Cohomology computations for Coxeter groups and their relatives

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Poznan
June 16, 2010

Manor Kindergarten 1954-55

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We want to compute $H^*(G; M)$ or possibly $H^*_G(Y; M)$ for M a G-module and Y a G-space for $M = \mathbf{Z}G$, or $\ell^2 G$, the square summable functions on G, or $\mathcal{N}(G)$, an associated von Neumann algebra, or a "Hecke - von Neumann algebra" used for "weighted ℓ^2-cohomology".

Of course, this is the same thing as cohomology of a space X (namely $X = BG$) with local coefficients in a $\pi_1(X)$-module M, where $H^*(X; M) := H^*_G(\tilde{X}; M)$ is the cohomology of $C^*_G(\tilde{X}; M) := \mathrm{Hom}_G(C_*(\tilde{X}), M)$.

Topological interpretation of $H^*(X; \mathbf{Z}G)$

Suppose X is compact (i.e., a finite complex). Then $H^*(X; \mathbf{Z}G) = H^*_c(\tilde{X})$. Even if the G-action on $\tilde{X}$ is only assumed to be proper, $H^*_G(\tilde{X}; \mathbf{Z}G) = H^*_c(\tilde{X})$. Similarly, $H^*_G(\tilde{X}; \ell^2 G)$ just means that we are using square summable cochains on $\tilde{X}$

Why are we interested in $\mathbf{Z}G$ coeffiicents?

- The rank of $H^1(G; \mathbf{Z}G)$ tells us the number of ends of G.
- Suppose $H^*(G; \mathbf{Z}G)$ is concentrated in a single degree, say n. Then G is a PD group $\iff$ $H^n(G; \mathbf{Z}G) = \mathbf{Z}$ and G is a *duality group* $\iff$ $H^n(G; \mathbf{Z}G)$ is torsion-free.

Example

$H^*(\mathbf{Z}^n; \mathbf{ZZ}^n) = H^*_c(\mathbf{R}^n)$, which is concentrated in degree $* = n$, where it is $\cong \mathbf{Z}$.

Why are we interested in $\ell^2 G$ coefficients?

Because Hilbert G modules have a "dimension" with respect to the von Neumann algebra $\mathcal{N}(G)$. Hence we can define ℓ^2-Betti numbers:

$$\ell^2 b^i(Y, G) := \dim_{\mathcal{N}(G)} H^i_G(Y; \ell^2 G).$$

Example

If G is a (higher genus) surface gp, then $H^*(G; \ell^2 G) = H^*(\mathbf{H}^2; \ell^2 G)$ which is concentrated in degree 1 and $\ell^2 b^1(G) = -\chi(G)$.

Which groups G are we interested in?

- Coxeter groups
- Artin groups
- Bestvina-Brady groups
- graph product of groups.

1. Introduction

2. The groups
   - Coxeter groups
   - Artin groups
   - Graph products
   - Bestvina-Brady groups

3. Computations
   - Some previous results
   - Graph products
   - Artin groups and Bestvina-Brady groups
   - A spectral sequence

Coxeter groups

$M = (m_{st})$ a symmetric $S \times S$ matrix with 1's on the diagonal and off-diagonal entries integers ≥ 2 or ∞. (M is called a *Coxeter matrix*.)

$$W := \langle S \mid \ (st)^{m_{st}} \rangle_{(s,t) \in S \times S} \rangle$$

(W, S) is called a *Coxeter system*. W is *right-angled* (a RACG) if each off-diagonal $m_{st} = 2$ or ∞.

Notation

$$\mathcal{S} := \{T \subset S \mid |W_T| < \infty\}$$
$$= \text{ the poset of } \textit{spherical subsets}$$

$L = L(W, S)$ is the *nerve* of (W, S), ie, the simplicial complex with vertex set S and simplices the nonempty elements of $\mathcal{S}$.
$K =$ geometric realization of $\mathcal{S} \cong$ the cone on L.
$K_s =$ the geometric realization of $\mathcal{S}_{\geq\{s\}} \cong \mathrm{Cone}(\mathrm{Lk}(s))$, where $\mathrm{Lk}(s)$ denotes the link of s in L.

$$K^{S-T} := \bigcup_{s \in S-T} K_s, \quad \partial K := K^S, \quad K_T := \bigcap_{s \in T} K_s$$

Artin groups

As before, (m_{st}) is a Coxeter matrix. Introduce generators $\{g_s\}_{s \in S}$ and for each $s \neq t$ with $m_{st} < \infty$, relations

$$g_s g_t \cdots = g_t g_s \cdots$$

setting equal the alternating words of length m_{st}. (NB each generator g_s has infinite order.) The result is the *Artin group* A. Let W be associated Coxeter gp. There is a a certain cell cx X' on which W acts freely. $X := X'/W$ is the *Salvetti cx*. $\pi_1(X) = A$.

The $K(\pi, 1)$-Conjecture

$X = BA$ (ie X is a $K(A, 1)$).

Definition

If each $m_{st} = 2$ or ∞, then A is *right-angled* (a RAAG).

Example

If A is a RAAG, then X is a certain union of subtori of T^S and the $K(\pi, 1)$-Conjecture is true.

The setup

Γ a graph with $\text{Vert}(\Gamma) = S$; L the flag cx determined by the graph and (W, S) the *RACS* with nerve L. Let $\{X_s\}_{s \in S}$ be a family of pointed spaces. Their *polyhedral product* is defined by

$$\mathcal{T}_L X_s := \bigcup_{T \in \mathcal{S}} X_T$$

where $X_T = \prod_{s \in T} X_s \subset \prod_{s \in S} X_s$.

Let $\{G_s\}_{s \in S}$ be a family of groups. Their *graph product* G is defined by

$$G = \prod_{\Gamma} G_s := \pi_1(\mathcal{T}_L BG_s)$$

Example

- If each $G_s = \mathbf{Z}/2$, then $G = \prod_\Gamma G_s$ is a RACG.
- If each $G_s = \mathbf{Z}$, then G is a RAAG.

Bestvina-Brady groups

Let A_L be the RAAG associated to a flag cx L. Let $\varphi : A_L \to \mathbf{Z}$ send each standard generator to 1. The *Bestvina-Brady group* is $BB_L := \operatorname{Ker} \varphi$.

Theorem (Bestvina-Brady)

If L is acyclic, then BB_L is type FP (or FL), but not finitely presented if $\pi_1(L) \neq 1$.

General form of the results

In every case, there is a Coxeter system (W, S) in the background. $\mathcal{S}$ is the poset of spherical subsets of S and K is the geometric realization of $\mathcal{S}$. There are explicit computations in almost all cases and they all have the same general form:

$$H^*(G; M) = \bigoplus_{\substack{T \in \mathcal{S} \\ p \leq *}} H^p(?, ?) \otimes M^{T,p},$$

where $(?, ?)$ is a pair of subcomplexes of K and $M^{T,p}$ is an abelian gp or G-module.

It turns out that there are two distinct possibilities for $(?, ?)$. In the first case (the locally finite case),

$(?, ?) = (K, K^{S-T})$, and there is no shifting of degrees in cohomology. (Remember $K^{S-T} = \bigcup_{s \in S-T} K_s$.) In the second case (the locally infinite case),

$(?, ?) = (K_T, \partial K_T)$,

and cohomology is shifted in degrees. (Remember $K_T = \bigcap_{s \in T} K_s$.)

Here

- ∂K_T is the (barycentric subdivision of) the link of the simplex T in L and $K_T = \text{Cone}(\partial K_T)$
- K^{S-T} (the union of mirrors indexed by $S - T$) is homotopy equivalent to the complement of the simplex T in L, and K is the cone on ∂K.

As an example of the first case:

Theorem (D)

$H^*(W; \mathbf{Z}W) = \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes M^T$, for a certain free abelian gp M^T.

Remarks

- (DDJMO) A similar formula holds for any locally finite bldg of type (W, S). (Jan Dymara will talk about this formula.)
- In particular since a graph product of finite groups is a locally finite *RAB*, a similar formula holds for such graph products.

The next two results are examples of the second case:

Theorem (D - Leary)

A the Artin gp associated to (W, S) and X its Salvetti cx. Then

$$H^*(X; \ell^2 A) = H^*(K, \partial K) \otimes \ell^2(A)$$

In particular, $\ell^2 b^i(X; A) = b^i(K, \partial K)$. If $K(\pi, 1)$-Conjecture holds for A, then we can replace the left hand side by $H^*(A; \ell^2 A)$.

I should be saying "reduced" ℓ^2-cohomology and writing $\mathcal{H}^*(X)$.

Theorem (Jensen-Meier)

If A is a RAAG, then

$$H^*(A;\mathbf{Z}A) = \bigoplus_{T\in\mathcal{S}} H^{*-|T|}(K_T,\partial K_T) \otimes \textit{free abelian gp}$$

This theorem was originally proved by using the first theorem and result of DJ that any *RAAG* is commensurable with a *RACG*.

Theorem

Suppose $G = \prod_{\Gamma} G_s$ is a graph product, where each G_s is infinite. Then

$$H^n(G; \mathbf{Z}G) = \bigoplus_{T \in \mathcal{S}} \bigoplus_{p+q=n} H^p(K_T, \partial K_T; H^q(G_T; \mathbf{Z}G))$$

Similarly,

Theorem

Still supposing each G_s is infinite,

$$\ell^2 b^n(G) = \sum_{T \in \mathcal{S}} \sum_{p+q=n} b^p(K_T, \partial K_T) \cdot \ell^2 b^q(G_T)$$

- Here G_T denotes the direct product $\prod_{s\in T} G_s$. So, ignoring torsion
$$H^*(G_T;\mathbf{Z}G_T) = \bigotimes_{\sum i_s=*} H^{i_s}(G_s;\mathbf{Z}G_T)$$
- I should be putting a Gr in front of the LHS for “associated graded”.

Artin groups

Suppose

- $A = A_L$ is the Artin group associated to (W, S), and X_L is the associated Salvetti complex.
- For each $T \subset S$, A_T is the subgp generated by T. When T is spherical $H^*(A_T; \mathbf{Z}A_T)$ is free abelian and concentrated in degree $|T|$ (ie A_T is a duality gp)

Theorem

$$H^n(X_L; \mathbf{Z}A_L) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes H^{|T|}(A_T; \mathbf{Z}A_L)$$

Bestvina-Brady groups

- Let A_L be the *RAAG* associated to the *RACS* (W, S), where $L =$ nerve of (W, S) (ie A_L is a graph product of $\mathbf{Z}$ s).
- $BB_L =$ kernel of $A_L \to \mathbf{Z}$ which sends each generator to 1.
- If L is acyclic, then BB_L is called a *Bestvina-Brady group*.

Theorem

Suppose BB_L is Bestvina-Brady. Then the cohomology of BB_L with group ring coefficients is isomorphic to that of A_L shifted up in degree by 1:

$$H^n(BB_L; \mathbf{Z}BB_L) = \bigoplus_{T \in \mathcal{S}_{>\emptyset}} H^{n-|T|+1}(K_T, \partial K_T) \otimes \mathbf{Z}(BB_L / BB_L \cap A_T).$$

L^2-cohomology of BB_L

Let $L^2 b^k(BB_L)$ be the k^{th} L^2-Betti number of BB_L.

Theorem

Suppose BB_L is Betvina-Brady. Then

$$L^2 b^k(BB_L) = \sum_{s \in S} b^k(K_s, \partial K_s)$$

where $b^k(K_s, \partial K_s)$ $(= \overline{b}^{k-1}(\mathrm{Lk}(s)))$ is the ordinary Betti number.

Idea of proof

- Suppose $\mathcal{P}$ is a poset, $\{X_a\}_{a\in\mathcal{P}}$ is a poset of spaces and $X = \bigcup_{a\in\mathcal{P}} X_a$
- There is a spectral sequence with

$$E_1^{p,q} = C^p(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^q(\mathcal{V}))$$

converging to $H^*(X)$ where the (nonconstant) coefficient system $\mathcal{H}^q(\mathcal{V})$ associates to a simplex $\sigma \in \mathrm{Flag}(\mathcal{P})$ the abelian gp $H^q(X_{\min a})$

- Want conditions to insure a decomposition:

$$E_2^{p,q} = E_\infty^{p,q} = \bigoplus_{a\in\mathcal{P}} H^p(\mathrm{Flag}(\mathcal{P}_{\le a}), \mathrm{Flag}(\mathcal{P}_{<a}); H^q(X_a))$$

Put $X_{<a} := \bigcup_{b<a} X_b$.

Main Lemma

The condition we need for this decomposition to hold is that $H^*(X_a) \to H^*(X_{<a})$ *is the* 0*-map,* $\forall a \in \mathcal{P}$

In all situations in which we will apply this lemma, $\mathcal{P} = \mathcal{S}$ so that $\text{Flag}(\mathcal{P}) = K$ and $\forall T \in \mathcal{S}$,

$$(\text{Flag}(\mathcal{P}_{\leq T}), \text{Flag}(\mathcal{P}_{<T})) = (K_T, \partial K_T).$$

The key point

for applying this to graph products is that when each G_s is infinite, $H^0(G_s; \mathbf{Z}G_s) = 0$, so by Künneth Formula, $H^*(G_T; \mathbf{Z}G_T) \to H^*(G_U; \mathbf{Z}G_T)$ is the 0-map whenever $U < T$.