1. cell complexes & Cube complexes

2. CAT(0)-inequality

3. Link condition

4. Flag complexes and Gromov's Lemma

1. Cell complexes and cube complexes

Cell = convex polytope in some $\mathbb{E}^m$

Cell complex = X is space formed by gluing together cells via isometries of their faces

Δ-complex : each cell is a

(Hatcher) simplex

Cube cx: each cell is a Euclidean cube $[0,1]^k$

<u>Example</u> A graph is a 1-cube cx

- Any cell cx X has a "length metric" d where

$d(x,y) = \inf\{\ell(\gamma) \mid \gamma$ a path (= broken geodesic) from x to $y\}$

<u>Links</u> $F < P$ face of polytope P

$N(F,P)$ = normal cone of inward pointing normal vectors

$Lk(F,P)$ = sphere of $N(F,P)$

$Lk(F,P)$ is a spherical polytope

X = cell cx., and $\tau < \sigma$

a cell and its face, then

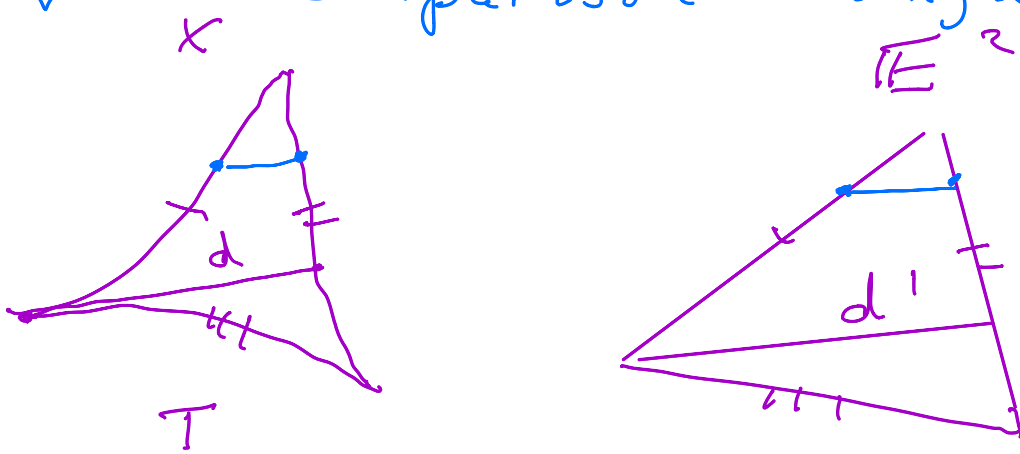
$Lk(\tau, \sigma)$ is spherical polytope

$$\boxed{Lk(\tau, X) = \bigcup_{\tau < \sigma} Lk(\tau, \sigma)}$$

2. CAT(0) - inequality

(X, d) = geodesic metric space

CAT(0) - inequality defined via comparison triangles



$$d \le d'$$

Def: X is CAT(0) '

if each triangle in X satisfies CAT(0)-inequality.

X is <u>NPC</u> if it satisfies CAT(0) locally.

Let (X, d) = complete geodesic space

<u>Thm:</u>

① NPC + $\pi_1(X) = 1 \iff$ CAT(0).

② CAT(1) defined using triangles in S^2 instead of $\mathbb{E}^2$.

$$\left(\begin{array}{l} \text{locally CAT(1)} + \\ \text{all geodesic loops} \\ \text{have } \ell \geq 2\pi \end{array}\right) \iff \text{CAT(1)}$$

Thm 2

$CAT(0) \Longrightarrow$ contractible

(geodesics are unique).

Cor NPC $\Longrightarrow$ aspherical

(meaning univ cover is contractible)

3. The link condition.

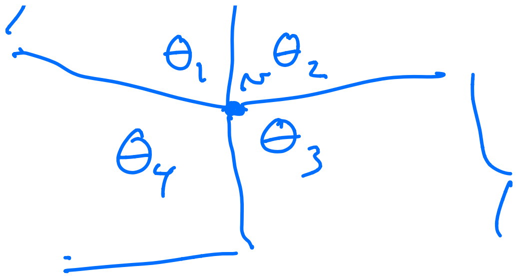
X = cell complex

Thm X is NPC $\Longleftrightarrow$

$\forall$ vertices v, $Lk(v, X)$ is

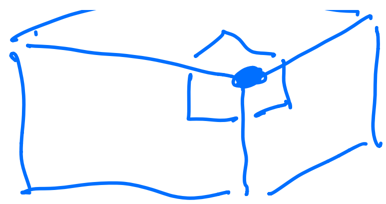
$CAT(1)$.

Ex X = polyhedral surface



Then $Lk(v,X)$ is S^1 of length $\sum \theta_i$. A surface X^2 is NPC $\iff$ the angle at each vertex is $\geq 2\pi$.

Further example Suppose each cell of X^2 is a Euclidean square, then X^2 is NPC $\iff$ At each vertex v there are > 3 squares



← NOT NPC

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4. <u>Flag complexes & Gromov's Lemma</u>

Goal:

<u>Thm.</u> X = cube cx. Then X is NPC $\Longleftrightarrow$ the link of each vertex is a flag cx.

<u>Def</u> A Δ-complex L is a <u>flag cx</u> if it is a simplicial cx and satisfies

(Flag) Any finite set of vertices of L that are pairwise connected by edges spans a simplex of L.

In other words, any clique in L^1 spans a simplex of L.

Remark: Gromov says L satisfies "no Δ-condition". I used to say "L is determined by its 1-skeleton". Term "flag cx" comes from Tits & incidence geometry.

- S^1

Ex a) If $L = m$-gon. Then L is flag $\Longleftrightarrow$ $m > 3$.

b) If P = poset and L is its order cx (i.e simplex is a chain), Then L is a flag cx. More generally a simplicial graph L' defines an "incidence relation" on Vert L'. If L = associated clique cx, then L is a flag cx (and conversely).

If X = cube cx and

$L_v = Lk(v, X)$. Then each simplex of L_v is an "all right spherical simplex": (meaning intersection of positive octant in $\mathbb{R}^n$ with S^{n-1})

<u>Gromov's Lemma</u>: Suppose L is an all right Δ-complex. Then L is CAT(1) $\iff$ L is a flag cx.

<u>Cor</u> a cube cx X is NPC $\iff$ link of each vertex is a flag cx.