$BG$ = classifying space for $G$

i.e., a CW cx s.t. $\pi_1(BG) = G$

& $\widetilde{BG}$ ($= EG$) is contractible

## Finiteness Properties of gps

$G$ is <u>type F</u> if
$BG$ = finite complex

i.e. $G \curvearrowright \widetilde{BG}$ cpt quotient.

$G$ is <u>type FH</u> if $\exists$ acyclic

cx $Y$, $G \curvearrowright Y$ freely

$Y/G$ cpt.

$G$ <u>type FL</u> : if $\exists$ acyclic

chain complex of finitely

generated free $\mathbb{Z}G$-modules

"  $0 \to C_n \to \quad \to C_0 \to \mathbb{Z}$.

$G$ <u>type FP</u> if " " " "

" " .. projective $\mathbb{Z}G$-modules

$$\boxed{F \Rightarrow FH \Rightarrow FL \Rightarrow FP}$$

Also can define, $F_m$, $FH_m$, $FL_m$, $FP_m$.

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1. Bestvina - Brady Groups

$L$ = flag cx, $\mathrm{Vert}\, L = S$

$A_L$ = RAAG

$T_L = BA_L$ = NPC cube cx

$X = \widetilde{T_L}$ : CAT(0) cube cx

Define $\varphi : A_L \longrightarrow \mathbb{Z}$ by

$$\varphi : a_s \longrightarrow 1$$

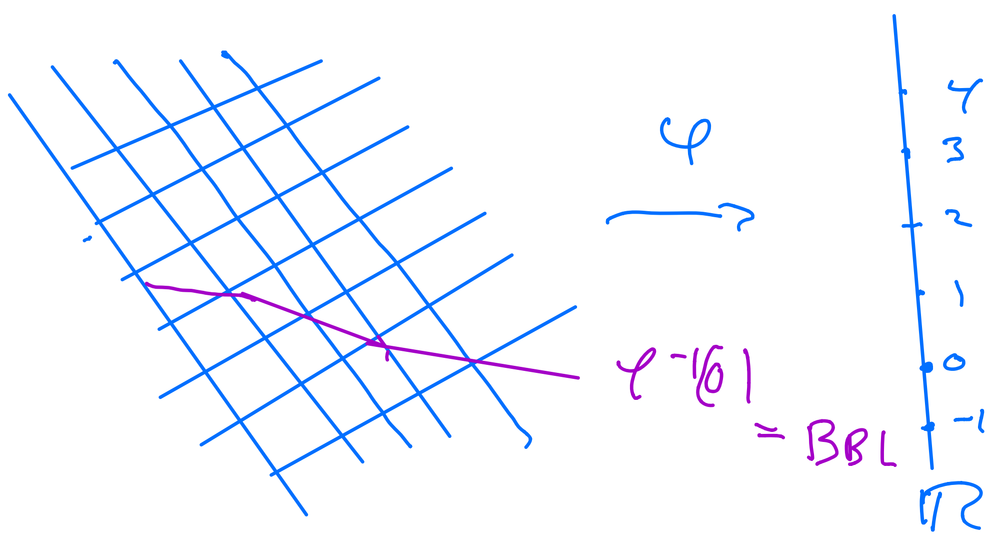
$\{a_s\}$ = standard generating set

$$\boxed{BB_L = \ker(\varphi : A_L \to \mathbb{Z})}$$

Extend $\varphi$ to "height function"

$\varphi : X \longrightarrow \mathbb{R}$ s.t.

- $\varphi|_{\mathrm{Vert}\,X} = \varphi|_{A_L} : A_L \to \mathbb{Z}$
- $\varphi|_{\mathrm{cube}}$ = linear & not constant (if dim cube $\neq 0$)




$$X_t = \varphi^{-1}(\{t\})$$

If $J \subset \mathbb{R}$ is interval then

$X_J = \varphi^{-1}(J)$.

Note: $BB_L \curvearrowright X_t$ freely and $X_t / BB_L$ is compact (it is a subcomplex of $T_L$]

However, $X_t$ need not be contractible.

Eventually, Bestvina-Brady prove the following

<u>Thm</u> $X_t$ is h.e to a wedge of copies of $L$, 1 for each vertex not in $X_t$

$$X_t = \bigvee_{\substack{v \text{ vertex} \\ \varphi(v) \neq t}} L$$

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<u>Cor</u> $\tilde{H}_*(X_t) = \bigoplus H_*(L)$

<u>Cor</u> $X_t$ is simply connected iff $\pi_1(L) = 1$.

<u>Cor</u>

- BBL type F $\Longleftrightarrow$ $L$ is contractible
- BBL type FH $\Longleftrightarrow$ $L$ is acyclic

<u>Pf</u> $\Longleftarrow$ direction requires more work

<u>Cor</u> $\exists$ gps of type FP not

of type F

## Morse Theory

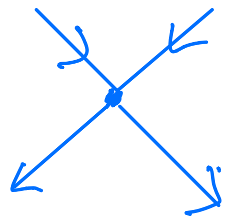
The link of each vertex in $X$ is iso $OL$, the octahedralization of $L$

$$OL = \bigcup_{\sigma \in L} O(\sigma) \quad \text{, where}$$

$$O(\sigma) = \mathop{*}_{\mathrm{vert}(\sigma)} S^0$$

$L$ is the descending link $Lk_{\downarrow}$ for the height function $\varphi$ on $X$. ($L$ is also the ascending link)

$\varphi$



__Lemma__ $J \subset J' \subset \mathbb{R}$ nonempty closed intervals s.t. $J' - J$ contains only one number $m = \varphi(\text{vertex})$. If $\inf J = \inf J'$

Then $X_{J'}$ is h.e to $X_J$ with copies of $Lk_{\downarrow}(v)$, $v \in \varphi^{-1}(m)$ coned off.

# Combining Bestvina-Brady and Reflection Group Trick.

A group of type FP is a <u>Poincaré duality gp of dim n</u> if

$$H^*(G;\mathbb{Z}G) = \begin{cases} 0 & * \neq n \\ \mathbb{Z} & * = n \end{cases}$$

For example, if $BG = M^n$ a closed manifold, then $G$ is a $PD^n$-group.

**Lemma** If $G \curvearrowright N^n$ freely on acyclic mfld $N$ (w/o bdry), then $G$ is a $PD^n$-group.

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Suppose $L$ = acyclic flag cx with $\pi_1(L) \neq 1$. Then level set $X_t$ is acyclic for $H = BB_L$, $X_t/H$ is a finite cx.

Now apply reflection gp trick to $X_t/H$., i.e., Thicken to a mfld $N$

$\exists$ epimorphism

$f: \pi_1(N) = \pi_1(X_*) \longrightarrow H = \pi_1(X_*/H)$

$\therefore \exists$ a covering space $\tilde{N} \to N$ with $\tilde{N}$ acyclic & gp of covering transformations $H$.

Triangulate $\partial N$ as flag cx giving RACG $W_{\partial N}$.

Lift triangulation to $\partial \tilde{N}$ giving a RACG $\tilde{W}$ $\curvearrowright$ $\mathcal{U}(\tilde{W}, \tilde{N}) = \mathcal{U}$.

Then $\mathcal{U}$ is an acyclic mfld. Moreover

$\widetilde{W} \rtimes H \curvearrowright \mathcal{U}$

Taking appropriate torsion free subgp

$\pi = \widetilde{\Gamma} \rtimes H$ acts on $\mathcal{U}$ with quotient a closed mfld.

<u>Thm</u>. In each dimension $\geq 4$ $\exists$ a $PD^4$-group $\pi$ which is not finitely presented.

<u>Remark</u>: Therefore not every $PD^n$ gp $\pi$ is the fundamental

group of a closed aspherical mfld.

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Further Application: finiteness properties for $\underline{E}G$.

$G$ = discrete gp, possibly with torsion.

$\exists$ CW complex $\underline{E}G$ with proper action $G \curvearrowright \underline{E}G$.

s.t $\forall$ finite subgp $F < G$

$(\underline{E}G)^F$ is contractible.

In particular, $\underline{E}G$ is contractible. Put

$\underline{B}G = \underline{E}G / G$

Say $G$ is type VF

if $\underline{B}G$ is finite cx

Similarly, VFH, VFL, VFP

etc.