(Joint with Jingyin Huang)

1. Manifolds with faces
2. BS bordifications
3. The classical examples
4. Blowing up the complement of hyperplane arrangement
5. Irreducible complex
6. Curve cx of a spherical Artin gp
7. Braid groups

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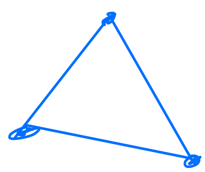
1. X^n is a manifold with corners if it is locally modeled on $[0,\infty)^n$. For $x \in X$

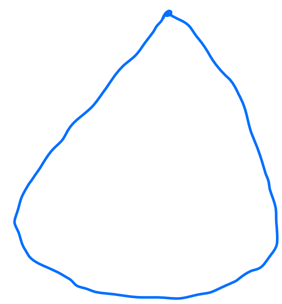
$c(x) = \# \{ i \mid x_i = 0 \}$

doesn't depend on local chart

<u>Def</u> X^n is <u>mfld with faces</u> if each x is in closure of precisely $c(x)$ codim 1 strata



Good



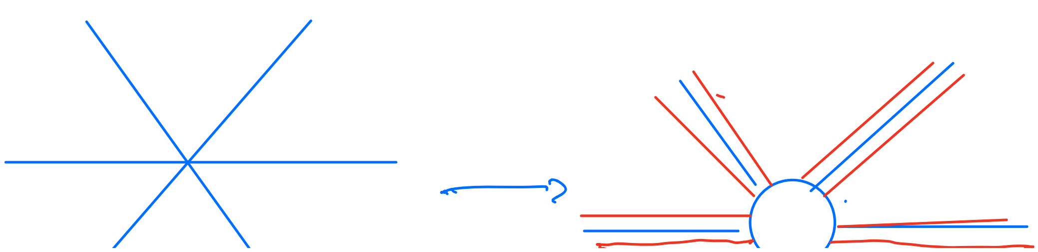
Bad

Deleting Tubular Nbhds

$Y^{n-k} \subset X^n$

$$X \odot Y = (X - Y) \cup SN$$

sphere bundle



2. BS bordifications

N = aspherical mfld

M = univ. cover

$G = \pi_1(N)$

Def^n A BS bordification of M is a mfld with faces $\overline{M}$

(i) $M = \operatorname{int}(\overline{M})$

(Usually $\overline{N} = \overline{M}/G$ is compact)

(ii) Each face of $\overline{M}$ is contractible

(iii) $\exists$ simpl. cx $\mathcal{C}$

$\{\text{simplices of } \mathcal{C}\} \longleftrightarrow \{\text{faces of } \overline{M}\}$

k-simplex $\longleftrightarrow$ codim$(k+1)$ face

(so $\mathcal{C}$ is nerve of covering of $\partial\overline{M}$ by its codim-1 faces)
Often $\mathcal{C}$ will have another interesting description

(iv) $\mathcal{C} \sim \bigvee S^m$

Remark (iv) can be used to show G is a virtual duality gp. (Meaning that $H^*(G; \mathbb{Z}G)$ is torsion free and concentrated in a single dimension)

Proof

$$H^*(G; \mathbb{Z}G) = H_c^*(M)$$

$$\underset{P.D}{=} H_{n-*}(\overline{M}, \partial\overline{M}) = \overline{H}_{n-*-1}(\partial\overline{M})$$

$$= \overline{H}_{n-*-1}(\mathcal{C})$$

So $\overline{H}_*(\mathcal{C})$ concentrated in degree $m \Rightarrow H^*_c(M)$ concentrates in degree $n-m-1$. $\square$

3. The classical examples:

a) Arithmetic gps:

$$G = SL(n, \mathbb{Z})$$

$$M = SO(n) \backslash SL(n, \mathbb{R})$$

$$N = M / G$$

$\mathcal{C}$ = spherical bldg for $GL(n, \mathbb{Q})$

= $(n-2)$-dim simpl. cx

Simpl$(\mathcal{C})$ = chains of subspaces of $\mathbb{Q}^n$.

Vert$(\mathcal{C})$ = {max parabolic subgps}

i.e. conjugate of

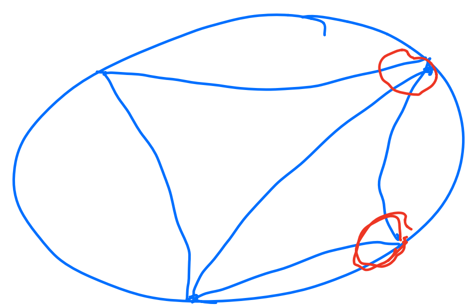
$$\left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right]$$

Codim 1 face

= (Sym space) $\times$ (Sym space) $\times$ Nilpotent Lie gp

for example $SL(2, \mathbb{Z})$

N



M

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b) Teichmuller Space

G = MCG

M = Teichmuller space

N = Moduli space

$\mathcal{C}$ = curve cx

$\mathrm{Vert}(\mathcal{C})$ = $\{$ Isotopy classes of simple closed curves

codim 1 face of $\overline{M}$ = Teichmuller space for model surface $\times \mathbb{R}$

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c) Pure Spherical Artin gps & hyperplane Arrangements

W = finite Coxeter gp

$W \curvearrowright \mathbb{R}^n$

in $W \curvearrowright \mathbb{C}^n$

$\mathcal{A} = \{$ reflecting hyperplanes in $\mathbb{C}^n \}$

$$N = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} H$$

or $$S^{2n-1} - \bigcup_{H \in \mathcal{A}} H$$ to make N compact

$A = \pi_1(N/W)$

$PA = \pi_1(N)$

M = univ. cover of N

<u>Goal</u>: Describe $\overline{N}$, $\overline{M}$ & $\mathcal{C}$.

4) <u>Blowing up hyperplane arrangements</u> (Deleting tubular nbhds)

$\mathcal{A}$ = complexification of real simplicial arrangement in $V = \mathbb{C}^n$

Subspace = intersection of hyperplanes in $\mathcal{A}$.

$Q(\mathcal{A}) = \{\text{subspaces}\}$ = intersection poset

For $F \in \mathcal{C}$

$$\mathcal{A}_E = \{ H \in \mathcal{A} \mid E \leq H \}$$

= arrangement on V/E

= "normal arrangement"

$$\mathcal{A}^E = \{ E \cap H \mid H \in \mathcal{A} - \mathcal{A}_E \}$$

= arrangement in E

$$V_0 = V - \bigcup_{H \in \mathcal{A}} H$$

$$S(V/E)_0 = S(V/E) - \bigcup_{H \in \mathcal{A}_E} S(H/E)$$

$\pi_1(S(V/E)_0)$ = parabolic subgp.

$\mathcal{E}$ = family of subspaces

Delete Tubular nbhds

of $E \in \mathcal{E}$, $\boxed{V_{\odot} = V - \bigcup_E \text{tubular nbhds of } E}$

Codim 1 faces will have

form: $E_{\odot} \times S(V/E)_{\odot}$

Fast def'n of blowup
following (De Concini Procesi)

$$\rho : V_\circ \to V \times \prod_{E \in \mathcal{E}} (S(V/E))$$

$$V_{\mathcal{O}} = \overline{\operatorname{Im} \rho} \, .$$

5) <u>Irreducible Subspaces</u>

<u>Def</u> $E \in Q(\mathcal{A})$ is <u>reducible</u> if $\mathcal{A}_E$ splits as a direct sum

$$V/E = V/F_1 \oplus V/F_2$$

$$\mathcal{A}_E = \mathcal{A}_{F_1} \oplus \mathcal{A}_{F_2} \, .$$

Def of irreducible complex $\mathcal{I}$

$\mathrm{Vert}\,\mathcal{I} = \{E \in Q(A) \mid E \text{ is irreducible}\}$
$\shortparallel$
$Q^{(0)}$

Two types of edges:

$\{E, F\}$.

comparable if $E < F$ or $F < E$

commuting: $E \cap F$ is reducible

$V/E \cap F = V/F \oplus V/E$

$\mathcal{I}$ = associated flag cx

Basically can define everything

$\overline{N} = V_{\odot}$ (or $SV_{\odot}$)

$G = PA = \pi_1(V_{\odot})$

$\overline{M}$ = univ. cover

$\hat{}^{(0)}$

For each $E \in \mathcal{J}^{(1)}$ we have

codim 1 face:

$$\partial_E \overline{N} = E_{\odot} \times S[V/E]_{\odot}$$

Could define $\mathcal{C}$

$$\mathcal{C}^{(0)} = \pi_0(p^{-1}(\partial_E \overline{N}))$$

$$= \{\text{codim 1 faces of } \overline{M}\}$$

So $\mathcal{C}/G = \mathcal{I}$

<u>Thm</u> $\overline{M}$ & $\mathcal{C}$ have all properties of BS compactification.

In particular, faces are contractible, $\mathcal{C} \sim \vee S^m$

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6) $\mathcal{C}$ as a curve ex

$$\mathcal{C}^{(0)} = \{\text{irreducible parabolics}\}$$
$$= \text{conjugates of } \pi_1(S(V/E)_\odot)$$
$$= \{\text{centers of irreducible parabolics}\}$$

$G \curvearrowright \mathcal{C}$ by conjugation

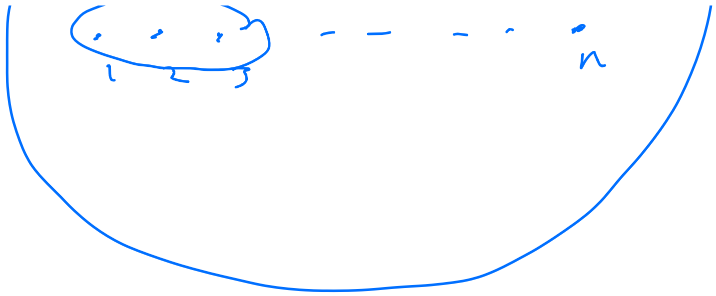
Stabilizer of vertex $=$ centralizer of center of P_E

$$\cong \pi_1(E_\odot) \times \pi_1(S(V/E)_\odot)$$

So $\mathcal{C}$ is a (simple) cx of gps $\mathcal{C}/G = \mathcal{I}$.

7) Braid groups

Relationship between curve cx for PB_n & $\mathcal{C}$ & $\mathcal{I}$



simple closed curve surrounds a set of indices $J \subset I^{(0)}$

= irreducible subspace $= \{ x_i = x_j \mid i, j \in J \}$

Normal rep = sub braid gp.

Pair of nested loops = 2 comparable elements

Disjoint loops $\longleftrightarrow$ reducible subspace