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Schreve, Le                      arXiv

Group theory  $\subset$  Topology

Given a group  $G$  find a nice space on which  $G$  acts.

The classifying space  $BG$  is

a CW complex s.t

- $\pi_1(BG) = G$
- Its universal cover  $EG$  is contractible

$\exists$  such a  $BG$  for any  $G$ .

its universal property: Given any connected

Space  $X$  and homomorphism  $\varphi: \pi_1(X) \rightarrow G$

$\exists$  map  $f: X \rightarrow BG$  s.t

$f_* = \varphi: \pi_1(X) \rightarrow G$  and  $f$

is unique up to homotopy

$\therefore BG$  is unique up to homotopy equivalence.

$\therefore$  Properties of  $BG$  are properties of  $G$ .

### Examples

$G$	$BG$
$\mathbb{Z}$	$S^1$
$\mathbb{Z}^n$	$T^n$
$F_n$ . non abelian	

• in ' free gp	
$\pi_1(M)$ , $\text{sec}(M) \leq 0$	$M$
a discrete torsion-free subgp of $\text{Lie gp } G$	$K \backslash G / G$
a mapping class gp (torsion free subgp)	Moduli space on $T/G$
a RAAG	

RAAGs  $L'$  a simplicial graph

$L$  = associated flag complex  
 $(= \text{clique cx})$

Define group  $A_L$ .

Generators  $\{x_n\}_{n \in V(L')}$

Relations:

$$[x_n, x_\omega] = 1 \quad \text{if } \{n, \omega\} \in E(L')$$

Examples:  $\mathbb{Z}^n$  or  $F_n$

Its classifying space:

$$BA_L = \bigcup_{\sigma \in L} T^\sigma$$

where  $T^\sigma$  is torus generated by the  $S'_n$  ( $= T^n$ ),  $n \in \sigma$ .

Remark Not obvious that univ cover  $EA_L$  is contractible

Geometric dimension and

Action dimension

$gdim G$  = minimum dim of model for  $BG$

$actdim G$  = minimum dim of model for  $BG$  by a manifold  
(i.e. a thickening of  $BG$ )

On general principles

$$\boxed{\text{actdim } G \leq 2 \text{ gdim } G}$$

- If  $G = \pi_1$  (closed aspherical mfld) then

$$\text{gdim } G = \text{actdim } G$$

- Sometimes action of  $G$  on contractible mfld is God-given

- $G \subset \mathcal{G}$  torsion-free lattice  
then  $\text{actdim } G \leq \dim \mathbb{K}/\mathcal{G}$

- $G \subset \text{Mod } \mathcal{G}$ , then

$$\text{actdim } G \leq \dim (\text{Teschmiller space})$$

- $G = F_2$   $\text{gdim } G = 1$   
 $\text{actdim } G = 2$

## (Co)homology of groups

$C_i = C_i(\Sigma G)$  = cellular chain  $\propto$

$$C_{i+1} \rightarrow C_i \rightarrow C_{i-1} \rightarrow \dots \rightarrow C_0 \rightarrow \mathbb{Z} \rightarrow 0$$

free resolution of  $\mathbb{Z}$  by free  $\mathbb{Z}G$ -modules

$cd(G)$  = shortest length of projective resolution of  $\mathbb{Z}$ .

$$\text{So } cd(G) \leq gdim(G)$$

Eilenberg-Ganea Thm (+ Stallings)

$$cd(G) = gdim(G)$$

(Except possibly  $cd(G) = 2$   
 $gdim(G) = 3$ )

Remark. Almost certainly }  
counterexamples with  $\text{cd} = 2$   $\text{gdim} = 3$

### Simple complexes of groups

$Q$  a poset

$GQ$ : a functor from  $Q$  to  
category of gps &  
monomorphisms  
 $\sigma \longrightarrow G_\sigma$

Remark: Usually  $Q$  will be  
the poset of simplices  $\mathcal{S}(L)$   
(including  $\emptyset$ ) in simplicial  $\text{cx } L$ .

Suppose we have a model for  
each  $G_\sigma$ . We can glue  
these together to get a space

$BGQ$  (the "aspherical realization"  
of  $GQ$ )

If  $|Q|$  is simply connected, then

$$\pi_1(BGQ) = \varinjlim G_\sigma = G$$

Example: If  $G = A_L$ , a RAAC

and each  $G_\sigma = \mathbb{Z}^\sigma = \pi_1(T^\sigma)$   
 $GQ =$

$ADS(L)$  is a complex a GPS

and  $BADS(L) = \cup$  tori.

K( $\pi_1, 1$ )-Question for  $GQ$ :

Is  $BGQ \sim BG$ ?

(i.e., Is  $BGQ$  aspherical?)

If yes, then

- $\text{gdim } G \leq \dim BGQ$ .

- We can get upper bound for

actdim  $G^\circ$  by "finding mflds with bdry which are models for  $BG_\sigma$  & then glue them together to get a model for  $BG$  by a mfd with bdry.

### General Artin groups

Associated to a Coxeter system  $(W, S)$

$$\begin{aligned} \mathcal{S}(L) &= \left\{ \sigma \subset S \mid W_\sigma \text{ is finite} \right\} \\ &= \left\{ \text{spherical subsets} \right\} \end{aligned}$$

$$AS(L) = \text{cx of gps}$$

We know good models for  $BA_\sigma$  by CW complex & mfd

Don't know

$K(\pi, 1)$  Conj for Artin gps:

$$BAS(L) = BA$$

## Obstructors & obstructor dimension

(Bestvina-Kapovich-Kleiner 2003)

Ingredients:

- coarse embeddings of metric spaces
- van Kampen's obstruction  $\text{rk}^m(K)$  for embedding simplicial  $\text{cx } K$  in  $\mathbb{R}^m$  or  $S^m$

(a certain cohomology class with  $\mathbb{Z}_2$  coefficients)

Def'n :  $X, Y$  metric spaces

$f: X \rightarrow Y$   $\hookrightarrow$  coarse embedding

if  $\exists$  functions  $\rho_-, \rho_+: (0, \infty) \rightarrow [0, \infty)$

s.t.  $\rho_-(t) \leq \rho_+(t) \longrightarrow \infty$

$$\rho_-(d(x,y)) \leq d(f(x), f(y)) \leq \rho_+(d(x,y)) .$$

### Idea

- If  $\vee k^m(K) \neq 0$ , then  $K \notin S^m$  and  $\text{Cone}_\infty(K)$  does not coarsely embed in any contractible  $(m+1)$ -mfld.
- If in addition  $\text{Cone}_\infty(K)$  coarsely embeds in  $E\mathcal{G}$  (or in  $\mathcal{G}$ ), then  $\mathcal{G}$  cannot act properly on any contractible  $(m+2)$ -mfld.

So

$$\text{actdim } \mathcal{G} \geq m+2$$

(As  $K$  varies,  $\sup \{m+2\}$  is "obstructor dimension")

Example.  $\mathcal{G} = F_n$ .  $K = \mathbb{S}^n$

$$\text{act dim} = m+2 = 2$$

•  $G = F_2 \times F_2 \dots K = K_{3,3}$

$$\text{act dim} = d+1$$

•  $G = \underbrace{F_2 \times F_2 \times \dots \times F_2}_{d+1} K = K_{3,3 \dots 3}$

$$\text{act dim} = 2(d+1)$$

How do you choose  $K$ ?

Configurations of subgroups of  $G$ .

- If  $H \subset G$ , then inclusion is coarse embedding, as  $\hookrightarrow$   
 $EH \hookrightarrow EG$ .
  - . Collection of subgps  $\{H_\alpha\}$   
 indexed by simplices  $\sigma$   
 in some simplicial cx
- $$\bigcup EH_\alpha \xrightarrow{\text{coarse embedding}} EG$$

- If  $H = \mathbb{Z}^d$ , then  
 $EH = \mathbb{R}^d = \text{Cone}_\infty(S^{d-1})$
- A collection of free abelian subgps gives the cone on a configuration of spheres

Example  $G = A_L$ , the RAAG associated to flag  $\mathcal{L}$ .

$$\bigcup_{\sigma \in L} \mathbb{Z}^\sigma \subset A_L$$

$$\bigcup_{\sigma \in L} \mathbb{R}^\sigma \subset EA_L$$

$\text{Cone}_\infty(\partial L)$ , where

$\partial L$  is obtained from  $L$

by replacing each vertex by  $S^0$ .

Thm (ADOS). Suppose  $\dim L = d$ .

1) If  $H_d(L; \mathbb{Z}_2) \neq 0$ , then

$$\text{actdim } A_L = 2d+2 \quad (= 2g \dim A_L)$$

2) If  $L$  embeds in a  
contractible  $d$ -complex (or  
if  $H_d(L; \mathbb{Z}_2) = 0$ ), Then

$$\text{actdim } A_L \leq 2d+1.$$