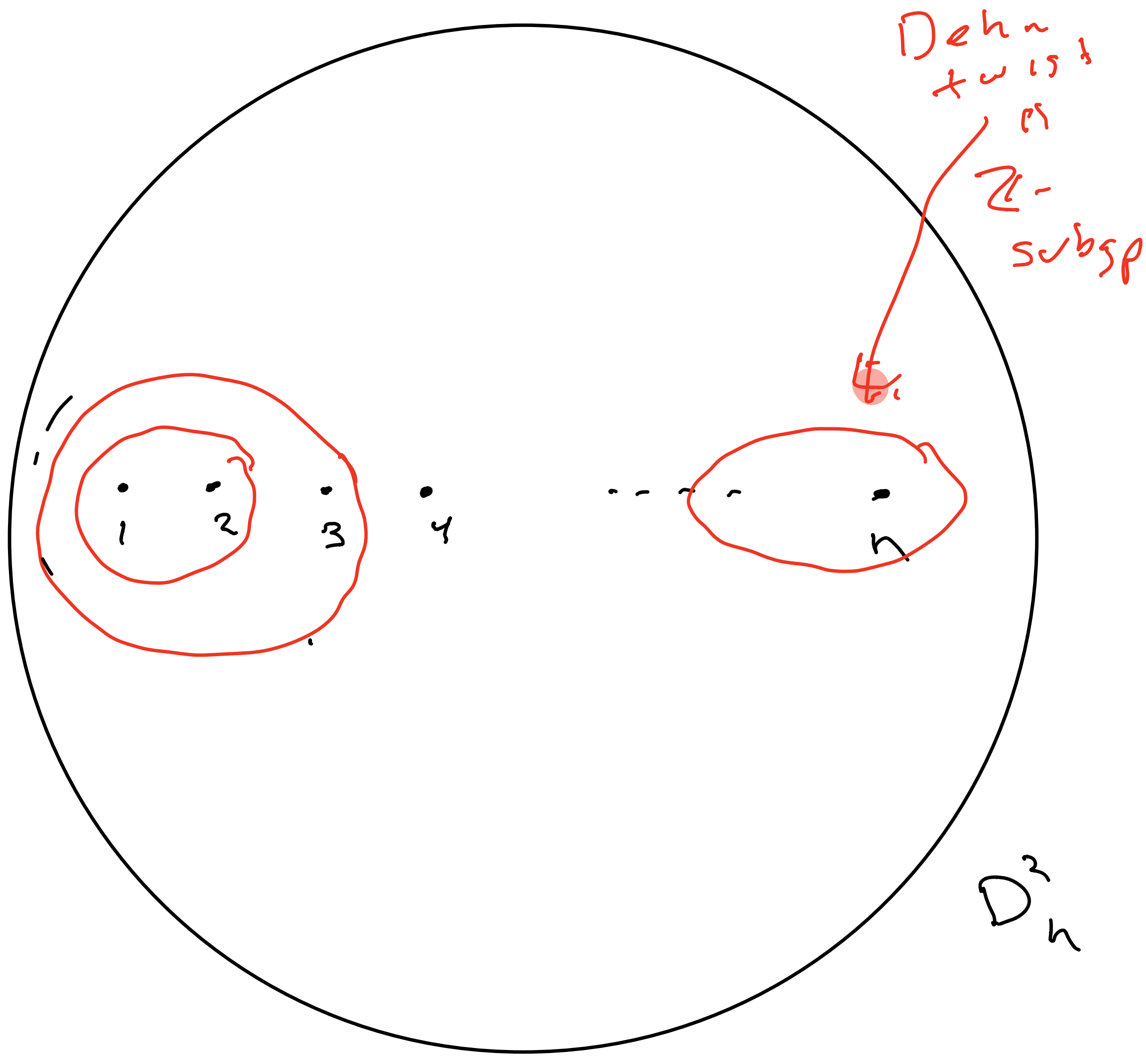
PB_n = pure braid group

$= \mathrm{Map}(D^2_n)$

$\mathcal{C}$ = curve complex

$\mathrm{Vert}(\mathcal{C}) = \Big\{$ isotopy classes of simple closed curves

$\{$ in D^2_n



A k-simplex is collection of $(k+1)$ disjoint curves

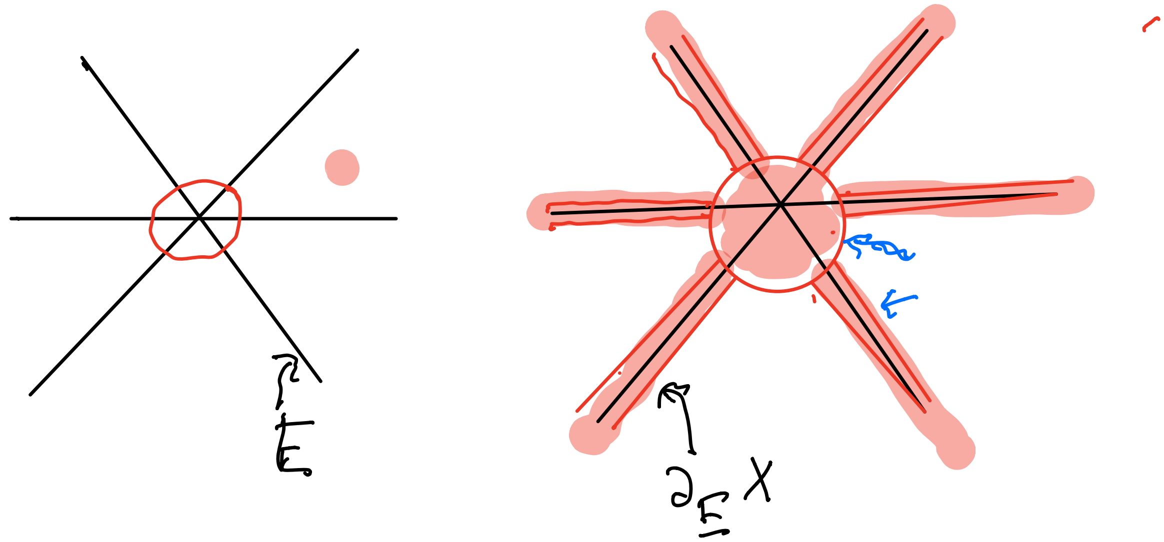
$$\mathring{X} = \text{Configuration space } C(D^2, n)$$

$$= \mathbb{C}^n - \bigcup_{i \neq j} \{x_i = x_j\}$$

$=$ complement of hyperplane arrangement $\mathcal{A}$

$PB_n = \pi_1(\mathring{X})$

$X =$ compactification



Hyperplane Arrangements

$V =$ complex vector space

$\mathcal{A} =$ finite set of hyperplanes in V

Subspace $=$ intersection of hyperplanes in $\mathcal{A}$

intersection

$Q(\mathcal{A})$ = poset of subspaces $\varepsilon\ldots$ poset

Example $W \curvearrowright \mathbb{R}^n$ as

finite reflection gp.

$W \curvearrowright V = \mathbb{R}^n \otimes \mathbb{C}$

Get Reflection Arrangement $\mathcal{A}_W$ in $\mathbb{C}^n$

Remark: The real hyperplanes cut S^{n-1} into simplices

So $\mathcal{A}_W$ is example of a "real, simplicial arrangement"

$$M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H$$

$$\overset{\circ}{X} \text{ or } \overset{\circ}{X} = S^{2n-1} \cap M(\mathcal{A})$$

The fundamental gp of $M(\mathcal{A}_W)$ is the <u>pure spherical Artin gp</u> PA_W.

More definitions

$$E \in Q(\mathcal{A})$$

$$\mathcal{A}_{V/E} = \{ H/E \mid E \leq H \}$$

$$\mathcal{A}^E = \{ H \cap E \mid E \not\leq H \}$$

<u>Fact</u> $M(\mathcal{A}_{V/E})$ is retract of $M(\mathcal{A})$. So

$M(\mathcal{A}_{V/E}) \longrightarrow M(\mathcal{A})$ is π_1-injective.

<u>Def</u> $\mathcal{A}$ is <u>reducible</u> if

$$\overset{V_1 \;+\; V_2}{\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2}, \quad \mathcal{A} \text{ subspace}$$

E is <u>irreducible</u> if $\mathcal{A}_{V/E}$ is irreducible.

<u>Def</u> A <u>parabolic</u> <u>subgp</u> of $G = \pi_1(M(\mathcal{A}))$ is

any conjugate of $\pi_1(M(\mathcal{A}$ ... $))$

$\ldots V/E')$

<u>Fact</u>: $\mathrm{Center}(\pi_1 M(\mathcal{A}_{V/E})) = \mathbb{Z}$ iff E is irreducible

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Two definitions of curve cx $\mathcal{C}$.

$\mathcal{C}_{alg}$ and $\mathcal{C}_{top}$

In both cases $G \curvearrowright \mathcal{C}$. Quotient is simplicial cx $\mathcal{D}$, the <u>irreducible cx</u>

$$\mathrm{Vert}\,\mathcal{D} = \left\{ \begin{array}{l} \text{non zero} \\ \text{irreducible subspaces} \end{array} \right\}$$

$$\mathrm{Edge}\,\mathcal{D} = \left\{ \{E,F\} \,\middle|\, \begin{array}{l} E \le F \text{ or } F \le E \\ \mathcal{A} = \mathcal{A}_{V/E} + \mathcal{A}_{V/F} \end{array} \right\}$$

<u>Algebraic Curve complex</u> $\mathcal{C}_{alg}$ appears in Cumplido, Gebhart, Gonzalez-Menedes, Wiest (Adv. in Math 2021)

Defined in terms of irreducible parabolic subgps = $\mathrm{Vert}(\mathcal{C}_{alg})$.

Alternatively, defined by set of centers of irreducible parabolic subgps ($\cong \mathbb{Z}$)
Such a $\mathbb{Z}$-subgp is analog of Dehn twist about a curve. So

$\mathrm{Vert}(\mathcal{C}_{alg}) = \{\mathbb{Z}\text{-subgps}\}$

k-simplices $= \{\mathbb{Z}^{k+1}\text{-subgps}\}$

<u>Topological Curve complex</u> $\mathcal{C}_{top}$

is defined by bordification of universal cover Y of X, where $\mathring{X} = M(\mathcal{A})$.

$\mathrm{Vert}(\mathcal{C}_{top}) = \{$codim 1 bdry faces of $\partial Y\}$

$\mathcal{C}_{top}$ = Nerve of covering by these faces

<u>Thm 1</u> $\mathcal{C}_{alg} = \mathcal{C}_{top}$.

<u>Thm 2</u> A $\overset{\text{essential}}{\text{hyperplane}}$ arrangement in $V = \mathbb{C}^n$. $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$ irreducible decomp. Then

$$|\mathcal{G}| \sim \bigvee S^{n-\ell-1}$$

Bordification of $\overset{\circ}{X}(\mathcal{A}) = M(\mathcal{A}) \cap S^{2n-1}$

Remove tubular nbhds of all irreducible subspaces in $Q(\mathcal{A})$ (starting with minimal subspaces)
The codim 1 ∂ face, correspondin to subspace E

$\partial_E X$ is trivial $S(\mathcal{A}_{V/E})_\odot$ -bundle over $S(\mathcal{A}^E)_\odot$

$$\partial_E = S(\mathcal{A}^E)_\odot \times S(\mathcal{A}_{V/E})_\odot$$

$$\pi_1(\partial_E) = \pi_1(S(\mathcal{A}^E)_\odot) \times \pi_1(S\mathcal{A}_{V/E})_\odot$$

$$\underset{\text{centralizer}}{\cap} \times \underset{\text{irreducible}}{\top}$$

of P_E | parabolic E
$\#$ | $\cup$
centralizer | $\mathbb{Z}$-subgp $\mathbb{Z}_E$

Need to show

- $\pi_1(\partial_E) \to G = \pi_1(X)$ is injective

- $N(\mathbb{Z}_E) = \underset{P_I}{\text{Centralizer}} \times \overset{\text{“}\pi_1(\partial_E)\text{”}}{P_E}$.

$\{$Normalizers of $\mathbb{Z}^{k+1}$-subgps$\}$ gives a complex of gps over $\mathcal{C}$

<u>Thm.</u> PA_W = pure spherical irreducible Artin gp. $X = X(\mathcal{A}_W)$

= compactified hyperplane complement

Y = univ. cover of X.

$\alpha \in \mathcal{C}$ a k-simplex

$\partial_\alpha Y$ = face of codim $k+1$

Then each

$\partial_\alpha Y$ is contractible

$\partial Y \sim |\mathcal{G}| =$ wedge of spheres

(each of dim $= n - \ell - 1$).

<u>Proof</u> that $|\mathcal{G}|$ is wedge of spheres.

$X = M(\mathcal{A}) \cap (S^{2n_1 - 1} \times \cdots \times S^{n_\ell - 1})$

$Y =$ univ cover, $\dim Y = 2n - \ell$.

Fact: $H^*_c(Y) = H^*(X; \mathbb{Z}G)$

$$= \begin{cases} \text{free abelian} & * = 2n-\ell \\ 0 & , \text{ otherwise} \end{cases}$$

Poincare duality

$H^*_c(Y) \cong H_{2n-\ell-*}(Y, \partial Y)$

So

$H^{2n-\ell}(Y) \cong H \quad (Y, \partial Y$

$2n-\ell$

$$= \bar{H}_{2n-\ell-1}(\partial Y)$$

$$= \bar{H}_{2n-\ell-1}(C)$$

$\square$