The cohomology of a Coxeter group with group ring coefficients as a W-module

Fayetteville

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Work with: J. Dymara, B. Okun and T. Januszkiewicz

Equivariant (co)homology. G a gp. Y a G- CW complex. M a G-module (right G-module for homology, left for cohomology).

$$C_*^G(Y;M) := M \otimes_{\mathbf{Z}G} C_*(Y)$$

$$C_G^*(Y;M) := \mathrm{Hom}_G(C_*(Y), M).$$

If G acts freely on Y, then M defines a system of local coefficients on Y/G and $H_G^*(Y;M)$ is the cohomology of Y/G with local coefficients. When the action is not free M still defines some sort of coefficient system on Y/G.

Suppose $M = \mathbf{Z}G$. Then

$$C_*^G(Y;\mathbf{Z}G) = C_*(Y)$$

$$C_G^*(Y;\mathbf{Z}G) = C_c^*(Y)$$

(provided G acts properly and cocompactly).

If Y is acyclic,

$$H_*(Y) = H_*(G; \mathbf{Z}G).$$

If, in addition, G acts properly (i.e., finite isotropy subgps) and Y/G is compact (i.e., a finite CW cx), then

$$H_c^*(Y) = H^*(G; \mathbf{Z}G).$$

Since $\mathbf{Z}G$ is a G-bimodule, both sides of these equations are G-modules.

Examples. If G is a virtual Poincaré duality gp (of dim $= n$), then $H^*(G; \mathbf{Z}G) \cong H_c^*(\mathbf{R}^n)$ which is concentrated in dim n (where it is $\mathbf{Z}$).

If G is virtually free, then $H^*(G, \mathbf{Z}G) \cong H_c^*(\text{tree})$ which is concentrated in dim 1 (where it is free abelian of countable rank).

Question. *Suppose G acts properly, cocompactly on a contractible Y. Is $H^*(G; \mathbf{Z}G)$ finitely generated as a G-module?*

Coxeter groups. (W, S) is a Coxeter system, i.e., S is set of generators, all relations are consequences of:

$$s^2 = 1 \quad \text{or} \quad (st)^{m(s,t)} = 1$$

for $m(s,t) \in \{2, 3 \ldots\} \cup \{\infty\}$. For $T \subset S$, $W_T := \langle T \rangle$.

$$\mathcal{S} := \{T \subset S \mid W_T \text{ is finite}\}$$

$$= \{\text{“spherical” subsets of } S\}.$$

A basic construction. X a CW cx.

$(X_s)_{s \in S}$ a family of subcomplexes (called *mirrors*).

$$S(x) := \{s \in S \mid x \in X_s\}$$

$$\mathcal{U} := (W \times X)/\sim$$

where $\sim$ is defined by $(w, x) \sim (v, y)$ iff $x = y$ and $w^{-1}v \in W_{S(x)}$.

$\mathcal{U}$ is a W CW cx. X is called the *fundamental chamber*.

Example. Suppose $K = |\mathcal{S}|$, the geom realization of the poset $\mathcal{S}$, and $K_s := |\mathcal{S}_{\geq \{s\}}|$. When $X = K$, write Σ instead of $\mathcal{U}$.

Facts. Σ is contractible. W acts properly on it.

For any $T \subset S$, put

$$X^T := \bigcup_{s \in T} X_s.$$

Theorem.

$$H_*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H_*(X, X^T) \otimes \widehat{H}^T$$

If X is compact and W-action is proper, then

$$H_c^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \widehat{A}^T$$

where free abelian groups $\widehat{H}^T$ and $\widehat{A}^T$ will be defined later.

Corollary.

$$H^*(W; \mathbf{Z}W) \cong \bigoplus_{T \in \mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{A}^T$$

These formulas are not W-equivariant. $\widehat{A}^T$ is a $\mathbf{Z}$-submodule of a certain W-module A^T and it maps isomorphically onto a quotient W-module $A^T/A^{>T}$, but A^T does not split as a direct sum.

The Equivariant Theorem. *There are decreasing filtrations of $H_*(\mathcal{U})$ and $H_c^*(\mathcal{U})$ as W-modules s.t. in filtration degree p the associated graded terms are*

$$\bigoplus_{\substack{T\in\mathcal{S}\\|T|=p}} H_*(X,X^T)\otimes(H^T/H^{>T}) \text{ and } \bigoplus_{\substack{T\in\mathcal{S}\\|T|=p}} H^*(X,X^{S-T})\otimes(A^T/A^{>T}).$$

Corollary. *There is a filtration of* $H^*(W;\mathbf{Z}W)$ *with associated graded term in degree* p*:*

$$\bigoplus_{\substack{T\in\mathcal{S}\\ |T|=p}} H^*(K,K^{S-T})\otimes(A^T/A^{>T}).$$

Corollary. $H^*(W;\mathbf{Z}W)$ *is finitely generated as a right* W*-module.*

Invariants. Let M be a left W-module. For any $T\subset S$,

$$M^T := \{m\in M \mid wm=m,\ \forall w\in W_T\},$$

the W_T-*invariants*. If $N \subset M$ is a $\mathbf{Z}$-submodule, $N^T := N \cap M^T$.

Coefficient systems. In our case this will mean a functor from the poset of cells of the orbit space X to the category of abelian gps. M gives such a functor $\mathcal{I}(M)$ by $\mathcal{I}(M) := M^{S(c)}$, where for any cell c, $S(c) := \{s \in S \mid c \subset X_s\}$; if $c < d$, $M^{S(c)} \to M^{S(d)}$ is the inclusion.

Equivariant cohomology & coefficient systems. We stick to cohomology for simplicity. There is are natural identifications:

$$C^i_W(\mathcal{U}; M) = \bigoplus_{c \in X^{(i)}} M^{S(c)} := C^i(X; \mathcal{I}(M)).$$

Notation. $A := \mathbf{Z}W$. $\forall T \in \mathcal{S}$, define elements $\tilde{a}_T$, $\tilde{h}_T$ in A:

$$\tilde{a}_T := \sum_{w \in W_T} e_w \quad \text{and} \quad \tilde{h}_T := \sum_{w \in W_T} (-1)^{l(w)} e_w.$$

We have $A^T = \mathbf{Z}(W/W_T) = \tilde{a}_T A$. Put $H^T := A\tilde{h}_T$.

Note: $A^T = 0$ and $H^T{=}0$ for $T \notin \mathcal{S}$. If $U \subset T$, then

$$A^T \subset A^U = A^T \quad \text{and} \quad H^T \subset H^U.$$

Another basis for A. $\forall w \in W$, put

$$\mathsf{In}'(w) := \{s \in S \mid l(sw) < l(w)\}.$$

Define $c_w := \tilde{a}_{\mathsf{In}'(w)} e_w$ and $\widehat{A}^T := \mathsf{Span}\{c_w \mid \mathsf{In}'(w) \subset T\}$.

Facts. $\{c_w\}_{T \supset \mathsf{In}'(w)}$ is a basis for A^T. In particular, $\{c_w\}_{w \in W}$ is basis for A and

$$A^T = \bigoplus_{\substack{U \in \mathcal{S} \\ U \supset T}} \widehat{A}^U.$$

Decompositions of coefficient systems. For $A = \mathbf{Z}W$,

$$\mathcal{I}(A) = \bigoplus_{T \in \mathcal{S}} \mathcal{I}(\widehat{A}^T)$$

For $c \in X^{(i)}$,

$$(A^T)^{S(c)} = \begin{cases} 0, & \text{if } c \subset X^{S-T}; \\ \widehat{A}^T, & \text{otherwise.} \end{cases}$$

Hence,

$$C^i(X; \mathcal{I}(\widehat{A}^T)) = \{f : X^{(i)} \to \widehat{A}^T \mid f(c) = 0 \text{ if } c \subset X^{S-T}\}$$

$$= C^i(X, X^{S-T}) \otimes \widehat{A}^T.$$

Theorem.

$$H^*_W(\mathcal{U};A) = \bigoplus_{T\in\mathcal{S}} H^*(X, X^{S-T}) \otimes \widehat{A}^T$$

Proof.

$$C^*_W(\mathcal{U};A) = C^*(X;\mathcal{I}(A)) = \bigoplus C^*(X;\mathcal{I}(\widehat{A}^T))$$

$$= \bigoplus C^*(X, X^{S-T}) \otimes \widehat{A}^T$$

$\square$

Filtration of the coefficient system. Define

$$A = F_0 \supset \cdots \supset F_p \supset \cdots \qquad \text{by}$$

$$F_p := \sum_{|T| \geq p} A^T.$$

Fact.

$$F_p/F_{p+1} = \bigoplus_{|T|=p} A^T/A^{>T}, \quad \text{where} \quad A^{>T} := \sum_{U \supset T} A^U.$$

This gives a filtration of $\mathcal{I}(A)$ and of cochains and cohomology.

Theorem. *The associated graded term of the filtration of* $H_c^*(\mathcal{U})$

is $H^*(X;\mathcal{I}(F_p))/H^*(X;\mathcal{I}(F_{p+1})) =$

$$\bigoplus_{|T|=p} H^*(X, X^{S-T}) \otimes (A^T/A^{>T}).$$

Weighted L^2-cohomology. Define an inner product $\langle\ ,\ \rangle_q$ on $\mathbf{R}W$ by $\langle e_w, e_v\rangle_q := q^{l(w)}\delta_{w,v}$. L^2_q $(= L^2_q(W))$ is its completion. A_q is the "Hecke algebra" with parameter q (a deformation of the group algebra $\mathbf{R}W$). For q sufficiently large we proved a Decomposition Theorem for L^2_q and a computation of the reduced L^2_q-cohomology of Σ as a module over A_q, analogous to above formula.

Hecke algebra coefficients. Same formula for $H^*(K;\mathcal{I}(A_q)$.

Buildings. Φ a building of type (W,S) with chamber transitive automorphism group G. $B =$ stabilizer of a chamber. $q =$ thickness of Φ (i.e. $q+1$ chambers meet along each mirror).

$$F(G/B) = \{f : G/B \to \mathbf{R} \mid f \text{ is finitely supported}\}$$

Lemma.

$$A_q^T \otimes_{A_q} F(G/B) = F(G/G_T)$$

Theorem.

$$C_c^*(\Phi;\mathbf{R}) = C^*(K;\mathcal{I}(A_q)) \otimes_{A_q} F(G/B).$$

Conjecture.

$$H_c^*(\Phi;\mathbf{R}) = H^*(K;\mathcal{I}(A_q)) \otimes_{A_q} F(G/B)$$

This implies

Conjecture. *The associated graded group for $H_c^*(\Phi)$ as a right G-module is*

$$\bigoplus_{|T|=p} H^*(K, K^{S-T}) \otimes (F^T/F^{>T})$$

where $F^T := F(G/G_T) \subset F(G/B)$.