Coxeter groups, Artin groups, buildings

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The basic object is a Coxeter system. To this one can associate various cell complexes: the Davis-Moussong complex, the Deligne complex of an Artin group, and the “standard realization” of any building whose type is the Coxeter system.

Chapter 4 of new book

Infinite group actions on polyhedra, to appear in Springer, 2024.

Outline

1. Coxeter groups
2. Artin groups
3. Buildings
4. Nonpositive curvature
5. The $K(\pi, 1)$ problem

Coxeter systems

$S =$ a set (of generators).
$M = m(s,t)_{s,t \in S}$ is a *Coxeter matrix*, ie, an $S \times S$ symmetric matrix with entries in $\mathbb{N} \cup \{\infty\}$, 1 *s* on the diagonal and off-diagonal entries ≥ 2. The *Coxeter group W* is defined by the presentation:

$$W = \langle S \mid (st)^{m(s,t)} = 1 \rangle_{(s,t) \in S \times S}$$

(W, S) is a *Coxeter system*

Alternate encoding of data

Graph L^1 with $\mathrm{Vert}\, L^1 = S$ and labelling of edges $m : \mathrm{Edge}\, L^1 \to \{2, 3, \dots\}$, where edge $\{s, t\}$ is labelled $m(s, t)$.

Special subgroups

For any $T \leq S$, put $W_T = \langle T \rangle$. When W_T is finite, it is called a *spherical subgroup* and T is a *spherical subset*.

Poset of spherical subsets

$\mathcal{S} = \{$spherical subsets of $S\}$. $\mathcal{S}^{op}$ is the opposite poset.
$L(W, S)$ is the simplicial complex with vertex set S and $\{$nonempty simplices$\} = \{T \in \mathcal{S} \mid T \neq \emptyset\}$. It is called *nerve* of (W, S).
$K(W, S) = |\mathcal{S}|$, the cone on the barycentric subdivision of L.
Called the standard *fundamental chamber*.

A simple complex of groups

$W\mathcal{S}^{op} = (W_T)_{T \in \mathcal{S}}$

A simple complex of groups is associated to group action with a strict fundamental domain.

Poset of spherical cosets of W

$\mathrm{Coset}(W) = \coprod_{T \in \mathcal{S}} W/W_T$,
called the *development* of $W\mathcal{S}^{op}$

Strict fundamental domains

Two possibilities:

A simplex Δ

The codimension-one faces of Δ are indexed by S. Faces are indexed by subsets $T < S$.The face $\Delta_T = \cap_{s \in T} \Delta_s$ has codimension $\operatorname{Card} T$.

The chamber $K(W, S)$

$K(W, S)$ is the geometric realization of $\mathcal{S}$ (or $\mathcal{S}^{op}$). Its k-simplices are chains $T_0 < \cdots < T_k$. These can be assembled into faces or dual cells.

$$K_T = |\mathcal{S}_{\geq T}| \qquad K(T) = |\mathcal{S}_{\leq T}|$$

$K(T)$ is a combinatorial cube of dimension $\operatorname{Card} T$.

For $x \in K$, put $S(x)$ be the smallest T where $x \in K_T - \partial K_T$ (Here $\partial K_T = |\mathcal{S}_{>T}|$.)

Davis-Moussong complex $\Sigma(W, S)$

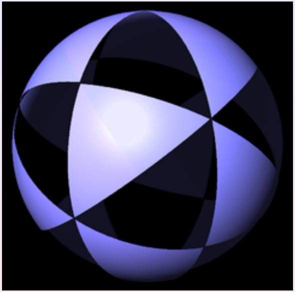
$\Sigma(W, S) = (W \times K)/\sim$, also denoted by $D(W, K)$, where

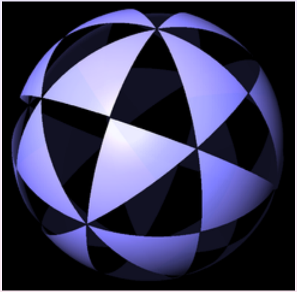
$$(w, x) \sim (w', x') \iff x = x' \text{ and } wW_{S(x)} = w'W_{S(x)}.$$

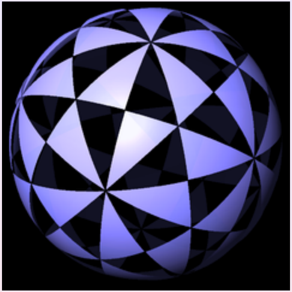
The subspace $W_T K(T) \subset \Sigma(W, S)$ is a cell called a *Coxeter zonotope*. The poset of such zonotopes is $\cong \mathrm{Coset}(W)$

Coxeter complex

$D(W, \Delta) = (W \times \Delta)/\sim$, where $\sim$ is defined as above.







	Coxeter system	Artin group	building
Notation	(W, S)	A	$\mathcal{C}$
spherical subsets	$\mathcal{S} = \{T < S \mid W_T \text{ is finite}\}$	*same*	*same*
fund. chamber	$K(W, S) = \lvert\mathcal{S}\rvert$	same	same
cell cx	Davis-Moussong cx $\Sigma(W, S)$	Deligne cx Λ	realization $\lvert\mathcal{C}\rvert$
simple cx gps	$(W_T)_{T\in\mathcal{S}}$	$(A_T)_{T\in\mathcal{S}}$	$(G_T)_{T\in\mathcal{S}}$
spherical cosets	$\coprod_{T\in\mathcal{S}} W/W_T$	$\coprod_{T\in\mathcal{S}} A/A_T$	$\coprod_{T\in\mathcal{S}} \mathcal{R}(T)$
CAT(0)?	yes	?	yes
contractible?	yes	?	yes
$K(\pi, 1)$ ques?	yes	?	yes

Definition of Artin group

(W, S) as before. For letters a, b and $m \in \{2, 3, \dots\}$, put

$$\mathrm{prod}(a, b; m) = \underbrace{ab\cdots}_{m \text{ terms}}$$

Let $\{a_s\}_{s\in S}$ be new symbols for generators. Define

$$A = A(W, S) = \langle \{a_s\} \mid \mathrm{prod}(a_s, a_t; m) = \mathrm{prod}(a_t, a_s; m)\rangle,$$

where $s \in S$ and $\{s, t\} \in \mathrm{Edge}\, L^1$. For $T \subset S$, put $A_T = \langle \{a_s\}_{s\in T}\rangle$.

Simple complex of groups

$A\mathcal{S}^{op} = \{A_T\}_{T \in \mathcal{S}}$. If, instead, the underlying poset is the set of proper subsets of S, then $\mathcal{S}^{op} \cong \{\text{faces of } \Delta\}$.

Poset of spherical cosets of A

$\mathrm{Coset}(A) = \coprod_{T \in \mathcal{S}(W,S)} A/A_T$, is the *development* of $A\mathcal{S}^{op}$. The corresponding cell complex is the *Deligne complex*. If we use the proper subsets, the corresponding poset of cosets is called the *Artin complex*.

Deligne complex

$\Lambda(W, S) = D(A, K) = (A \times K)/\sim$, as before

$$(a, x) \sim (a', x') \iff x = x' \text{ and } aA_{S(x)} = a'A_{S(x)}.$$

When fund chamber is simplex Δ, as before, define the *Artin complex* to be $D(A, \Delta)$.

The Deligne cx is similar to Davis-Moussong cx except that along each codimension 1 face, instead of 2 chambers meeting, we have a an infinite cyclic group worth of chambers.

Buildings

Combinatorially, a "building" is a set $\mathcal{C}$ of "chambers" with extra structure. In particular, each building will have an associated Coxeter system (W, S).

Chamber systems

A *chamber system* over S is a set $\mathcal{C}$ together with a family of equivalence relations in indexed by S. Each s-equivalence class must have at least 2 elements. Two s-equivalent chambers are *s-adjacent* if they are not equal.

Example

The Coxeter group W is a chamber system over S. Two elements are s-equivalent if they determine the same coset in $W/W_{\{s\}}$. (This is the "thin building" of type (W, S).)

Example

The Artin group $A = A(W, S)$ is a chamber system over S; it is usually not a building.

Galleries

A *gallery* in $\mathcal{C}$ is a sequence of adjacent chambers $C_0, C_1, \ldots, C_k$. If C_{i-1} is s_i-adjacent to C_i, then the gallery has *type* $(s_1, s_2, \ldots, s_k)$. If each $s_i \in T \subset S$, then the gallery is a *T-gallery*.

Residues

A *T-residue* is a T-gallery connected component. For example, the $\{s\}$-residue containing a chamber C is the s-equivalence class containing C (analogous to a coset).

Examples of rank 2 buildings, $S = \{s, t\}$

Trees

The set of edges in a tree (without a terminal vertex) is a building of type $S = \{s, t\}$ and a building of type (D_∞, S).

Generalized *m*-gons

Given $m \in \mathbb{N}$, $m \geq 2$, a finite bipartite graph Γ is called a *generalized m-gon* if it has girth $2m$ and diameter m. $\mathcal{C} = \mathrm{Edge}\,\Gamma$ is a chamber system over S and a building of type (D_m, S).

Chamber systems of type (W, S)

Let $\mathcal{C}$ be a (gallery connected) chamber system over S and $m(s,t)$ a Coxeter matrix. Then $\mathcal{C}$ has *type* $m(s,t)$ (or type (W,S)) if each $\{s,t\}$ residue is a generalized $m(s,t)$-gon. The chamber system is *thick* if each s-residue has more than 2 elements.

Feit-Higman Theorem

Finite, thick generalized m-gons exist only for $m \in \{2,3,4,6,8\}$.

W-distance

Define $\delta : \mathcal{C} \times \mathcal{C} \to W$ as follows. Suppose $C, D \in \mathcal{C}$ and $C = C_0, \cdots, C_k = D$ is a minimal gallery between them. Let $(s_1, \dots, s_k)$ be its type and let $w = s_1 \cdots s_k$ be the associated element of W. Then $\delta(C,D) = w$.

Definition of building

A chamber system $\mathcal{C}$ of type (W, S) equipped with a W-distance function $\delta : \mathcal{C} \to \mathcal{C}$ is a *building* if δ satisfies certain axioms (which we won't state).

Geometric realization

This is a space $|\mathcal{C}|$ where there is a copy of $K(W, S)$ for each chamber in $\mathcal{C}$. In other words, $|\mathcal{C}| = (\mathcal{C} \times K(W, S))/\sim$, where as before,

$$(C, x) \sim (C', x') \iff x = x' \text{ and } C, C' \in \text{same } S(x)\text{-residue.}$$

Chamber-transitive group actions

Suppose G is a chamber-transitive group of automorphisms of $\mathcal{C}$. Fix $C \in \mathcal{C}$ and let B (or $G_\emptyset$) denote the stabilizer of C. For $T \subset S$, let $G_T =$ stabilizer of T-residue containing C. Then $G\mathcal{S}^{op} = \{G_T\}_{T \in \mathcal{S}}$ is a simple complex of groups. Moreover, $G = \lim G_T$.

Recovering the building

$\mathcal{C} = G/B$. $\quad \mathrm{Coset}(G) = \coprod_{T \in \mathcal{S}} G/G_T$ is the poset of spherical cosets in $G\mathcal{S}^{op}$. A coset of G_T is the same thing as a T-residue. The development $\mathrm{Coset}(G)$ is $D(G, K) = (G \times K)/\sim$. $(= |\mathcal{C}|)$.

Right-angled buildings

RABs

Suppose (W, S) is right-angled. $(G_s)_{s \in S}$ for each $T \in \mathcal{S}$, let G_T be the direct product of the G_s, $s \in T$. The direct limit G is the *graph product* and $G\mathcal{S}^{op} = \{G_T\}_{T \in \mathcal{S}}$ defines a *right-angled building* with $D(G, K) = (G \times K)/\sim$.

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Notation	(W, S)	A	$\mathcal{C}$
spherical subsets	$\mathcal{S} =$ $\{T < S \mid W_T \text{ is finite}\}$	*same*	*same*
fund. chamber	$K(W, S) = \lvert\mathcal{S}\rvert$	same	same
cell cx	Davis-Moussong cx $\Sigma(W, S)$	Deligne cx Λ	realization $\lvert\mathcal{C}\rvert$
simple cx gps	$(W_T)_{T\in\mathcal{S}}$	$(A_T)_{T\in\mathcal{S}}$	$(G_T)_{T\in\mathcal{S}}$
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CAT(0)?	yes	?	yes
contractible?	yes	?	yes
$K(\pi, 1)$ ques?	yes	?	yes

CAT(0) spaces

Gromov defined what it means for a complete geodesic metric space to be "CAT(0)" by comparing its triangles with triangles in $\mathbb{R}^2$. A space is "nonpositively curved" (abbreviated NPC) if it is locally CAT(0).

Basic facts

1. Simply connected and NPC $\implies$ CAT(0).
2. CAT(0) $\implies$ contractible.
3. A piecewise euclidean polyhedron is NPC if the link of each of each cell (a piecewise spherical polyhedron) is CAT(1).

Theorem (Moussong 1988)

$\Sigma(W,S)$ is $\mathrm{CAT}(0)$.

Corollary (D.)

If $\mathcal{C}$ is a building of type (W,S), then $|\mathcal{C}|$ is $\mathrm{CAT}(0)$. If $\mathcal{C}$ is a spherical building, then the link of the cone point, $D(\mathcal{C},\Delta^n)$ is $\mathrm{CAT}(1)$.

Spherical Coxeter groups

Suppose W is finite and acts as a reflection group on S^n with fundamental chamber a spherical simplex Δ^n. Then the Coxeter complex $D(W,\Delta^n) \cong S^{n-1}$; hence, is $\mathrm{CAT}(1)$.

Conjecture (Charney-Davis)

When (W, S) *is not spherical, the Deligne complex,* $D(A, K)$*, is* $\mathrm{CAT}(0)$*. If*

Conjecture (Charney-Davis)

When (W, S) *is spherical, the Artin complex* $D(A, \Delta^n)$ *is* $\mathrm{CAT}(1)$*.*

This implies the previous conjecture for general Artin groups. (Since the link of a cell in Λ corresponds to a spherical Artin subgroup.)

Suppose $GQ = \{G_T\}_{T \in Q}$ is a simple complex of groups over a poset Q. Each group G_T has a classifying space BG_T which is aspherical, i.e., is a $K(G_T, 1)$

Using the injections $G_T \to G_{T'}$ we can glue together the BG_T to form a new space BGQ, called the *aspherical realization* of GQ. Its homotopy type is well-defined. Its fundamental group is G.

$K(\pi, 1)$-problem

Is $BGQ = BG$, i.e., is BGQ aspherical?

Theorem

If $D(G, |Q|)$ is contractible, then the $K(\pi, 1)$-question for GQ has a positive answer.

Proof

Suppose $D(G, |Q|)$ is contractible. $D(G, |Q|) \times_G EG$ has two projections p_1, p_2 to $|Q|$ and BG, respectively. The fiber of p_1 over $|Q|_T$ is BG_T. So, $(D(G, |Q|) \times_G EG) \sim BGQ$. The fiber of p_2 is $D(G, |Q|)$; so, p_2 is a homotopy equivalence.

Corollary

If G is a Coxeter group or a chamber transitive group on a building, then the $K(\pi, 1)$-question for $G\mathcal{S}^{op}$ has a positive answer.

Theorem [Charney-D]

The answer is also positive for RAAGs and for Artin groups with $\dim K \leq 2$

The $K(\pi, 1)$-question for general Artin groups is an important open question in geometric group theory.

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