Betti numbers of residual towers of covers of reflection groups

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Fifteen years ago Boris Okun and I were trying to prove the Singer Conjecture for right-angled Coxeter groups (RACGs). One of our ideas was to use Lück’s Approximation Theorem. We had a specific tower of covers for which we could compute the Betti numbers. The only problem was we got the wrong answer: instead of proving vanishing results we were getting that all $\ell^2$-Betti numbers not in the top or bottom degree were nonzero! This was the wrong answer for hyperbolic space, $\mathbb{H}^n$, $n > 2$. Eventually we realized that the reason was that the subgroups were not normal.
1. Introduction
   - Towers of reflection groups
   - Small covers
   - Convex hulls in $\mathbb{H}^n$

2. The $h$-vector

3. Morse theory of small covers
Suppose $W$ is a group generated by reflections across the facets of a right-angled fundamental polytope $P$ on $\mathbb{H}^n$. Then $W$ is a RACG and $\mathbb{H}^n/W = P$.

More generally, given a RACG $W$, there is a Davis-Moussong complex $\Sigma$, tessellated by copies of a fundamental chamber $P$, so that $W$ acts as a reflection group on $\Sigma$. $\Sigma$ has the structure of a CAT(0) cube complex dual to the tessellation by chambers. (So, $\Sigma$ is contractible.) If $\Sigma$ is a $n$-manifold, then $P$ is “polytope-like”. We may as well assume $P$ is a simple polytope.
For $X \subset \Sigma$, let $\mathcal{H}(X)$ be the set of half-spaces of $\Sigma$ containing $X$ and put

$$\text{Conv}(X) := \bigcap_{H \in \mathcal{H}(X)} H.$$ 

If $X$ is finite, then $\text{Conv}(X) = P'$ is polytope-like and the subgroup $W' \leq W$ generated by reflections across the facets of $P'$ is a RACG.

It follows that $W$ is residually finite. (If we regard $W$ as the set of centers of chambers in $\Sigma$ and $X$ is a finite subset of $W$, then $X$ embeds in $W/W' \subset \Sigma/W' = P'$.)
So, we can find

\[ P = P_0 \subset P_1 \subset \cdots P_i \subset \cdots \quad \text{with} \quad \bigcup P_i = \Sigma. \]

where each \( P_i \) is the convex hull of a finite set. This gives a residual tower of RACGs

\[ W = W_0 > W_1 > \cdots W_i > \cdots \]

N.B. The subgroup \( W_i \) is *not* normal in \( W \).
Remark

Lück’s Approximation Theorem does not work in this generality. Indeed, since $P_i$ is contractible, $\overline{H}_*(W_i; \mathbb{Q}) = 0$. So, $b_k(W_i) = 0$ and similarly, for the normalized Betti numbers, $\frac{b_k(W_i)}{[W : W_i]} = 0$.

(Of course, if $\Sigma = \mathbb{H}^{2k}$, then $\beta_k^{(2)}(W) \neq 0$.)

At one point we thought the problem might be caused by the fact that the $W_i$ were not torsion-free. However, this was not the problem.
∃ torsion-free subgroups $\pi_i < W_i$ of index $2^n$ so that $M_i = \Sigma/\pi_i$ is an $n$-manifold with nonzero Betti numbers. The Betti numbers depend on the combinatorics of the polytope $P_i$. Moreover, the normalized Betti numbers do not limit to 0.

More detail

Given a RACG $W$ with fundamental polytope $P$, $\exists$ a homomorphism $W \to (\mathbb{Z}/2)^n$ with torsion-free kernel $\pi$ gives a small cover $M = \Sigma/\pi \to \Sigma/W = P$. These can occur as the real points of a toric variety over $P$. Calculating the cohomology of $M$ was the topic of my paper with Januszkiewicz twenty five years ago. It turns out that $M$ has a perfect cell structure (over $\mathbb{Z}/2$) in the sense of Morse theory with

$$\#\{k\text{-cells}\} = h_k(P),$$

where $(h_1, \ldots, h_n)$ is the so-called $h$-vector of $P$. 
When $W$ is a reflection group on $\mathbb{H}^n$ (and in many other cases) the $k^{th}$ normalized Betti number satisfies:

$$\frac{b_k(M_i; \mathbb{F}_2)}{[W : \pi_i]} \geq C > 0,$$

for all $k \neq 0, n$. In favorable cases we can replace $\mathbb{F}_2$ by $\mathbb{Z}$. So, the conclusion of Lück’s Theorem again fails but for a different reason!
Theorem (Benjami-Eldan 2012)

Suppose \( X \subset \mathbb{H}^n \) and \( \#X = N \).

\[
\text{vol}(\text{Conv}(X)) \leq C_n N
\]

In fact,

\[
C_n = \frac{2(2\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)}.
\]

Conv(\(X\)) is a polytope: we can take \(X\) to be its vertex set.
Let $P \subset \mathbb{R}^n$ be a simple polytope. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a generic linear form ("generic" means $f|_{\text{edge}} \neq \text{constant}$). Then $f$ induces an (upward-pointing) orientation on each edge of $P$. For each $v \in \text{Vert}(P)$, its index, $\iota(v)$, is defined by

$$\iota(v) = \#\{\text{inward-pointing edges at } v\},$$

and

$$h_k(P) := \#\{v \in \text{Vert}(P) \mid \iota(v) = k\}$$

- $h_0 = 1 = h_n, \quad \sum h_i = N = \# \text{Vert}(P)$
- Each $v \in \text{Vert}(P)$ of index $k$ determines a unique $k$-face $F_v$ of $P$ s.t. $v \in F_v$ and the maximum of $f$ on $F_v$ is at $v$. 
Suppose $P = P_0 < P_1 < \cdots P_i \cdots$ is the tower of reflection polytopes. Let

$m_i = \#\{\text{chambers in } P_i\}.$

If $P \subset \mathbb{H}^n$, then

$m_i = \text{vol}(P_i)/ \text{vol}(P_0).$

Let $N(P_i)$ be the number of vertices in $P_i$. By Benjamini-Eldar,

\[
\frac{N(P_i)}{m_i} \geq \frac{\text{vol}(P_0)}{C_n} = C > 0.
\]

In fact, for each $k \neq 0, n$, \( \frac{h_k(P_i)}{m_i} \) is bounded away from 0.
Proposition

For $k \neq 0, n$, $h_k \geq CN$

Proof.

$$f_i = \#\{\text{faces of codim } i + 1 \text{ in } P\}$$
$$= \#\{i\text{-simplices in the dual triangulation of } S^{n-1}\}$$

This follows from two facts:

- $f_{k-1} > Cf_{n-1} = CN$, where $C = 1/\binom{n}{k}$.
- The $h_i$ are linear combinations of the $f_i$, eg, $h_1 = f_0 - n$. 
This is based on a paper of D, Januszkiewicz from twenty five years ago.

**Small covers**

- $P$ a simple polytope, $\mathcal{F} = \{\text{facets of } P\}$ ($= \mathcal{F}(P)$)
- Let $\lambda : \mathcal{F} \to (\mathbb{Z}/2)^n - 0$ be a function such that if $F_1, \ldots, F_n$ meet at a vertex, then $\lambda(F_1), \ldots, \lambda(F_n)$ is a basis for $(\mathbb{Z}/2)^n$. $\lambda$ is called a *characteristic function*.
- The characteristic function induces a homomorphism $\overline{\lambda} : W \to (\mathbb{Z}/2)^n$ with torsion-free kernel $\pi$. Put $M = \Sigma/\pi$. The group $(\mathbb{Z}/2)^n \cong M$ with quotient $P$. The projection $p : M \to P$ is called a *small cover*. 
Suppose \( p : M \to P \) a small cover. Then \( P \subset M \) is a fundamental domain for \((\mathbb{Z}/2)^n\)-action.

Given a \( k \)-face \( F \) of \( P \), let \( M_F = p^{-1}(F) \). It is a \( k \)-manifold with \((\mathbb{Z}/2)^k\)-action and a small cover of \( F \).

Let \( \varphi = f \circ p : M \to P \to \mathbb{R} \), where \( f \) is the height function on \( P \). Then \( \varphi \) is a Morse function. The critical points are at the vertices of \( P \).

The index of the critical point at \( v \) is \( \iota(v) \).
The Morse function $\varphi : M \to \mathbb{R}$. Given $v \in \text{Vert}(P)$, let $F_v$ be the union of faces of $F$ which contain $v$. Put

$$C_v = (\mathbb{Z}/2)^k \tilde{F}_v,$$

where $k = \iota(v)$

Then $C_v$ is a $k$-cell, the *ascending submanifold at* $v$. Moreover,

$$\overline{C_v} = M_{F_v} := M_v$$

is a (possibly non-orientable) $k$-manifold.
Proposition

\( \varphi \) is perfect in the sense of Morse theory (homology with coefficients in \( \mathbb{F}_2 \)), i.e.,

\[
b_k(M; \mathbb{F}_2) = h_k(P).
\]

Proof.

Each \( \overline{C}_v (= M_v) \) is a manifold; hence, a mod 2 cycle. So, all incidence numbers are 0 mod 2.

Remark

If all \( M_v \) are orientable manifolds, then the above proposition is true with coefficients in \( \mathbb{Z} \).
The way to insure all the $M_F$ are orientable is to assume that the characteristic function $\lambda : \mathcal{F} \rightarrow (\mathbb{Z}/2)^n - 0$ has image lying in $\{e_1, \ldots, e_n\}$ (the standard basis). In other words, the facets of $P$ are colored by $n$ colors. $P$ may not always admit such a coloring; however, some simple polytopes do have such colorings. If $P$ has such a coloring, the orientability of the $M_F$ is assured. Also, colorability is inherited by towers $P > P_1 > \cdots$. 
Review of construction

- Start with increasing sequence of convex polytopes $P < P_1 < \cdots P_i < \cdots$, which exhaust $\Sigma$ and give a residual chain $W > W_1 > \cdots W_i > \cdots$, where 
  $[W : W_i] = m_i = \#$ of copies of $P$ in $P_i$. For each $i$, glue together $2^n$ copies of $P_i$ giving a manifold $M_i$ with fundamental group $\pi_i < W_i$ and a residual tower $M \leftarrow M_1 \leftarrow \cdots$ and a chain $W > \pi > \pi_1 > \cdots$.

- The normalized Betti numbers satisfy:
  \[
  \frac{b_k(M_i; \mathbb{F}_2)}{2^n m_i} = \frac{h_k(P_i)}{2^n m_i} \geq \frac{C'}{2^n} \geq C.
  \]

- In particular,
  \[
  \frac{1}{2^n} \sum \frac{b_k(M_i; \mathbb{F}_2)}{m_i} = \frac{1}{2^n} \frac{N(P_i)}{m_i} \geq \frac{C}{2^n}.
  \]