Cohomology of random graph products

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I will discuss

Asymptotic properties which hold *asymptotically almost surely* ($=$ *a.a.s.*) for random right-angled Coxeter gps ($=$ RACGs) or more generally for random graph products of groups.

Let $\Gamma(n, p)$ denote the Erdős-Rényi random graph, i.e., $\Gamma(n, p)$ is the probability space of all graphs on vertex set $[n]$ where each edge is inserted with uniform probability p.

Theorem

Fix a positive integer $d > 1$. Suppose $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$. Then a.a.s. the dimension of the flag cx, $X(n,p)$, associated to the graph $\Gamma(n,p)$ is d. If $W(n,p)$ is the RACG *associated to $\Gamma(n,p)$ then a.a.s. $W(n,p)$ is a duality group over $\mathbb{Q}$ of dimension $\lfloor d/2 \rfloor + 1$.*

Random graphs

- $\Gamma(n,p) :=$
  $\{$graphs of vertex set $[n]$, each edge has probability $p\}$
- p induces a probability measure on $\Gamma(n,p)$ (where p will be given as a function of n).
- A property of graphs holds *a.a.s.* if the measure of the subset of $\Gamma(n,p)$ having the property $\to 1$ as $n \to \infty$. ("$\Gamma(n,p)$ has the property *a.a.s.*")

Theorem (Erdős-Rényi)

Suppose $p \geq \frac{\log n + \omega(n)}{n}$*, where* $\lim_{n\to\infty} \omega(n) = \infty$*. Then* $\Gamma(n,p)$ *is connected a.a.s.*
If $p \leq \frac{\log n - \omega(n)}{n}$*, then* $\Gamma(n,p)$ *is not connected a.a.s.*

If Γ is any simplicial graph, then it determines a simplicial cx X called its *flag complex* (or *clique complex*). The 1-skeleton of X is Γ; a simplex of X is a complete subgraph of Γ.
Let $X(n,p)$ denote the flag cx associated to $\Gamma(n,p)$ (Call $X(n,p)$ the *random flag cx*.)

Theorem

Fix a positive integer $d > 1$*. Suppose* $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$*. Then* $\dim X(n,p) = d$*, a.a.s.*

The Erdős-Rényi Thm is a computation of the 0^{th} Betti number of $\Gamma(n,p)$. It was extended to a calculation of the Betti numbers of $X(n,p)$ by Kahle in his thesis (based on ideas of Meshulam and Linial) and further extended in later work (based on ideas of Garland). Roughly, Kahle's result is the following.

Theorem (Kahle (in part with Hoffmann, Paquette))

Fix d. Suppose $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$. Then a.a.s.,

$$\widetilde{H}_k(X(n,p);\mathbb{Q}) \neq 0 \text{ iff } k = \lfloor d/2 \rfloor .$$

Graph products of groups

Let $\underline{G} = \{G_i\}_{i \in \mathbb{N}}$ be a sequence of nontrivial, discrete groups. Let Γ be a graph with vertex set $[n]$ (where $[n] := \{1, \dots, n\}$). Define a new group $\mathcal{G}$ $(= \mathcal{G}(\Gamma, \underline{G}))$, called the *graph product*, by taking the free product of the G_i, $i \in [n]$, and then imposing the relations that elements in G_i commute with elements of G_j whenever $\{i, j\} \in \text{Edge}(\Gamma)$. (We are mainly interested in case where G_i is the constant sequence $G_i = G$.)

Example

- If $G = \mathbb{Z}/2$, then $\mathcal{G}$ is the RACG associated to Γ.
- If $G = \mathbb{Z}$, then $\mathcal{G}$ is the RAAG associated to Γ.

With coauthors Dymara, Januszkiewicz, Okun and Okun separately I have been calculating the cohomology of graph products of groups with various coefficients. Today I will stick to coefficients in the group ring $\mathbb{Z}\mathcal{G}$ or $\mathbb{Q}\mathcal{G}$. The answers are in terms of the ordinary cohomology groups of the associated flag cx, $X = X(\Gamma)$, as well as, the cohomology of subcomplexes $X - \sigma$ or $\mathrm{Lk}(\sigma, X)$ where σ ranges over $\mathcal{S}(X)$, the poset of simplices in X (including the empty simplex). There are essentially 2 different formulas depending on whether all G_i are finite or all are infinite.

Remark

Of course, the reason for studying $H^*(\mathcal{G}; \mathbb{Z}\mathcal{G})$ is that it gives info about 1) the number of ends of $\mathcal{G}$, 2) $\mathrm{vcd}\,\mathcal{G}$, or 3) the question of whether $\mathcal{G}$ is a virtual duality gp.

Cohomology of graph products of finite gps

There are essentially 2 different formulas depending on whether all G_i are finite or all are infinite. Given a simplex σ, put $I(\sigma) := \{i \in [n] \mid i \text{ is a vertex of } \sigma\}$.

Theorem (DDJO)

Suppose each G_i is finite. Then, for $\mathcal{G} = \mathcal{G}(\underline{G}), \Gamma)$,

$$\operatorname{Gr} H^*(\mathcal{G}; \mathbb{Z}\mathcal{G}) = \bigoplus_{\sigma \in \mathcal{S}(X)} H^*(\operatorname{Cone} X, X - \sigma) \otimes \hat{A}^{I(\sigma)},$$

where $\hat{A}^{I(\sigma)}$ is a certain (free abelian) subgroup of $\mathbb{Z}(\mathcal{G}/G_\sigma)$ (where G_σ denotes the $I(\sigma)$-fold product of the G_i).

Cohomology of graph products of infinite gps

Theorem (D - Okun)

Suppose each G_i is infinite. Then

$$\operatorname{Gr} H^m(\mathcal{G}; \mathbb{Z}\mathcal{G}) = \bigoplus_{\substack{\sigma \in \mathcal{S}(X) \\ i+j=m}} H^i(\operatorname{Cone} \operatorname{Lk}(\sigma), \operatorname{Lk}(\sigma); H^j(G_\sigma; \mathbb{Z}\mathcal{G})),$$

where $\operatorname{Lk}(\sigma) = \operatorname{Lk}(\sigma, X(\Gamma))$.

Theorem (Kahle)

Fix $d > 1$ and suppose $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$. Then the random flag cx, $X = X(n,p)$, satisfies the following a.a.s :

- $\dim X = d$,
- *X is $\lfloor (d-2)/4 \rfloor$-connected (which implies* $\widetilde{H}_i(X;\mathbb{Z}) = 0$ *for $i > \lfloor d/2 \rfloor$), and*
- $\widetilde{H}_i(X;\mathbb{Q}) \neq 0$ *iff $i = \lfloor d/2 \rfloor$.*

Moreover, these properties hold a.a.s. for $X - \sigma$ for every simplex σ of X.

Remark

We don't know if $H_i(X;\mathbb{Z})$ can have torsion for $\lfloor (d-2)/4 \rfloor < i \leq \lfloor d/2 \rfloor$.

Putting together the results of DDJO with theorem of Kahle we get:

Theorem

Suppose Γ *is a random graph product of finite gps. As before,* $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$. *Then the following properties hold a.a.s.*

- *For* $i \leq \lfloor (d-2)/4 \rfloor + 1$ *or* $i > \lfloor d/2 \rfloor + 2$, $H^i(\mathcal{G}; \mathbb{Z}\mathcal{G}) = 0$ *a.a.s. Moreover, for* $i = \lfloor d/2 \rfloor + 1$, $H^i(\mathcal{G}; \mathbb{Z}\mathcal{G}) \neq 0$.
- $H^i(\mathcal{G}; \mathbb{Q}\mathcal{G}) \neq 0$ *iff* $i = \lfloor d/2 \rfloor + 1$ *(ie* $\mathcal{G}$ *is a duality gp over* $\mathbb{Q}$ *of formal dim* $\lfloor d/2 \rfloor + 1$*).*

Corollary

As before, suppose each G_i is finite. Then a.a.s

- $$\operatorname{Ends} \mathcal{G} = \begin{cases} \infty, & \text{if } d = 1; \\ 1, & \text{if } d \geq 2. \end{cases}$$
- *The cohomological dimension over $\mathbb{Q}$ of $\mathcal{G}$ is given by* $\operatorname{cd}_{\mathbb{Q}} \mathcal{G} = \lfloor d/2 \rfloor + 1$. *Over $\mathbb{Z}$, the virtual cohomological dimension of $\mathcal{G}$ is either $\lfloor d/2 \rfloor + 1$ (if $H_{\lfloor d/2 \rfloor}(X - \sigma)$ is torsion-free for all $\sigma \in \mathcal{S}(X)$) or $\lfloor d/2 \rfloor + 2$ (if $H_{\lfloor d/2 \rfloor}(X - \sigma)$ has nontrivial torsion).*

Links in random flag complexes

Lemma

Suppose k be a positive integer $< \lfloor d/2 \rfloor$. *Then a.a.s. for every* $(k-1)$*-simplex* τ *in* X, $\mathrm{Lk}(\tau, X)$ *is the random flag complex* $X(n', p')$ *where* $n' \sim p^k n$ *and* $p' = p$. *Hence, if* $\frac{1}{n^{2/d}} \ll p \ll \frac{1}{n^{2/(d+1)}}$, *then* $\frac{1}{(n)^{2/d'}} \ll p' \ll \frac{1}{(n)^{2/(d'+1)}}$, *where* $d' = d - 2k$.

Proof.

The expected degree at each vertex is $p(n-1)$. Hence, if $\dim \tau = k-1$, the expected number of vertices in $\mathrm{Lk}(\tau, X)$ is $n'\ p^k n$. Thus, *a.a.s.* $\mathrm{Lk}(\tau, X) = X(n', p)$. □

Suppose each G_i is infinite and $\mathcal{G} = \mathcal{G}(\Gamma(n,p), \underline{G})$ Putting the previous lemma together with the results of D-Okun gives:

Theorem

The following hold a.a.s.

- *For* $i \leq \lfloor (d-2)/4 \rfloor + 1$, $\widetilde{H}^i(\mathcal{G}; \mathbb{Z}\mathcal{G}) = 0$.
- *For* $i < \lfloor d/2 \rfloor + 1$, $\widetilde{H}^i(\mathcal{G}; \mathbb{Q}\mathcal{G}) = 0$ *and* $H^{\lfloor d/2 \rfloor + 1}(\mathcal{G}; \mathbb{Q}\mathcal{G}) \neq 0$.
- $$\mathrm{Ends}\, \mathcal{G} = \begin{cases} \infty, & \text{if } d = 1; \\ 1, & \text{if } d \geq 2. \end{cases}$$

Corollary

For the random right-angled Artin group A_Γ, the following properties hold a.a.s.

- $\mathrm{cd}(A_\Gamma) = d + 1$.
- $H^i(A_\Gamma; \mathbb{Q}\Gamma) = 0$ *for* $i < \lfloor d/2 \rfloor + 1$ *or* $i > d + 1$.
- $L^2 b_i(A_\Gamma)$ *is nonzero if and only if* $i = \lfloor d/2 \rfloor + 1$, *in which case,* $L^2 b_{\lfloor d/2 \rfloor + 1}(A_\Gamma) = b_{\lfloor d/2 \rfloor + 1}(\mathrm{Cone}\, X, X) = \tilde{b}_{\lfloor d/2 \rfloor}(X)$.

This is a method for proving the statement in Kahle's Thm with coefficients in $\mathbb{Q}$. Recall this is the statement

$$\widetilde{H}_i(X;\mathbb{Q}) \neq 0 \quad \text{iff} \quad i = \lfloor d/2 \rfloor .$$

The *normalized Laplacian* $\Delta : C^0(X;\mathbb{R}) \to C^0(X;\mathbb{R})$ is defined by

$$A(\varphi)(v) := \frac{1}{m(v)} \sum \varphi(w) \quad \text{and} \quad \Delta := 1 - A,$$

Δ is positive semidefinite and its eigenvalues lie in $[0,2]$.

Theorem (Garland, Ballmann-Świątkowski, Zuk)

Suppose X is a finite simplicial complex and k is a positive integer $< \dim X$. Assume that there is an $\varepsilon > 0$ so that for each $(k-1)$-simplex σ in X

- $\mathrm{Lk}(\sigma, X)$ *is connected, and*
- $\kappa_\sigma \geq \frac{k}{k+1} + \varepsilon$ *(where κ_σ denotes the smallest positive eigenvalue of the normalized Laplacian on $C^0(\mathrm{Lk}(\sigma, X); \mathbb{R})$).*

Then $H^k(X; \mathbb{R}) = 0$.

Theorem (Hoffman-Kahle-Paquette)

(Rough version) Let $\Gamma = \Gamma(n, p)$ *and let* κ_Γ *be the smallest positive eigenvalue of* Δ*. If* $\frac{\log n}{n} \ll p$*, then* $\kappa_\Gamma > 1 - \varepsilon$ *a.a.s.*