Cohomology of Coxeter groups and buildings

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HAPPY BIRTHDAY BOB!

Nonclassical buildings

We shall be concerned with buildings where the associated Coxeter system (W, S) is arbitrary, i.e., W need not be a spherical or Euclidean reflection group.

Such buildings arise in geometric group theory (e.g., when W is a “right-angled” Coxeter group) and in the theory of Kac-Moody groups over finite fields.

We want to compute

- $H^*(W; \mathbb{Z}W)$, or
- $H^*_c(\mathcal{U})$, where $\mathcal{U}$ is a suitable topological realization of a bldg Φ, or
- $L^2H^*(\mathcal{U})$ (when Φ is sufficiently thick), or
- $L^2_{\mathbf{q}}H^*(W)$ (when $\mathbf{q}$ is sufficiently large).
(Here the inner product depends on $\mathbf{q}$, a multiparameter of positive real numbers.)

The four answers are essentially the same. The proofs are also basically the same.

All answers have the same form:

$$H^*(\ ) = \bigoplus_{\substack{T \subset S \\ |W_T| < \infty}} H^*(K, K^{S-T}) \otimes \hat{A}^T.$$

Here (K, K^{S-T}) is a pair of finite simplicial complexes depending only on (W, S). $H^*(K, K^{S-T})$ denotes ordinary cohomology. $\hat{A}^T$ is an abelian gp or Hilbert space of functions on W or Φ.

Idea of proof

The cohomology gp on left can be computed by a non-constant coefficient system $\mathcal{I}(A)$ which associates to each simplex σ of K an abelian gp $A^{T(\sigma)}$.
The key is to prove that $\mathcal{I}(A)$ decomposes as $\bigoplus \mathcal{I}(\hat{A}^T)$.

Some history

When Φ is an irreducible affine bldg, K is a simplex Δ and $H_c^*(\mathcal{U})$ was calculated by Borel-Serre in 1976.

Theorem of Borel–Serre

If Φ is an irreduible affine bldg, then $H_c^*(\mathcal{U}(\Phi, \Delta))$ is free abelian and concentrated in the top degree ($= \dim \Delta$).

In contrast, to the Borel-Serre result, in the general case, $H_c^*(\mathcal{U}(\Phi, K))$ usually is not concentrated in a single degree.

Their method was to compactify $\mathcal{U}(\Phi, \Delta)$ by adding a spherical bldg at ∞.

They used this theorem to calculate the virtual cohomological dimension of S-arithmetic groups. The idea is that if a gp Γ acts properly and cocompactly on a contractible cell complex X, then $H^*(\Gamma; \mathbb{Z}\Gamma) \cong H_c^*(X)$.

more history

(D. - 1998) Computed $H^*(W;\mathbb{Z}W)$.
(DDJO - 2006) as a W-module.

(Dymara - Januszkiewicz 2002) Formula for ordinary L^2-cohomology, $L^2H^*(\mathcal{U}(\Phi, K))$, for very large thickness.

(D. - Meier 2002) Formula for $H^*_c(\mathcal{U}(\Phi, K))$ - mistake in proof.

(DDJO - 2007) Weighted $L^2_{\mathbf{q}}$-cohomology, $L^2_{\mathbf{q}}H^*(W)$, $\mathbf{q} >> 1$.
This also gave ordinary L^2-cohomology, $L^2H^*(\mathcal{U}(\Phi, K))$.

(DDJMO -2009) Correct proof for $H^*_c(\mathcal{U}(\Phi, K))$.

Books

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- M.W. Davis, J. Dymara, T. Januszkiewicz, J.Meier and B. Okun, *The compactly supported cohomology of buildings*, Comment. Math. Helv., to appear.
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- J. Dymara and D. Osajda, *Boundaries of right-angled hyperbolic buildings*, Fund. Math. **197** (2007), 123–165.
- J. Dymara and T. Januszkiewicz, *Cohomology of buildings and their automorphism groups*, Invent. Math. 150 (2002), 579–627.

Coxeter groups

A *Coxeter matrix* on a set S is an $S \times S$ symmetric matrix (m_{st}) with 1's on the diagonal and off-diagonal entries integers ≥ 2 or the symbol ∞. The associated *Coxeter group* is the group W with generating set S and relations:

$$(st)^{m_{st}} = 1$$

(Note: these relations imply that each $s \in S$ is an involution.)

The pair (W, S) is a *Coxeter system*.

For each $T \subset S$, $W_T := \langle T \rangle$. Also, W_s is the subgroup $\langle s \rangle$ of order 2 generated by s.

T is a *spherical* subset of S if $|W_T| < \infty$.
$\mathcal{S} :=$ the poset of spherical subsets.

Reduced expressions

Given a word $\mathbf{s} = (s_1, \dots, s_n)$ in S, its *value* is $w(\mathbf{s})$ is the element $w \in W$ defined by $w := s_1 \cdots s_n$.
$\mathbf{s}$ is a *reduced expression* for w if $l(w) = n$.

Chamber systems

Definition

A *chamber system* is a set Φ (of “chambers”) together with a family of equivalence relations indexed by some set S

Two chambers c and d are *s-adjacent* if they are s-equivalent and not equal.

A *gallery* in Φ is a finite sequence $(c_0, \dots, c_n)$ of adjacent chambers. It is *type* $(s_1, \dots, s_n)$ if c_i and c_{i-1} are s_i-adjacent. (So, its type is a word in S.)

Definitions

- Φ is *connected* if any two chambers can be connected by a gallery.
- Given a subset $T \subset S$, a gallery of type $(s_1, \ldots, s_n)$ is a *T-gallery* if each $s_i \in T$.
- A *residue* of *type* T is a T-gallery connected component of Φ.

Suppose (W, S) is a Coxeter system. A *bldg* of type (W, S) is

- a chamber system Φ over S st $\forall s \in S$, each s-equivalence class contains at least 2 elements.
- a *W-valued distance function* $\delta : \Phi \times \Phi \to W$ (st φ, φ' are connected by a minimal gallery of type $(s_1, \dots, s_n) \iff \delta(\varphi, \varphi') = s_1 \cdots s_n$).

Example

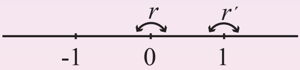
W has the structure of a chamber system over S.

- $v, w \in W$ are s-equivalent $\iff$ they belong to the same coset of W_s.
- A gallery in W is an edge path in $\mathrm{Cay}(W, S)$. (Here a "chamber" is a vertex of $\mathrm{Cay}(W, S)$.)
- Its type is a word in S.
- A T-residue is a coset of W_T in W.

Define $\delta : W \times W \to W$ by $\delta(w, v) = w^{-1}v$. This gives W the structure of a bldg (called the *thin building*).

Geometric and topological realizations

The notion of a Coxeter group arose from the study of (cocompact) reflection groups on spheres and Euclidean spaces. When the action is irreducible, the fundamental domain is a simplex Δ. We can reconstruct the action by pasting together copies of Δ, one for each element of the group W.



We will write $\mathcal{U}(W, \Delta)$ for the result of the pasting construction applied to the fund domain Δ

There are different ways to associate a topological space to a bldg Φ.

A *mirror structure* on a CW complex X is a family of subcomplexes $\{X_s\}_{s \in S}$. For each $x \in X$, put

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Define

$$\mathcal{U}(\Phi, X) = (\Phi \times X)/\sim$$

where $(c, x) \sim (c', x') \iff x = x'$ and c and c' belong to the same $S(x)$-residue.

Two choices for X stand out:

The classical chamber Δ

Given a Coxeter system (W, S), let Δ be a simplex of dimension $|S| - 1$ with its codim 1 faces indexed by S (ie, Δ_s denotes a codim 1 face of Δ). (Tits' original choice)

The standard chamber K

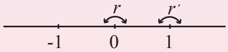
$\mathcal{S} :=$ the poset of spherical subsets of S.
$K :=$ the geometric realization of $\mathcal{S}$.
When W is infinite, K is the subcomplex of the barycentric subdivision of Δ spanned the barycenters of the faces of Δ st the corresponding subgp of W is finite.

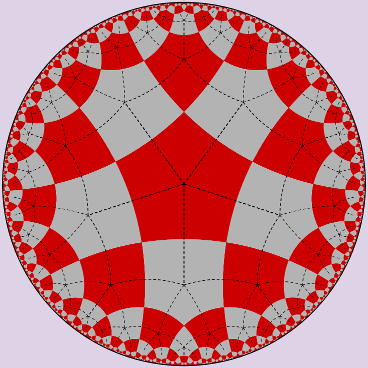
$$K_s := K \ \cap \ \Delta_s.$$

The standard realization (when $X = K$) is the most important for geometric group theory. The reason is that if $\Gamma \subset \mathrm{Aut}(\Phi)$ is a cocompact lattice, then $H^*_c(\mathcal{U}(\Phi, K)) = H^*(\Gamma; \mathbb{Z}\Gamma)$. Moreover, the cohomology of Γ with $\mathbb{Z}\Gamma$ coefficients tells us a good deal about Γ, for example, its vcd ($=$ virtual cohomological dimension), its number of ends and if it is a duality gp.

Example (The edge set of a tree)

Let $\mathcal{T}$ be a tree w/o terminal vertices. Any tree is bipartite. Call the colors s, t. Let W be the infinite dihedral group on $\{s, t\}$. Then $\Phi = \text{Edge}(\mathcal{T})$ is a building of type $(W, \{s, t\})$.
The W-distance between b,$c \in \Phi$ is defined as follows. Take the edge path w/o backtracking from b to c. Its type gives a word in $\{s, t\}$ and hence, an element $w \in W$ and $\delta(b, c) = w$.
$K = \Delta =$ an interval. $\mathcal{U}(W, K)$ is the real line. $\mathcal{U}(\Phi, K)$ is the original tree $\mathcal{T}$.





$\mathcal{U}(W, K)$, with K a pentagon

Notation

Suppose $\{X_s\}_{s \in S}$ is a mirror structure. A *face* of X is a subset

$$X_T := \bigcap_{s \in T} X_s.$$

Given $U \subset S$,

$$X^U := \bigcup_{s \in U} X_s.$$

The abelian group A

Suppose Φ is a bldg of type (W, S). Let A be either

- {finitely supported $\mathbb{Z}$-valued functions on Φ}, ie, the free abelian group on Φ (when $\Phi = W$ this is the group ring $\mathbb{Z}W$), or
- The Hilbert space, {**R**-valued, L^2-functions on Φ}, or
- When $\Phi = W$ and $q \in (0, \infty)$, we can have $A = \{$"q-weighted" L^2-functions on $W\}$.

In second and third cases the associated cohomology groups have dimensions, ie, the "L^2-Betti numbers"

To fix ideas, let's concentrate on the case
$A = \{$finitely supported $\mathbb{Z}$-valued functions on $\Phi\}$

For each $T \subset S$, put

$$A^T := \{f \in A \mid f \text{ is constant on each } T\text{-residue}\}.$$

In the finitely supported case, note that $A^T = 0$ whenever T is not spherical.

Example (The thin bldg)

If $\Phi = W$, then $A = \mathbb{Z}W$ (the group ring) and $A^T = (\mathbb{Z}W)^{W_T}$ $(\cong \mathbb{Z}[W/W_T])$.

Suppose X is a CW complex with mirror structure $\{X_s\}_{s\in S}$. For each cell c, put

$$S(c) := \{s \in S \mid c \subset X_s\}.$$

Define a (nonconstant) coefficient system $\mathcal{I}(A)$ on X by

$$c \mapsto A^{S(c)}$$

If X is a finite complex and $X_T \ (= \bigcap_{s\in T} X_s)$ is empty when T is not spherical, then

$$\begin{aligned} H^*(X : \mathcal{I}(A)) &= H^*_c(\mathcal{U}(\Phi, X)), \text{ or} \\ &= L^2H^*(\mathcal{U}(\Phi, X)), \text{ as the case may be} \end{aligned}$$

$A^U \subset A^T$ whenever $U \supset T$. So, put

$$A^{>T} := \sum_{U \supsetneq T} A^U.$$

Fact

$A^T/A^{>T}$ is free abelian.

Let $\hat{A}^T$ be a complementary summand for $A^{>T}$ in A^T.

Main Theorem

Suppose X is a mirrored CW complex. Then

$$H^*(X;\mathcal{I}(A)) = \bigoplus_{T\in\mathcal{S}} H^*(X, X^{S-T}) \otimes \hat{A}^T.$$

(In the L^2 cases there is a restriction on the thickness of Φ.)

Corollary

$$H^*_c(\mathcal{U}(\Phi, K)) = \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes \hat{A}^T.$$

Decomposition Theorem

For any spherical subset T,

$$A^T = \bigoplus_{U \subset T} \hat{A}^U.$$

In particular, for $T = \emptyset$, we have

$$A = \bigoplus_{U \in \mathcal{S}} \hat{A}^U.$$

This leads to a decomposition of the coefficient system $\mathcal{I}(A)$ and a decomposition of cochain complexes

$$C^*(X;\mathcal{I}(A)) = \bigoplus_{T\in\mathcal{S}} C^*(X, X^{S-T}; \hat{A}^T).$$

where the coefficients on the RHS vanish on cells in X^{S-T} and are constant on the other cells.

In each of the cases (eg compactly supported or weighted L^2-cohomology) the proof of the Decomposition Theorem is different. In the case at hand, you need the following

Proposition

Let Δ be the simplex of dimension $n = \mathrm{Card}(S) - 1$. Then $\mathcal{H}^*(\Delta; \mathcal{I}(A))$ (which is $= H^*_c(\mathcal{U}(\Phi, \Delta^f)))$ is free abelian concentrated in dimension n.

(You also need some similar statements corresponding to certain subcomplexes of Δ.)

Why is the proposition plausible?

Let Δ^f denote the union of the spherical faces of Δ (ie, Δ^f is a simplex with some faces deleted).

When W is infinite, $\mathcal{U}(W, \Delta^f)$ can be identified with the image of the interior of a convex cone in real projective space, ie, $\mathcal{U}(W, \Delta^f) \cong \mathbf{R}^n$, where $n = \dim \Delta$. (So, $H^*(\Delta^f; \mathcal{I}(\mathbb{Z}W)) = H^*_c(\mathcal{U}(W, \Delta^f) = H^*_c(\mathbf{R}^n)$, which is concentrated in dimension n.

The proof of the proposition uses PL Morse theory and $\mathrm{CAT}(0)$ geometry. In particular, there is the following new result

Theorem

$\mathcal{U}(\Phi, \Delta^f)$ has a complete, piecewise Euclidean $\mathrm{CAT}(0)$ metric.