Compactly supported cohomology of buildings

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Combinatorial approach to buildings

The data for a bldg is first of all a Coxeter system (W, S). Then

- a set Φ of "chambers,"
- a family of equivalence relations on Φ ("adjacency relations") indexed by the set S (of fundamental generators for the Coxeter gp),
- a "W-valued distance function $\delta : \Phi \times \Phi \to W$.

Of course, these objects must satisfy certain axioms.

Classically, one is interested in the case where W is a finite reflection gp (leading to "spherical bldgs") or a cocompact Euclidean reflection gp ("affine bldgs").

Geometric and topological realizations

We want to associate a topological space to Φ. There is a certain freedom of choice in the construction. Choose

- a space X (as a "model chamber")
- a family of subspaces $\{X_s\}_{s \in S}$ indexed by S (called a "mirror structure").

Let $\mathcal{U}(\Phi, X)$ be the result of pasting together copies of X, one for each element of Φ, so that copies corresponding to s-adjacent elements of Φ are pasted together along X_s.

Let X^f be the points of X with a locally finite nbhd in $\mathcal{U}(W, X)$.

Two choices for X stand out:

- $X = \Delta$, a simplex with its codim 1 faces indexed by S. (This was Tits' original choice - the resulting space, $\mathcal{U}(\Phi, \Delta)$, is called the "classical realization" of Φ.)
- $X = K$, the geometric realization of the poset of "spherical subsets" of S. $K \subset \Delta^f$ as its "compact core." (K is the so-called "Davis chamber." The resulting space, $\mathcal{U}(\Phi, K)$, is called the "standard realization" of Φ.)

All 3 spaces $\mathcal{U}(\Phi, K)$, $\mathcal{U}(\Phi, \Delta^f)$ and $\mathcal{U}(\Phi, \Delta)$ are contractible. The first 2 are locally finite, the third usually is not. Δ^f is not compact (unless it is $= \Delta$).

The standard realization (when $X = K$) is the most important for geometric group theory. The reason is that if $\Gamma \subset \mathrm{Aut}(\Phi)$ is a cocompact lattice, then $H_c^*(\mathcal{U}(\Phi, K)) = H^*(\Gamma; \mathbb{Z}\Gamma)$. Moreover, the cohomology of Γ with $\mathbb{Z}\Gamma$ coefficients tells us a good deal about Γ, for example, its vcd (= virtual cohomological dimension), its number of ends and if it is a duality gp.

When Φ is an irreducible affine bldg, $K = \Delta = \Delta^f$ and $H_c^*(\mathcal{U}(\Phi, \Delta))$ was calculated by Borel-Serre in 1976.

Theorem of Borel–Serre

$H_c^*(\mathcal{U}(\Phi, \Delta))$ is free abelian and concentrated in the top degree.

From this, they derived cohomological properties of "S-arithmetic gps."

Our main result will be a computation of $H_c^*(\mathcal{U}(\Phi, X^f))$ for any space X with mirror structure. ($X = K$ is the case of interest.) However, as a first step we will need to calculate $H_c^*(\mathcal{U}(\Phi, \Delta^f))$. The result is similar to the Borel–Serre Theorem: it is free abelian and concentrated in the top degree. In contrast, $H_c^*(\mathcal{U}(\Phi, K))$ usually is not concentrated in a single degree.

History

- In 1998 I calculated the compactly supported cohomology of (the complex associated to) any Coxeter system (W, S).
- In 2002, John Meier and I gave a similar calculation for the compactly supported cohomology of the standard realization of any building of type (W, S). However, there was a mistake in the proof.
- In 2006, Dymara, Januszkiewicz, Okun and I gave an equivariant version of the formula for Coxeter gps as well as a proof for right-angled buildings.
- Now we can do the general case using an idea from our 2007 paper on weighted L^2-cohomology.

Books

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Chamber systems

Definition

A *chamber system* is a set Φ (of "chambers") together with a family of equivalence relations indexed by some set S

Two chambers c and d are *s-adjacent* if they are s-equivalent and not equal.

A *gallery* in Φ is a finite sequence $(c_0, \dots, c_n)$ of adjacent chambers. It is *type* $(s_1, \dots, s_n)$ if c_i and c_{i-1} are s_i-adjacent. (So, its type is a word in S.)

Definitions

- Φ is *connected* if any two chambers can be connected by a gallery.
- Given a subset $T \subset S$, a gallery of type $(s_1, \ldots, s_n)$ is a *T-gallery* if each $s_i \in T$.
- A *residue* of *type* T is a T-gallery connected component of Φ.

Coxeter groups

A *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix (m_{st}) with 1's on the diagonal and off-diagonal entries integers ≥ 2 or the symbol ∞. The associated *Coxeter group* is the group W with generating set S and relations:

$$(st)^{m_{st}} = 1$$

(Note: these relations imply that each $s \in S$ is an involution.)

The pair (W, S) is a *Coxeter system*.

For each $T \subset S$, $W_T := \langle T \rangle$. Also, W_s is the subgroup $\langle s \rangle$ of order 2 generated by s.

Example

W has the structure of a chamber system over S.

- $v, w \in W$ are s-equivalent $\iff$ they belong to the same coset of W_s.
- A gallery in W is an edge path in $\mathrm{Cay}(W, S)$. (Here a "chamber" is a vertex of $\mathrm{Cay}(W, S)$.)
- Its type is a word in S.
- A T-residue is a coset of W_T in W.

T is a *spherical* subset of S if $|W_T| < \infty$.
$\mathcal{S} :=$ the poset of spherical subsets.

Suppose (W, S) is a Coxeter system. A *bldg* of type (W, S) is

- a set Φ (of chambers)
- a family of equivalence relations indexed by S ("adjacency relations") st $\forall s \in S$, each s-equivalence class contains at least 2 elements.
- a *W-valued distance function* $\delta : \Phi \times \Phi \to W$ (st φ, φ' are connected by a minimal gallery of type $(s_1, \ldots, s_n) \iff \delta(\varphi, \varphi') = s_1 \cdots s_n$).

Example (Thin bldgs)

W is itself a bldg. w, w' are *s-adjacent* if $w' = ws$.
$\delta : W \times W \to W$ is defined by $\delta(w, w') = w^{-1}w'$.
A T-residue is a coset of W_T (where $W_T := \langle T \rangle$.)

There are different ways to associate a topological space to a bldg Φ.

A *mirror structure* on a CW complex X is a family of subcomplexes $\{X_s\}_{s \in S}$. For each $x \in X$, put

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Define

$$\mathcal{U}(\Phi, X) = (\Phi \times X)/\sim$$

where $(c, x) \sim (c', x') \iff x = x'$ and c and c' belong to the same $S(x)$-residue.

Example (The classical realization of Tits)

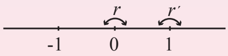
Let Δ be a simplex, of dimension $\text{Card}(S) - 1$.
$\{\Delta_s\} = \{\text{codim 1 faces}\}$.
Then $\mathcal{U}(\Phi, \Delta)$ is the *classical realization*.

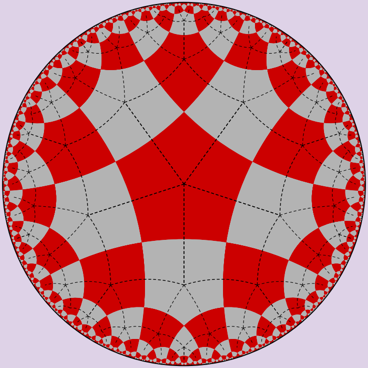
Example (The standard realization)

Let K be the geometric realization of $\mathcal{S}$, the poset of spherical subsets of S. K is a subset of the barycentic subdivision of Δ.
Put $K_s = K \cap \Delta_s$. $\mathcal{U}(\Phi, K)$ is the *standard realization*.
Only the spherical faces of K are nonempty and only spherical residues contribute to $\mathcal{U}(\Phi, K)$.

Example (The edge set of a tree)

Suppose W is the infinite dihedral group with generators s, t. Let $\mathcal{T}$ be a tree w/o terminal vertices. Any tree is bipartite. Call the colors s, t. Let W be the infinite dihedral group on $\{s, t\}$. Then $\Phi = \text{Edge}(\mathcal{T})$ is a building of type $(W, \{s, t\})$. The W-distance between b,$c \in \Phi$ is defined as follows. Take the edge path w/o backtracking from b to c. Its type gives a word in $\{s, t\}$ and hence, an element $w \in W$ and $\delta(b, c) = w$. $K = \Delta =$ an interval. $\mathcal{U}(W, K)$ is the real line. $\mathcal{U}(\Phi, K)$ is the original tree $\mathcal{T}$.





$\mathcal{U}(W, K)$, with K a pentagon

Notation

Suppose $\{X_s\}_{s \in S}$ is a mirror structure. A *face* of X is a subset

$$X_T := \bigcap_{s \in T} X_s.$$

Given $U \subset S$,

$$X^U := \bigcup_{s \in U} X_s.$$

X^f denotes the complement of the spherical faces of X.

Suppose Φ is a bldg of type (W, S).
Let A be the set of finitely supported $\mathbb{Z}$-valued functions on Φ, ie, A is the free abelian gp on Φ. For each $T \subset S$, put

$$A^T := \{f \in A \mid f \text{ is constant on each } T\text{-residue}\}.$$

Note that $A^T = 0$ whenever T is not spherical. Also,

Example (The thin bldg)

If $\Phi = W$, then $A = \mathbb{Z}W$ (the group ring) and $A^T = (\mathbb{Z}W)^{W_T}$ $(\cong \mathbb{Z}[W/W_T])$.

$A^U \subset A^T$ whenever $U \supset T$. So, put

$$A^{>T} := \sum_{U \supsetneq T} A^U.$$

Fact

$A^T / A^{>T}$ is free abelian.

Let $\hat{A}^T$ be a complementary summand for $A^{>T}$ in A^T.

Main Theorem

Suppose X is a finite CW complex. Then

$$H_c^*(\mathcal{U}(\Phi, X^f) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \hat{A}^T.$$

X a mirrored CW complex, $X^{(k)} := \{k\text{-cells}\}$. Define a cochain complex and "coefficient system" $\mathcal{I}(A)$ on X by

$$\mathcal{C}^k(X; \mathcal{I}(A)) := \{f : X^{(k)} \to \mathbb{Z} \mid f(c) \in A^{S(c)}\}.$$

Proposition

If X is compact, then $\mathcal{H}^*(X;\mathcal{I}(A)) = H_c^*(\mathcal{U}(\Phi, X^f))$.

Strategy for proving Main Thm

Prove a decomposition result for for the coefficient system $\mathcal{I}(A)$ so that cohomology with coefficients in any given summand can be computed.

Decomposition Theorem

For any spherical subset T,

$$A^T = \bigoplus_{U \subset T} \hat{A}^U.$$

In particular, for $T = \emptyset$, we have

$$A = \bigoplus_{U \in \mathcal{S}} \hat{A}^U.$$

To prove this we are led back to classical realizations of bldgs.

Proposition

Let Δ be the simplex of dimension $n = \mathrm{Card}(S) - 1$. Then $\mathcal{H}^*(\Delta; \mathcal{I}(A))$ (which is $= H_c^*(\mathcal{U}(\Phi, \Delta^f))$) is free abelian concentrated in dimension n.

(You also need some similar statements corresponding to certain subcomplexes of Δ.)

Why is the proposition plausible?

Because when W is infinite $\mathcal{U}(W, \Delta^f)$ can be identified with the image of the interior of a convex cone in real projective space, ie, $\mathcal{U}(W, \Delta^f) \cong \mathbf{R}^n$, where $n = \dim \Delta$.

Ingredients for proof of the proposition

- $\mathcal{U}(\Phi, \Delta^f)$ has a piecewise Euclidean $\mathrm{CAT}(0)$ metric (extending Moussong's $\mathrm{CAT}(0)$ metric on $\mathcal{U}(\Phi, K)$).
- Theorem of Brady-McCammond-Meier on when $\mathrm{CAT}(0)$ complexes are highly connected at ∞. (Sufficient condition: complements of balls of radius $\frac{\pi}{2}$ in links of cells are sufficiently highly connected).
- Theorem of Dymara-Osajda and independently Schulz: complements of $\frac{\pi}{2}$-balls in spherical bldgs are highly connected (homology is concentrated in top degree).

Outline of argument

$\mathcal{U}(\Phi, \Delta^f)$ is $\mathrm{CAT}(0)$.

$\Longrightarrow$ $H^*(\mathcal{U}(\Phi, \Delta^f))$ is free abelian and concentrated in top degree.

$\Longrightarrow$ decompositon of coefficient system $\mathcal{I}(A)$.

$\Longrightarrow$ Main Thm (computation of $H_c^*(\mathcal{U}(\Phi, X^f))$.)