Coxeter Groups and CAT(0) metrics

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The plan:

- First, explain Gromov's notion of a nonpositively curved metric on a polyhedral complex. Then give a simple combinatorial condition for nonpositive curvature on a cubical cell complex.The condition is that the link of each vertex is a "flag complex."
- Next, explain a construction for producing examples of cubical complexes where the link of each vertex is arbitrary.
- Give applications in topology and geometric group theory.

1. CAT(0) geometry
   - The CAT(κ)-inequality
   - Polyhedra of piecewise constant curvature
2. Cubical complexes and simplicial complexes
   - Simplicial complexes
   - Cubical cell complexes
   - The complex P_L
3. A digression: moment angle complexes
4. Construction of examples
   - The basic construction
   - Cohomological dimension
   - Aspherical mflds not covered by $\mathbf{R}^n$
   - The reflection group trick

"$\mathrm{CAT}(0)$-space" is a term invented by Gromov.
Also, called "Hadamard space." Roughly, a space which is "nonpositively curved" and simply connected.

$$\begin{aligned} \text{C} &= \text{“Comparison” or “Cartan”} \\ \text{A} &= \text{“Aleksandrov”} \\ \text{T} &= \text{“Toponogov”} \end{aligned}$$

In the 1940's and 50's Aleksandrov introduced the notion of a "length space" and the idea of curvature bounds on length space. He was primarily interested in lower curvature bounds (defined by reversing the $\mathrm{CAT}(\kappa)$ inequality). He proved that a length metric on S^2 has nonnegative curvature iff it is isometric to the boundary of a convex body in $\mathbb{E}^3$. First Aleksandrov proved this result for nonnegatively curved piecewise Euclidean metrics on S^2, i.e., any such metric was isometric to the boundary of a convex polytope. By using approximation techniques, he then deduced the general result (including the smooth case) from this.

One of the first papers on nonpositively curved spaces was a 1948 paper of Busemann. The recent surge of interest in nonpositively curved polyhedral metrics was initiated by Gromov's seminal 1987 paper.

Definitions

Let (X, d) be a metric space. A path $c : [a, b] \to X$ is a *geodesic* if $d(c(s), c(t)) = |s - t|, \quad \forall\, s, t \in [a, b]$. (X, d) is a *geodesic space* if any two points can be connected by a geodesic segment.

Given a path $c : [a, b] \to X$, its *length*, $l(c)$, is defined by

$$l(c) := \sup\{\sum_{i=1}^{n} d(c(t_{i-1}, t_i)\},$$

where $a = t_0 < t_1 < \cdots t_n = b$ runs over all possible subdivisions.

For $\kappa \in \mathbf{R}$, $\mathbb{X}^2_\kappa$ is the simply connected, complete, Riemannian 2-manifold of constant curvature κ:

- $\mathbb{X}^2_0$ is the Euclidean plane $\mathbb{E}^2$.
- If $\kappa > 0$, then $\mathbb{X}^2_\kappa = \mathbb{S}^2$ with its metric rescaled so that its curvature is κ (i.e., it is the sphere of radius $1/\sqrt{\kappa}$).
- If $\kappa < 0$, then $\mathbb{X}^2_\kappa = \mathbb{H}^2$, the hyperbolic plane, with its metric rescaled.

A *triangle* T in a metric space X is a configuration of three geodesic segments (the "edges") connecting three points (the "vertices") in pairs. A *comparison triangle* for T is a triangle T^* in $\mathbb{X}^2_\kappa$ with the same edge lengths. When $\kappa \leq 0$, a comparison triangle always exists. When $\kappa > 0$, a comparison triangle exists $\iff$ $l(T) \leq 2\pi/\sqrt{\kappa}$, where $l(T)$ denotes the sum of the lengths of the edges. (The number $2\pi/\sqrt{\kappa}$ is the length of the equator in a 2-sphere of curvature κ.)

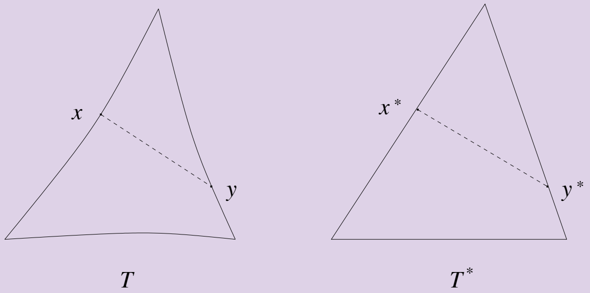
If T^* is a comparison triangle for T, then for each edge of T there is a well-defined isometry, $x \mapsto x^*$, which takes the given edge of T onto the corresponding edge of T^*.

A metric space X is a CAT(κ)-*space* if:

- If $\kappa \leq 0$, then X is a geodesic space, while if $\kappa > 0$, it is required there be a geodesic segment between any two points $< \pi/\sqrt{\kappa}$ apart.
- (*The* CAT(κ) *inequality*). For any triangle T (with $l(T) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and any two points $x, y \in T$, we have

$$d(x, y) \leq d^*(x^*, y^*),$$

  where x^*, y^* are the corresponding points in the comparison triangle T^* and d^* is distance in $\mathbb{X}^2_\kappa$.



$$d(x,y) \le d^*(x^*,y^*),$$

Definition

A metric space X *has curvature* $\leq \kappa$ if the $\mathrm{CAT}(\kappa)$ inequality holds locally.

Facts

- If $\kappa' < \kappa$, then $\mathrm{CAT}(\kappa') \implies \mathrm{CAT}(\kappa)$.
- $\mathrm{CAT}(0) \implies$ contractible.
- curvature $\leq 0 \implies$ aspherical.

Theorem (Aleksandrov and Toponogov)

A Riemannian mfld has sectional curvature $\leq \kappa$ *iff* $\mathrm{CAT}(\kappa)$ *holds locally.*

A convex polytope in $\mathbb{X}^n_\kappa$ is an $\mathbb{X}_\kappa$-*polytope*.

Definition

Suppose $\mathcal{F}$ is the poset of faces of a cell complex. An $\mathbb{X}_\kappa$-*cell structure* on $\mathcal{F}$ is a family $(C_F)_{F\in\mathcal{F}}$ of $\mathbb{X}_\kappa$-polytopes s.t. whenever $F' < F$, $C_{F'}$ is isometric to the corresponding face of C_F.

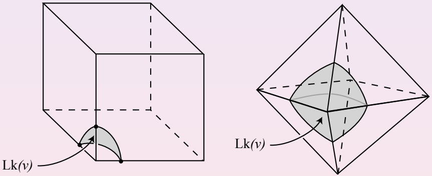
Suppose Λ is a $\mathbb{X}_\kappa$-polyhedral complex. A path $c : [a, b] \to \Lambda$ is *piecewise geodesic* if there is a subdivision $a = t_0 < t_1 \cdots < t_k = b$ s.t. for $1 \leq i \leq k$, $c([t_{i-1}, t_i])$ is contained in a single (closed) cell of Λ and s.t. $c|_{[t_{i-1}, t_i]}$ is a geodesic. The *length* of the piecewise geodesic c is defined by $l(c) := \sum_{i=1}^{n} d(c(t_i), c(t_{i-1}))$.
Λ has a natural length metric:
$d(x, y) := \inf\{l(c) \mid c \text{ is a piecewise geodesic from } x \text{ to } y\}$.

Definition

The geodesic space Λ is a *piecewise constant curvature polyhedron*. As $\kappa = +1, 0, -1$, it is, respectively, *piecewise spherical*, *piecewise Euclidean* or *piecewise hyperbolic*.

Piecewise spherical ($=$ *PS*) polyhedra play a distinguished role in this theory. In any $\mathbb{X}_\kappa$-polyhedral cx each "link" naturally has a *PS* structure.

- The link of a vertex v in a cell is the intersection of a small sphere about the vertex with the cell.
- For example, the link of a vertex in a cube is a (spherical) simplex.



Similarly, the link, Lk(v), of a vertex in a cubical cell complex is a simpicial cx.

Theorem

Let Λ be an $\mathbb{X}_\kappa$-polyhedral complex. TFAE

- $\mathrm{curv}(\Lambda) \leq \kappa$.
- $\forall v \in \mathrm{Vert}\,\Lambda), \quad \mathrm{Lk}(v, \Lambda)$ *is* $\mathrm{CAT}(1)$.

Definition

An affine *simplex* is the convex hull of a finite set T of affinely independent points in some Euclidean space. Its *dimension* is $\mathrm{Card}(T) - 1$. For S a finite set, Δ^S, the *simplex on* S, is the convex hull of the standard basis of $\mathbf{R}^S$ (where $\mathbf{R}^S := \{x : S \to \mathbf{R}\}$).

Example

A 1-*simplex* is an interval; a 2-*simplex* is a triangle; a 3-*simplex* is a tetrahedron.

Definition

An *abstract simplicial complex* consists of a set S (of *vertices*) and a poset $\mathcal{S}$ of finite subsets of S s.t.

- $\emptyset \in \mathcal{S}$,
- $\{s\} \in \mathcal{S}$, $\forall s \in S$.
- If $T \in \mathcal{S}$ and $U \subset T$, then $U \in \mathcal{S}$.

Definition

A (geometric) *simplicial complex* is a cell complex in which all cells are geometric simplices.

Suppose L is a geometric simplicial cx. Put

$$S := \mathrm{Vert}(L), \quad \text{and}$$
$$\mathcal{S}(L) := \{T \subset S \mid T \text{ is the vertex set of a simplex in } L\}.$$

Definition

A *geometric realization* of an abstract simplicial cx $\mathcal{S}$ is a geometric simplicial cx L s.t. $\mathcal{S} = \mathcal{S}(L)$.

Theorem

Every abstract simplicial cx $\mathcal{S}$ has a geometric realization.

Given a set S and a function $x : S \to \mathbf{R}$, $\text{Supp}(x) := \{s \in S \mid x_s \neq 0\}$. $\mathbf{R}^S$ denotes the Euclidean space of finitely supported functions $x : S \to \mathbf{R}$ and Δ^S is the simplex on S. We want to prove:

Theorem

Every abstract simplicial cx $\mathcal{S}$ has a geometric realization.

Proof.

Given $\mathcal{S}$, define a subcx $L \subset \Delta^S$ by

$$L := \{x \in \Delta^S \mid \text{Supp}(x) \in \mathcal{S}\} = \bigcup_{T \in \mathcal{S}} \Delta^T$$

Clearly, $\mathcal{S}(L) = \mathcal{S}$. □

All right simplicial complexes

Definition

A spherical simplex σ is *all right* if it is isometric to the intersection of *sphere*^{n} with the “quadrant”, $[0, \infty)^{n+1}$. Equivalently, σ is all right if each of its edge lengths is $\pi/2$ (or if each of its dihedral angles is $\pi/2$.

A simplicial complex L is *all right* if each of its simplices is all right. Thus, every simplicial complex has an all right PS structure (defined by declaring each of its simplices to be all right).

Definition

The *standard cube on a set S* is: $\square^S := [-1,1]^S \subset \mathbf{R}^S$. Its dimension is $\mathrm{Card}(S)$.

For each $T \subset S$, put

$$\square^T := [-1,1]^T \times \{0\}^{S-T} \subset \mathbf{R}^S.$$

The link of a vertex in a cube is a (spherical) simplex. Similarly, the link, $\mathrm{Lk}(v)$, of a vertex in a cubical cell complex is a simpicial cx.

Suppose v is a vertex in a cubical cell complex P.

- As an abstract simplicial complex $\mathrm{Lk}(v; P)$ is isomorphic to the poset of cells of P which contain v.
- A neighborhood of v in P is homeomorphic to the cone on $\mathrm{Lk}(v)$.

Definition

A simplicial cx L is a *flag complex* iff any finite set of vertices which are pairwise connected by edges spans a simplex of L.

Examples

- $\partial\Delta^n$ is not a flag cx for $n \geq 2$
- A k-gon (i.e. a triangulation of S^1) is a flag cx iff $k \geq 4$
- The barycentric subdivision of any cell complex is a flag cx. (This shows that the condition of being a flag cx does not restrict the topological type of L: it can be any polyhedron.)

Gromov's Lemma

A simplicial complex with its all right PS structure is $\mathrm{CAT}(1)$ $\iff$ *it is a flag complex.*

Corollary

A cubical complex with its natural piecewise Euclidean metric is nonpositively curved $\iff$ *the link of each vertex is a flag complex.*

Recall $\square^T := [-1,1]^T \times \{0\}^{S-T} \subset \mathbf{R}^S$.
A face of $\square^S$ is *parallel* to $\square^T$ if it has the form $[-1,1]^T \times \{\varepsilon\}$ for some $\varepsilon \in \{\pm 1\}^{S-T}$.

The cubical complex P_L

Given a simplicial complex L with vertex set S, define a subcomplex P_L of $\square^S$ by

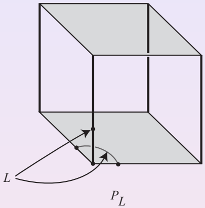
$$P_L := \bigcup_{T \in \mathcal{S}(L)} \text{all faces parallel to } \square^T$$

Main Property

For each vertex v, $\mathrm{Lk}(v, P_L) = L$

Example

Suppose S consists of 3 points and L is the union of an interval and a point. Then $\square^S$ is a 3-cube and P_L is indicated subcx.



More examples

- If $L = \Delta^{n-1}$, then $P_L = \square^n$
- If $L = \partial\Delta^{n-1}$, then $P_L = \partial\square^n = S^{n-1}$.
- If L is a set of n points, then P_L is the 1-skeleton of $\square^n$, eg, if $L = S^0$, then $P_L = \partial\square^2 = S^1$.

Joins

If T and U are disjoint sets, then $\Delta^T * \Delta^U = \Delta^{T\cup U}$. (Note: $\dim(\Delta^T * \Delta^U) = \mathrm{Card}(T \cup U) - 1 = \dim(\Delta^T) + \dim(\Delta^U) + 1$.) Similarly, If L_1 and L_2 are simplicial complexes, then $L_1 * L_2$ is defined by taking the joins of simplices in L_1 with those in L_2 (including the two empty simplices). We have: $\mathcal{S}(L_1 * L_2) = \mathcal{S}(L_1) \times \mathcal{S}(L_2)$

More

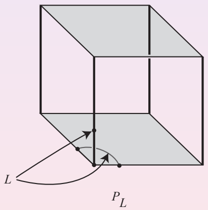
- $P_{(L_1 * L_2)} = P_{L_1} \times P_{L_2}$, eg, if $L = S^0 * S^0$, then $P_L = S^1 \times S^1 = T^2$, or if L is the n-fold join $S^0 * \cdots * S^0$ (the bdry of an n-dim octahedron), then $P_L = T^n$.
- If L is a k-gon, then P_L is the orientable surface of Euler characteristic $2^{k-2}(4-k)$

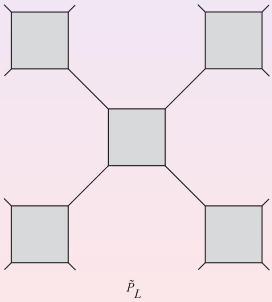
If L is a triangulation of S^{n-1}, then P_L is an n-mfld.

The $(\mathbb{Z}/2)^S$-action

- Let $\{r_s\}_{s\in S}$ be standard basis for $(\mathbb{Z}/2)^S$. Represent r_s as reflection on $\square^S$ across the hyperplane $x_s = 0$, ie, r_s changes the sign of the s^{th}-coordinate. This defines a $(\mathbb{Z}/2)^S$-action on $\square^S$.
- The subcomplex P_L is $(\mathbb{Z}/2)^S$-stable.
- $[0,1]^S$ is a (strict) fundamental domain for $(\mathbb{Z}/2)^S$-action on $\square^S \quad (= [-1,1]^S)$.
- $K := P_L \cap [0,1]^S$ is a (strict) fundamental domain for $(\mathbb{Z}/2)^S$-action on P_L.

The universal cover of P_L is denoted $\widetilde{P}_L$. The cubical structure on P_L lifts to one on $\widetilde{P}_L$.





- Let W_L be the gp of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{P}_L$. Let $\varphi : W_L \to (\mathbb{Z}/2)^S$ be the projection. We have the short exact sequence
- $$1 \to \pi_1(P_L) \to W_L \to (\mathbb{Z}/2)^S \to 1.$$
- $(\mathbb{Z}/2)^S$ acts simply transitively on $\mathrm{Vert}(P_L)$, so, W_L acts simply transitively on $\mathrm{Vert}(\widetilde{P}_L)$.
- Let $v \in K$ be the vertex $(1, \dots, 1)$. Choose a lift $\widetilde{K}$ of K in $\widetilde{P}_L$ (N.B. K is a cone) and let $\tilde{v}$ be the lift of v in $\widetilde{K}$. The 1-cells at v or $\tilde{v}$ correspond to elements of S. The reflection r_s flips the 1-cell at v labeled by s. Let $\tilde{s}$ be the unique lift of r_s which stabilizes the corresponding 1-cell at $\tilde{v}$. (Eventually, I will drop the ˜ from $\tilde{s}$.)

A presentation for W_L

- Since $\tilde{s}^2$ fixes $\tilde{v}$ and covers the identity on $\widetilde{P}_L$, it follows that $\tilde{s}^2 = 1$.
- Since W_L is simply transitive on $\mathrm{Vert}(\widetilde{P}_L)$, the 1-skeleton of $\widetilde{P}_L$ is the Cayley graph of $(W_L, \widetilde{S})$.
- Suppose $\{s, t\}$ is an edge of L. The corresponding 2-cell at $\tilde{v}$ has edges labeled successively by $\tilde{s}, \tilde{t}, \tilde{s}, \tilde{t}$. It follows that $(\tilde{s}\tilde{t})^2 = 1$.
- Since the 2-skeleton of $\widetilde{P}_L$ is simply connected, it is the Cayley 2-complex of a presentation. Therefore, W_L has a presentation with generating set $\widetilde{S} = \{\tilde{s}\}_{s \in S}$ and relations: $\tilde{s}^2 = 1$ and $(\tilde{s}\tilde{t})^2 = 1$, $\forall \{s, t\} \in \mathrm{Edge}(L)$.
- W_L is a *right-angled Coxeter group*.

Theorem

$\widetilde{P}_L$ is contractible iff L is a flag cx.

Proof.

Gromov: a cubical complex is $\mathrm{CAT}(0) \iff$ it is simply connected and the link of each vertex is a flag cx. Since the 2-skeleton of $\widetilde{P}_L$ is the Cayley two complex for (W_L, S), $\widetilde{P}_L$ is simply connected. □

Here is a generalization of the construction of P_L which has received a good deal of recent interest.

- Let (X, A) be a pair of spaces and L a simplicial cx with $\mathrm{Vert}(L) = S$. We define certain subspaces of the product $\prod_{s\in S} X$.
- For each $T \in \mathcal{S}(L)_{>\emptyset}$, let X^T be the set of $(x_s)_{s\in S}$ in the product defined by

$$\begin{cases} x_s \in X & \text{if } s \in T, \\ x_s \in A & \text{if } s \notin T. \end{cases}$$

- $$Z(L; X, A) := \bigcup_{T\in\mathcal{S}(L)_{>\emptyset}} X^T$$

Examples

- $(X, A) = ([-1, 1], \{\pm 1\})$. Then $Z(L; [-1, 1], \{\pm 1\}) = P_L$.
- $(X, A) = (S^1, \{1\})$. Then the fundamental gp of $Z(L; S^1, \{1\})$ is the right-angled Artin gp determined by the 1-skeleton of L. If L is a flag cx, then $Z(L; S^1, \{1\})$ is the standard $K(\pi, 1)$ for the Artin gp.
- $(X, A) = (D^2, S^1)$. Then $Z(L; D^2, S^1)$ is the *moment angle cx* of L. The group $(S^1)^S$ acts on $(Z(L; D^2, S^1)$. The quotient space is the same space K $(\subset [0, 1]^S)$ considered earlier. If K is a n-dim convex polytope and L is the bdry cx of its dual, then $Z(L; D^2, S^1)$ is a smooth mfld, and if T is an appropriate subgp of codim n in $(S^1)^S$, then $Z(L; D^2, S^1)/T$ is a "toric variety".

L is a flag cx with vertex set S and W_L is associated right-angled Coxeter gp. S is its fundamental set of generators..

The basic construction

A *mirror structure* on a space X is a family of closed subspaces $\{X_s\}_{s \in S}$. For $x \in X$, put $S(x) = \{s \in S \mid x \in X_s\}$. Define

$$\mathcal{U}(W, X) := (W \times X)/\sim,$$

where $\sim$ is the equivalence relation: $(x, w) \sim (x', w') \iff x = x'$ and $w^{-1}w' \in W_{S(x)}$ (the subgp generated by $S(x)$). $\mathcal{U}(W, X)$ is formed by gluing together copies of X (the *chambers*). The gp W_L $(= W)$ acts on it.

Another construction of $\widetilde{P}_L$

Recall $K := P_L \cap [0,1]^S$. For each $s \in S$, K_s is the intersection of K with the hyperplane $x_s = 0$. This is a mirror structure on K.

Theorem

The natural maps $\mathcal{U}((\mathbb{Z}/2)^S, K) \to P_L$ and $\mathcal{U}(W_L, K) \to \widetilde{P}_L$ are homeomorphisms.

The basic idea

The topology of the simplicial cx L is reflected in properties of the gp W_L.

- Suppose π is a torsion-free gp. Its *cohomological dimension*, $\mathrm{cd}(\pi)$ is defined to be the maximum integer k st $H^k(\pi; M) \neq 0$ for some π-module M.
- Its *geometric dimension*, $\mathrm{gd}(\pi)$ is the smallest dimension of a $K(\pi, 1)$ complex. Obviously, $\mathrm{cd}(\pi) \leq \mathrm{gd}(\pi)$.
- Eilenberg-Ganea proved equality if $\mathrm{cd}(\pi) \geq 3$ and Stallings, Swan proved it for $\mathrm{cd}(\pi) = 1$.

The Eilenberg-Ganea Problem

Is there a gp π with $\mathrm{cd}(\pi) = 2$ and $\mathrm{gd}(\pi) = 3$?

Conjectured answer

Yes.

- Suppose L is a flag triangulation of an acyclic 2-complex with $\pi_1(L) \neq 0$. Put $\pi_L := \pi_1(P_L) = \mathrm{Ker}(\varphi : W_L \to (\mathbb{Z}/2)^S$.
- π_L is torsion-free. It is a conjectured Eilenberg-Ganea counterexample.
- Put $\partial K := K - \overset{\circ}{K}$. In our case it is acyclic. It follows that $\mathcal{U}(W_L, \partial K)$ is acyclic (but not simply connected). Hence, $\mathrm{cd}(\pi_L) = 2$.
- $\dim \widetilde{P}_L = \dim L + 1 = 3$ and the only contractible complex which π_L seems to act on is $\widetilde{P}_L$ $(= \mathcal{U}(W_L, K))$.

Remark

Brady, Leary, Nucinkis proved these W_L are counterexamples to the version of the Eilenberg-Ganea Problem for groups with torsion.

Example (Different cd over $\mathbb{Z}$ than $\mathbb{Q}$)

- Suppose L is a flag triangulation of $\mathbf{R}P^2$. Then
- $H^3(\pi_L; \mathbb{Z}\pi_L) = H^3(\widetilde{P}_L; \mathbb{Z}) = H^2(\mathbf{R}P^2) = \mathbb{Z}/2$.
- $H^3(\widetilde{P}_L; \mathbb{Q}) = 0$ and $H^2(\widetilde{P}_L; \mathbb{Q})$ is a countably generated $\mathbb{Q}$ vector space. Hence,
- $\mathrm{cd}_{\mathbb{Z}}(\pi_L) = 3$ and $\mathrm{cd}_{\mathbb{Q}}(\pi_L) = 2$

Facts

- A closed m-mfld M^m, $m \geq 3$, with the same homology as S^m need not be homeomorphic to S^m, because it need not be simply connected. However,
- If $\pi_1(M^m) = 1$, then $M^m \cong S^m$ (Poincaré Conjecture).
- Similarly, a contractible open mfld Y^m, $m \geq 3$, is homeomorphic to $\mathbf{R}^m$ iff it is simply connected at ∞. (Stallings, Freedman Perelman).
- Every such homology m-sphere M^m (simply connected or not) bounds a contractible $(m+1)$-mfld.

- Suppose L^{n-1} is a non simply connected homology $(n-1)$-sphere triangulated as a flag cx.
- Then $\widetilde{P}_L$ is a contractible n-dim homology mfld (the non manifold points are the vertices) and $\widetilde{P}_L$ is not simply connected at ∞. (Its fundamental gp at ∞ is the inverse limit of free products of an increasing number of copies of $\pi_1(L)$.)
- $\widetilde{P}_L$ can be modified to be a contractible n-mfld. Let C be a contractible n-mfld bounded by L $(= \partial K)$. Remove $\overset{\circ}{K}$ and replace it by $\overset{\circ}{C}$. Then $Y^n := \mathcal{U}(W_L, C)$ is a contractible n-mfld $\not\cong \mathbf{R}^n$ and $M^n := Y^n/\pi$ (where $\pi = \pi_1(P_L)$) is a closed aspherical mfld with universal cover Y^n.

Reflection Group Trick

- Given a group π which has a finite $K(\pi, 1)$ complex, this is a technique for constructing an aspherical mfld M which retracts back onto $K(\pi, 1)$. (*Aspherical* means its universal cover is contractible.) In a nutshell the trick goes as follows:
- Thicken $K(\pi, 1)$ to a X, a compact mfld with bdry. (X is homotopy equivalent to $K(\pi, 1)$.)
- Put $L := \partial X$. Triangulate L as a flag cx and let W $(= W_L)$ be the corresponding right-angled Coxeter gp. As before modify P_L to a mfld by removing each copy of $\overset{\circ}{K}$ and replacing it by $\overset{\circ}{X}$
- $M := \mathcal{U}((\mathbb{Z}/2)^S, X)$ is the desired aspherical mfld.

Sample applications

Point: many interesting gps have finite $K(\pi, 1)$-complexes (even 2-dimensional ones).

- By choosing π a Baumslag-Solitar gp, we can get $\pi_1(M)$
  - to be non-residually finite, or
  - to have an infinitely divisible subgp ($\cong \mathbb{Z}[1/2]$).
- By choosing π to have unsolvable word problem (can do this with a 2-dim $K(\pi, 1)$), we get $\pi_1(M)$ with unsolvable word problem.

$M := \mathcal{U}((\mathbb{Z}/2)^S, X)$.

- As before, we can construct $\mathcal{U}(W_L, X)$. It is not contractible. But it is aspherical. Hence, M is aspherical (since it is covered by $\mathcal{U}(W_L, X)$). (Pf: It is a union of copies of X glued together along contractible pieces.)
- $\widetilde{M} =$ (univ cover of M). We can explicitly describe $\widetilde{M}$ as follows. Let $\widetilde{X} =$ (univ cover of X) and $\pi = \pi_1(X)$.
- $\widetilde{L}$ is the induced triangulation of $\partial\widetilde{X}$ and $\widetilde{S} = \text{Vert}(\widetilde{L})$. $\widetilde{W}$ $(= W_{\widetilde{L}})$ the corresponding right-angled Coxeter gp. Give $\widetilde{X}$ the induced mirror structure (indexed by $\widetilde{S}$). Then $\widetilde{M} = \mathcal{U}(\widetilde{W}, \widetilde{X})$.

The gp $\widetilde{W} \rtimes \pi$ acts on $\mathcal{U}(\widetilde{W}, \widetilde{X})$ with quotient space X and if Γ is the inverse image of the commutator subgp of W_L in $\widetilde{W}$, then $\Gamma \rtimes \pi$ acts freely with quotient space M.

References

- M.W. Davis, Exotic aspherical manifolds, *School on High-Dimensional Topology*, ICTP, Trieste, 2002.
- M.W. Davis, *The Geometry and Topology of Coxeter Groups*, London Math. Soc. Monograph Series, vol. 32, Princeton Univ. Press, 2007.
- M.W. Davis *The cohomology of a Coxeter group with group ring coefficents*, Duke Math. J. **91** (1998), 297–314.