Hyperbolization

This refers to various functorial constructions for converting a cell cx J into a NPC cube cx $\mathcal{H}(J)$. If J is a manifold, then $\mathcal{H}(J)$ will also be one.

Procedures are by induction on $\dim J$. Assume $\mathcal{H}(J^{n-1})$ has been defined, σ = n-cell.

We need a way to define $\mathcal{H}(\sigma)$ an NPC mfld with bdry, $\partial \mathcal{H}(\sigma) = \mathcal{H}(\partial \sigma)$.

Then $\mathcal{H}(J^n)$ is defined by gluing the $\mathcal{H}(\sigma^n)$ onto $\mathcal{H}(J^{n-1})$ in same combinatorial pattern as the σ^n are glued onto J^{n-1}.

The cross with interval construction

Define a sequence of functors, $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_n, \dots$ where $\mathcal{D}_n$ is defined on all cell complexes of dim n.

Note: Functor means we have a category $\mathcal{B}$ of cell complexes, morphism means combinatorial equivalence onto subcomplex, $\mathcal{C}$ = category of NPC cube complexes

Morphism = isometric embedding onto a sub cx. Hyperbolization is functor from $\mathcal{B}$ to $\mathcal{C}$.

Definition of $\mathcal{O}_n$:

$\mathcal{O}_0$: If $\dim J = 0$, $\mathcal{O}_0(J) = J$

(i.e., $\mathcal{O}_0(v) = v$)

Assume by induction that $\mathcal{O}_{n-1}$ has been defined for all J, $\dim J \leq n-1$.

Define

$$\mathcal{O}_n(J^{n-1}) = \mathcal{O}_{n-1}(J^{n-1}) \times \{\pm 1\}$$

& if $\sigma \in J^{(n)} = \{n\text{-cells}\}$

$$\mathcal{O}_n(\sigma) = \mathcal{O}_{n-1}(\partial\sigma) \times [-1, 1]$$

Then

$$\mathcal{O}_n(J^n) = \mathcal{O}_n(J^{n-1}) \cup \bigcup_\sigma \mathcal{O}_n(\sigma)$$

with canonical gluing.

$$\mathcal{O}(J) = \mathcal{O}_{\dim J}(J).$$

<u>Remarks</u> 1) Each component

of $\mathcal{D}_n(J)$ corresponds to a unique vertex v.

$\exists$ 2^n vertices in such a comp. ,

2) If $\mathcal{D}_{n-1}(J^{n-1})$ is a cube cx (by induction) then so is $\mathcal{D}_n(J)$

3) The component of $\mathcal{D}_n(J)$ containing v depends only on a nbhd of v in J. In fact only on the poset of cells containing v. Hence we might as well assume

$J = \mathrm{Cone}(\overline{L})$ for some cell complex $\overline{L}$.

4) Let L = barycentric subdivision of $\overline{L}$. It turns out that $\mathcal{U}(\mathrm{Cone}\,\overline{L})$ is closely related to RACG W_L & is covered by polyhedral product

$$P_L = [-1, 1]^L.$$

Before examining cubical structure on $\mathcal{U}(\mathrm{Cone}\,\overline{L})$ more closely we turn to a relative version.

Relative Hyperbolization:

Suppose B is a connected NPC cell complex and $B \subset J$ another cell cx. (In practice J will be a mfld.) Assume

(*) for any cell $\sigma \in J$, $\sigma \cap B$ is either empty or a single cell of B.

Let $P(J, B)$ = poset of cells in J which have nonempty intersection with B and are not $\subset B$.

Put $\dim(J, B)$ = largest dim of any $\sigma \in \mathcal{P}(J, B)$.

<u>Goal</u>: Define a NPC polyhedron $\mathcal{P}(J, B)$ s.t

- $\mathcal{P}(J, B)$ contains 2^n copies of B, $n = \dim(J, B)$, each isometrically embedded as a totally geodesic subcx
- $\mathcal{P}(J, B)$ depends only on $P(J, B)$

So we can replace J by a regular nbhd of B in J.

- If σ is a k-cell, then $\mathcal{P}(\sigma, \sigma \cap B)$ is a NPC k-mfld with totally geodesic ∂.

- For $n = \dim(J,B)$, $(C_2)^n \curvearrowright \mathcal{P}(J,B)$ as a reflection gp. Fundamental chamber $K(J,B)$ is identified with 1^{st} derived nbhd of B
- $\mathcal{P}(J,B)$ retracts onto $K(J,B)$.

In other words, $\mathcal{P}(J,B)$ is a version of reflection gp trick with nonpositive curvature.

As before, there is a quick definition of $\mathcal{P}(J,B)$ by induction on $\dim(J,B)$.

Here it is:

$\dim(J,B) \le 0$. Then

$$\mathcal{D}_0(J,B) = B$$

Suppose $\mathcal{D}_{n-1}(J,B)$ defined for $\dim(J,B) < n$. So

$\mathcal{D}_{n-1}(J^{n-1}\cup B, B)$ and n-cell σ

$\mathcal{D}_{n-1}(\partial\sigma \cup B, B)$ are defined

As before, put

$$\mathcal{D}_n(J^{n-1}\cup B, B) = \mathcal{D}_{n-1}(J^{n-1}\cup B, B) \times \{\pm 1\}$$

$$\mathcal{D}_n(\sigma \cup B, B) := \mathcal{D}_n(\sigma, \sigma \cap B)$$

$$= \mathcal{D}_{n-1}(\partial\sigma \cup B, B) \times [-1, 1]$$

Then for each $\sigma \in P^{(n)}(J,B)$

glue $\mathcal{D}_n(\sigma \cup B, B)$ onto

$\mathcal{D}_n(J^{n-1} \cup B, B)$ by natural identification. $\square$

Each cell of $\mathcal{D}(J \sim B, B)$ will have the form

$$(m\text{-cell of } B) \times [-1,1]^k$$

where k corresponds to a chain of length k, in $P(J,B)$ containing a given cell τ in B

$$\tau < \sigma_1 < \sigma_2 < \dots < \sigma_k .$$

Here is one consequence.

B = NPC cell cx

underlying cell cx can be

PL - embedded in mfld

(say $\mathbb{R}^N$). Let

J = reg nbhd.

<u>Thm</u> (Bizhong Hu). Let B be any NPC cell cx. Then B isometrically embeds as a totally geodesic subspace of some NPC mfld M.

<u>Remark</u> In contrast, a totally geodesic subspace of a NPC Riemannian mfld must be a submfld.

Reflection gp action on $\mathcal{O}(J, v)$, $J = \text{Cone}\, L$

$C_2 \curvearrowright \mathcal{O}_n(J) =$

$= \mathcal{O}_{n-1}(J^{n-1}) \times \{\pm 1\} \cup \bigcup \mathcal{O}'_{n-1}(\partial\sigma) \times [-1, 1]$

by multiplying by (-1) on n^{th} interval. So if $\dim J = n$ $(C_2)^n$ acts. Fundamental Chamber $= K$ is a reg nbhd. of v in $\mathcal{O}_n J$

cubes in $\mathcal{O}_n J$ are k-faces of $[-1, 1]^n$ determined by equations $x_i = 0$

for some subset of $\{1, \dots n\}$

Each k-cube is determined

by a chain of cells

$\sigma_1 < \dots < \sigma_k$ of length k

So we have exactly

reflection gp construction

where chamber $K = K_L$,

L = barycentric subdivision of T

(so L is flag cx)

Here we have $(C_2)^n$-action

Previously, $(C_2)^S$-action

$S = \{\text{vertices of } L\}$

$= \{ \text{simplices } \sigma \text{ in } I \}$

Then $(C_2)^S \curvearrowright P_L = \mathcal{U}((C_2)^S, K)$

Here we take quotient of P_L by

$$H = \ker \{ (C_2)^S \to (C_2)^n \}$$

given by $\sigma \longrightarrow \dim \sigma + 1$

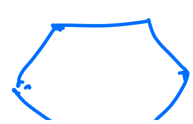
$s_\sigma \longrightarrow s_i$.

$$\ell_n(J) = P_L / H .$$

<u>Ex</u> $I =$ 3-gon

$L =$ 6-gon

$\ell_n(J) =$ 4-fold cover of



= surface genus 2

General Hyperbolization

Properties (= Axioms) : Funct

$\mathcal{H}$: {n-dim cell complexes} $\longrightarrow \mathcal{P}\Sigma_n$

NPC cell complex
cubically cube cx

(H0) Preserves local structure

(near $\sigma \in J$ looks like $\sigma \times \mathrm{Cone}(Lk(\sigma, J))$
in $\mathcal{H}(J)$ nbhd will be $\mathcal{H}(\sigma) \times L'$
L' = subdivision of $Lk(\sigma)$

(H1) $\mathcal{H}(pt) = pt$

(H2) $\mathcal{H}(\sigma)$ is mfld with boundary (basically $\partial(\sigma)$)

(H3) $\mathcal{H}(\sigma)$ orientable

Then we get a map (unique up to homotopy)

$c : \mathcal{H}(J) \longrightarrow J$ s.t

$c_{|}(\mathcal{H}(\sigma)) = \sigma$

(H3) $\Rightarrow$

$c_* : H_*(\mathcal{H}(J)) \to H_*(J)$

is onto. without (H3)

but with H2 we get onto with $\mathbb{Z}/2$ coefficient.

(H4) J = smooth triangulation of smooth mfld

$\mathcal{H}(J)$ = smooth mfld

(H3) The stable tangent bundle of $H(\sigma)$ is trivial

General method of Gromov (after Davis - Januszkiewicz)

Suppose J = n dim simpl complex. Taking barycentric subdiv. we can assume

$$J \xrightarrow{p} \Delta^n$$

Assume we have a method of ~~constructing~~ functorial metho $\mathcal{H}(\Delta^n)$

s.t $\mathcal{H}(\partial\Delta^{k}) = \partial\mathcal{H}(\Delta^{k}) \subset \mathcal{H}(\Delta^{k})$

$\mathcal{H}(J)$ is defined by pull-back

square

$$\begin{tikzcd} \mathcal{H}(J) \arrow[r] \arrow[d] & J \arrow[d, "p"] \\ \mathcal{H}(\Delta^n) \arrow[r, "e"'] & \Delta^n \end{tikzcd}$$

$H(J) \subset \mathcal{H}(\Delta^n) \times J$ $\lrcorner$

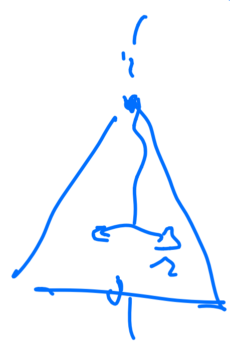
Gromov's cylinder construction

$\mathcal{H}(J^1) = J^1$

Define $\mathcal{H}(\partial\Delta^n)$

$\mathcal{H}(\Delta^2)$ take a reflection

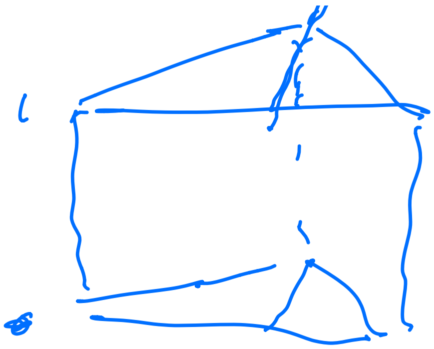
on r on $\partial\Delta^2$



$\mathcal{H}(r): \mathcal{H}(\partial\Delta^2) \to$

C = fundamental domain r

+



$\mathcal{H}(\partial\Delta^2) \times I$

identify $C_+ \times 0$ with $C_+ \times 1$

The bdry of what's left

is $\mathcal{H}(C_-) \times 0 \cup \mathcal{H}(C_-) \times 1$

$= \mathcal{H}(\partial\Delta^2)$

i.e Take $\mathcal{H}(\partial\Delta^n) \times S^1$

and slit open along

$\mathcal{H}(C_+) \times *$

Application: Non triangulable 4-mfld

<u>Rohlin</u>: M^4 = closed 4 mfld with $w_1 = 0$, then $\operatorname{sgn}(M^4) \equiv 0 \ (16)$.

E_8-homology mfld

$N^4 = E_8$-plumbing $\quad \operatorname{sgn}(N^4) = 8$

$\partial N^4 =$ homology 3-sphere Σ^3

$\Sigma^3 = \partial W^4 \leftarrow$ contractible

$N^4 \cup_{\Sigma^3} W^4$ not PL-mfld.

$N^4 \cup \operatorname{cone}(\partial N^4) = X$

$h(X^4) =$ mfld except at 1 pt

$\operatorname{sgn} h(X^4) = 8$

$$M^4 = \left(h(X^4) - \underset{(\text{cone pt})}{\text{nbhd}} \right) \cup W^4$$

aspherical 4-mfld with

can't be triangulated

$(\tilde{M}^4 \not\cong \mathbb{R}^4)$