1. Polyhedral Products

Def^{n} L = simp cx S = Vert L

$\underline{A} = \{(A_s, B_s)\}_{s \in S}$ collection

of pairs of subspaces,
$*_s \in B_s$ base point

$$\underline{A}^L \subset \prod_{s \in S} A_s \quad \text{s.t.}$$

(a) $x_s = *_s$ for all but finitely many s

(b) $\{s \in S \mid x_s \notin B_s\}$ is a simplex of L.

Examples.

(0) $(A_s, B_s) = ([0,1], 0)$

$\underline{A}^L = K_L$ = chamber

(1) $(A_s, B_s) = ([-1,1], \{\pm 1\})$

$\underline{A}^L = P_L$

(2) $(A_s, B_s) = (\mathrm{Cone}\, E_s, E_s)$

E_s a discrete set

$\underline{A}^L = Z_L$.

(3) $(A_s, B_s) = (S^1, *)$

$\underline{A}_L = T^L$

To get CAT(0) (NPC) examples

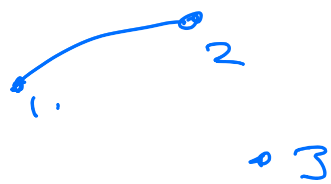
L should be a flag cx

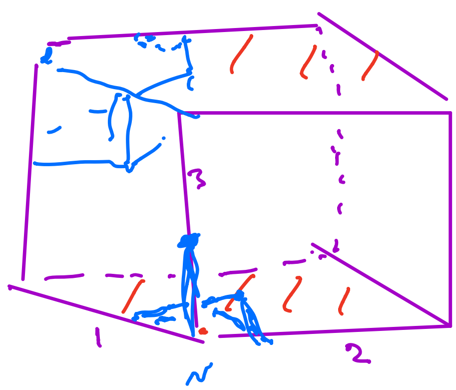
P_L : RACG's

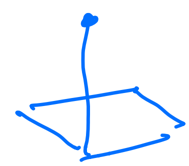
$(A_s, B_s) = ([-1,1], \{\pm 1\}) = (I, \partial I)$

$\prod A_2$ is a cube I^S

P_L is cubical sub cx of I^S

$L =$ 



$K_L =$ 

$Lk(v) = L$.

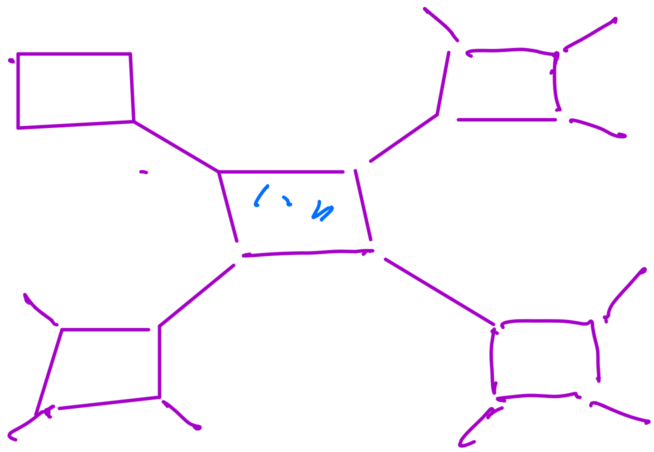
$C_2 = \mathbb{Z}/2 \qquad C_2^S \curvearrowright I^S$,

$C_2^S \curvearrowright P_L$

Fundamental domain is K_L

P_L is NPC $\iff$ L is flag (by Gromov's Lemma)

P_L = universal cover



$\widetilde{P}_L$ is CAT(0)

Let W_L = gp of all lifts of $(C_2)^S$ action to $\widetilde{P}_L$

<u>Claim</u> W_L is RACG.

<u>Pf</u> r_s = reflection on I^S

s = lift to $\widetilde{P}_L$ which fixes appropriate face $(x_s = 0)$ of a lift of K_L

$s^2 = 1$, $\{s, t\}$ = pair

$(st)^2 = 1 \iff I^{(s,t)}$ is a 2-cube in $\widetilde{P}_L$

$\iff \{s,t\} = \sigma^1 \in L.$

Presentation: generators $= S$

Relations

$s^2 = 1 \qquad \forall s$

$(st)^2 = 1 \qquad \{s,t\} \in \text{Edge } L.$

<u>Remarks</u>

1) If $L = S^{n-1}$, then P_L is an n-manifold.

2) Suppose $L = 4$-gon, then $W_L = D_\infty \times D_\infty$ and P_L is the standard flat T^2 embedded in $S^3 = \partial I^4$.

Suppose L = m-gon then P_L is a surface with

$$\chi(P_L) = 4\left(1 - \tfrac{1}{2}m + \tfrac{1}{4}m\right) = 4 - m$$

(This example is due to Coxeter)

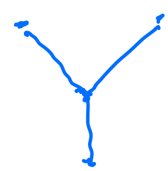
Z_L : RAB's

$$(A_s, B_s) = (\mathrm{Cone}\, E_s, E_s)$$

E_s a discrete set. $|E_s| > 1$

$$Z_L = \underline{A}^L = \underline{\mathrm{Cone}\, E}^L$$

$\mathrm{Cone}\, E_s$ =



Since $\prod \mathrm{Cone}\, E_s$ is cube complex, Z_L is a cube cx.

Lemma If L is flag cx, then Z_L is NPC

Pf: Based on 3 - facts:

- L flag cx $\Rightarrow Lk(\sigma)$ is flag cx
- Links in Z_L have form $Lk(\sigma) * \cancel{*} E_\lambda$.
- Join of flag complexes is a flag cx. $\square$

Remark: If each E_λ is a discrete gp G_λ, Then $G = \bigoplus G_\lambda$ acts on Z_L and K_L is fundamental domain.

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Universal Covers: Z_L = univ. cover

Thm. $\tilde{Z}_L$ is standard realization of a RAB associated to Coxeter gp W_L with fund chamber K_L. Each apartment is $\cong P_L$.

Graph Products of a family a gps $(G_s)_{s \in S}$ w.r.t a simpl. graph L^1. Form polyhedral product

$$Z_L = \underline{\text{Cone}\, G}^L \quad \text{where}$$

$$\underline{\text{Cone}\, G} = \{(\text{Cone}\, G_s, G_s)\}_{s \in S}$$

As before $\bigoplus_{s \in S} G_s \curvearrowright Z_L$

Let

G_L = gp of all lifts of

$\bigoplus G_s$ action to $\tilde{Z}_L$.

Presentation: G_L is quotient

of free product $\ast_{s \in S} G_s$

by relations:

$[g_s, g_t] = 1$ if $g_s \in G_s$, $g_t \in G_t$ and $\{s, t\} \in \text{Edge}\, L'$

Examples:

- If each $G_s = C_2$, then $G_L = W_L$, the RACG.

- If each $G_s = \mathbb{Z}$, then $G_L = A_L$ the RAAG

T_L : The classifying space of A_L

Put each $(A_s, B_s) = (S^1, \underset{*}{\overset{!}{\shortparallel}})$

and

$$T_L = \underline{A}^L \subset T^S = \prod_{s \in S} S^1$$

S^1 is a cube cx with 1 vertex and 1-edge.

Therefore, the product T^S is also a NPC cube cx.

The the link of vertex in S^1 is S^0. For $v \in \mathrm{Vert}(T^S)$

$Lk(v) = * S^0 =$ octahedron on Δ^S

Similarly.

$$Lk(v, T_L) = OL$$

$$= \bigcup_{\sigma \in L} O(\sigma) \leftarrow \text{octahedron on } \sigma$$

Exercise
<u>Lemma</u> L a flag cx $\Longrightarrow$ OL is flag cx.

$\therefore$ T_L is NPC cube cx

So $\widetilde{T}_L$ is CAT(0).

Obviously, $\pi_1(T_L) = A_L$

So, $T_L = BA_L$

Polyhedral Products of Classifying Spaces

Let $\{G_s\}_{s \in S}$ be a collection of gps. $(A_s, B_s) = (BG_s, *_s)$

[illegible]

and $X_L \subset \prod_{s \in S} BG_s$

corresponding polyhedral product

<u>Thm</u> 1) $\pi_1(X_L) = G_L$, the graph product of the BG_s.

2) $X_L \sim BG_L$

<u>Pf</u> Both statements follow by induction over subcomplexes of L

e.g. $\pi_1(X_L) = \lim_{\sigma \in L'} G_\sigma = G_L$

A different proof of 2) is as follows:

$X_L' := ((EG_s, G_s \ldots))^L$

[illegible]

Then $X_L' \longrightarrow X_L$ is a covering space. Also, since

$$f : (EG_n, G_n) \longrightarrow (\mathrm{Cone}\, G_n, G_n)$$

is a G_n-equivariant homotopy equivalence

it induces a homotopy equiv.

$$X_L' \longrightarrow Z_L$$

Taking universal covers we see that $\widetilde{X}_L \longrightarrow Z_L$ is h.e. $\therefore$ $\widetilde{X}_L$ is contractible.

- A different description of $\widetilde{P}_L$ $(= \Sigma_L)$

$$K = K_L \quad (= P_L \cap [0,1]^S)$$

For $s \in S$, put

$$K_s = K \cap \{t_s = 0\}$$

For $\sigma \in L$,

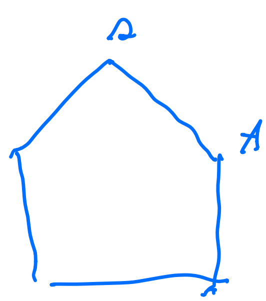
$$K_\sigma = \bigcap_{s \in \sigma} K_s$$

The stabilizer of K_σ is $C_2^\sigma \cong W_\sigma = \langle s \in \sigma \rangle$.

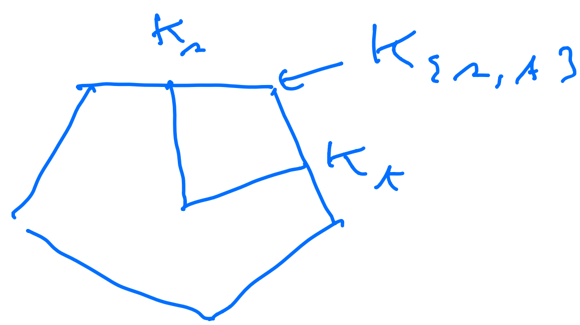
K_σ is "dual cone" of the simplex $\sigma \in L$

& $\partial K = \bigcup K_s \cong L$

<u>Ex</u> If L is a PL-triangulation of S^{n-1} then K_σ is dual cell to σ, & $K = \mathrm{Cone}(\partial K) \cong D^n$. We think of L as bdry cx of a simplicial polytope and K as the dual simple polytope.



L



K

For any $x \in K$. $\sigma(x) =$ largest σ

s.t $x \in K_\sigma$ $\left(\sigma = \{s \mid x_s = 0\}\right)$

<u>Alternate Description of</u> Σ_L

$$\mathcal{U} = (W \times K)/\sim$$

where $(w, x) \sim (w', x') \iff$ $x = x'$ and $wW_{\sigma(x)} = w'W_{\sigma(x)}$.

Then $W \curvearrowright \mathcal{U}$ as a reflection group.

with strict fundamental domain K.

<u>Note</u>: If $\Gamma = \ker(W \to C_2^S)$

Then Γ acts freely on $\mathcal{U}$

& $\mathcal{U}/\Gamma = P_1$.

The Reflection Group Trick.

Suppose G is a group of type F. This means $\exists$ a finite CW complex X s.t $X \sim BG$.

- Thicken X to a mfld with bdry N.
- Triangulate ∂N as a flag simplicial cx L.

Put dual cell structure on ∂N s.t. N_σ = dual cell to $\sigma \in L$.

- $W = W_L$ = associated RACG.
- As before, put

$$\mathcal{U} = (W \times N)/\sim$$

Then $\mathcal{U}$ is aspherical mfld and if $\Gamma = \ker W \to C_2^S$ then $\mathcal{U}/\Gamma = M$ is a closed aspherical mfld

Moreover, N is a strict fundamental domain for C_2^S action on M & the quotient map $M \to N \sim BG$ is a retraction.

__Cor__. Given any gp G of type F, there is a closed aspherical mfld M which retracts onto BG. (In particular G is a retract of $\pi_1 M$.)