WEIGHTED L^2-COHOMOLOGY

OF COXETER GROUPS

Bedlewo,

July 5, 2005

work with

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Cochains. Let Y be a cell complex, usually not compact, often contractible.

$$\mathcal{E}_n := \{n\text{-cells in } Y\}$$

$$C^n(Y) = \{\mathbf{R}\text{-valued cochains on } Y\} := \{f : \mathcal{E}_n \to \mathbf{R}\}$$

$$C^n_c(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid f \text{ is finitely supported}\}$$

$$L^2C^n(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid \sum f(\sigma)^2 < \infty\}$$

Cohomology. $\delta : C^n(Y) \to C^{n+1}(Y)$ is the coboundary map.

$$H^n(Y) := \ker \delta / \operatorname{im} \delta, \qquad H_c^n(Y) := \ker \delta / \operatorname{im} \delta.$$

$L^2C^n(Y)$ is a Hilbert space. In favorable situations δ is a bounded linear operator. So, $\ker \delta$ is a closed subspace of $L^2C^n(Y)$. However, $\operatorname{im} \delta$ might not be. Define the *reduced L^2-cohomology*:

$$\mathcal{H}^n(Y) := \ker \delta / \overline{\operatorname{im} \delta}.$$

Example. Suppose Y^n is a contractible n-mfld (e.g. $Y^n = \mathbf{R}^n$).

$$H^i(Y^n) = \begin{cases} \mathbf{R}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$H_c^i(Y^n) = \begin{cases} \mathbf{R}, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

What about $\mathcal{H}^*(Y^n)$? In many situations, $\mathcal{H}^i(Y^n) = 0$ for $i \neq \frac{n}{2}$ and can be nonzero for $i = \frac{n}{2}$.

Bringing a group into the picture. Suppose Γ is a countable discrete group.

$$\mathbf{R}\Gamma := \{f : \Gamma \to \mathbf{R} \mid f \text{ is finitely supported}\}$$

$$= \{\text{finite sums, } \sum f(\gamma) e_\gamma\} = \text{the gp algebra}$$

$$L^2(\Gamma) := \{f : \Gamma \to \mathbf{R} \mid \sum f(\gamma)^2 < \infty\}$$

$L^2(\Gamma)$ is the Hilbert space completion of $\mathbf{R}\Gamma$ with inner product: $e_\gamma \cdot e_{\gamma'} := \delta(\gamma, \gamma')$. There are two orthogonal Γ-actions on $L^2(\Gamma)$

(by left or right translation).

Γ-dimension. Let $\mathcal{N}(\Gamma)$ be the set of Γ-equivariant bounded linear endomorphisms of $L^2(\Gamma)$ (the *von Neumann algebra*). Given $\varphi \in \mathcal{N}(\Gamma)$, define $\mathrm{tr}_\Gamma(\varphi) := \varphi(e_1) \cdot e_1$. Suppose

$$V \subset \bigoplus_{\text{finite}} L^2(\Gamma)$$

is a closed, Γ-stable subspace (a *Hilbert Γ-module*).

Let $p_V : \oplus L^2(\Gamma) \to \oplus L^2(\Gamma)$ be orthogonal projection. Represent

p_V as a matrix (p_{ij}) with entries in $\mathcal{N}(\Gamma)$ and define

$$\dim_\Gamma V = \mathrm{tr}_\Gamma(p_V) := \sum \mathrm{tr}_\Gamma(p_{ii}) \in [0, \infty).$$

Some properties:

- $\dim_\Gamma V = 0 \iff V = 0$,

- $\dim_\Gamma L^2(\Gamma) = 1$,

- $\dim_\Gamma(V_1 \oplus V_2) = \dim_\Gamma V_1 + \dim_\Gamma V_2$,

- If F is a finite gp, then $\dim_F V = \frac{1}{|F|} \dim_{\mathbf{R}} V$,

- If $F \subset \Gamma$ is finite subgp, then $\dim_\Gamma L^2(\Gamma)^F = \frac{1}{|F|}$.

Suppose now that Γ acts on a cell cx Y by permuting the cells.

Always assume:

- $\forall$ cell σ, its stabilizer Γ_σ is finite,

- Y/Γ is compact.

There are only finitely many orbits of cells in Y. So,

$$L^2C^i(Y) = \bigoplus_{\text{orbits of cells}} L^2(\Gamma)^{\Gamma_\sigma} \subset \bigoplus L^2(\Gamma).$$

is a Hilbert Γ-module. So, $L^2C^i(Y)$ and $\mathcal{H}^i(Y)$ have Γ-dimensions.

The Γ-dimension of the second is called the i^{th} L^2-*Betti number*.

$$\dim_\Gamma L^2C^i(Y) \;=\; \bigoplus_{\text{orbits of } i\text{-cells}} \frac{1}{|\Gamma_\sigma|}$$

$$\dim_\Gamma \mathcal{H}^i(Y) := b^i(Y;\Gamma)$$

$$L^2\chi(Y;\Gamma) := \sum (-1)^i b^i(Y;\Gamma)$$

Atiyah's Formula.

$$L^2\chi(Y;\Gamma) = \sum_{\textit{orbits of cells}} \frac{(-1)^{\dim\sigma}}{|\Gamma_\sigma|} := \chi^{\text{orb}}(Y/\Gamma)$$

Goal: Define new groups $L^2_q\mathcal{H}^*(\ )$, depending on a positive real parameter q, which interpolate between $H^*(\ )$ and $H^*_c(\ )$ and which give $\mathcal{H}^*(\ )$ when $q = 1$.

Coxeter groups. S a finite set. (m_{st}) a *Coxeter matrix*, i.e., a symmetric $S \times S$ matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise.} \end{cases}$$

$$W := \langle S \mid (st)^{m_{st}} = 1,\ (s,t) \in S \times S \rangle.$$

(W, S) is a *Coxeter system*. For $T \subset S$, $W_T := \langle \{s\}_{s \in T} \rangle$.

If W_T is finite, T is a *spherical subset*.

$$\mathcal{S} := \{T \subset S \mid W_T \text{ is finite}\} = \{\text{spherical subsets}\}$$

Growth series.

$$W(t) := \sum_{w \in W} t^{l(w)}, \quad \text{where } l(w) \text{ denotes word length.}$$

It is a rational function in t, e.g.,

$$\frac{1}{W(t)} = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(t^{-1})},$$

$$\rho := \text{ radius of convergence of } W(t)$$

$$= \text{smallest root of } 1/W(t).$$

An inner product. Let $q \in (0, \infty)$.

$$\mathbf{R}^{(W)} := \{\text{finitely supported functions } W \to \mathbf{R}\}$$

$$\langle e_v, e_w \rangle_q := q^{l(w)} \delta(e_v, e_w)$$

$$L^2_q(W) := \text{completion of } \mathbf{R}^{(W)}.$$

The Hecke algebra. $\mathrm{R}_q W$ is a deformation of the group algebra.

Multiplication:

$$e_s e_w = \begin{cases} e_{sw}, & l(sw) > l(w); \\ q e_{sw} + (q-1) e_w, & l(sw) < l(w). \end{cases}$$

Define an anti-involution $*$ on $\mathrm{R}_q W$ by $(e_w)^* := e_{w^{-1}}$.

Key properties:

$$(xy)^* = y^* x^* \quad \text{and} \quad \langle xy, z \rangle_q = \langle y, x^* z \rangle_q.$$

The Hecke-von Neumann algebra

$$\mathcal{N}_q = \text{a completion of } \mathbf{R}_q W$$

$$:= \{\mathbf{R}_q W\text{-equivariant bounded linear operators on } L^2_q(W)\}$$

von Neumann trace. For $\varphi \in \mathcal{N}_q$, set

$$\mathrm{tr}_{\mathcal{N}_q}(\varphi) = \langle \varphi(e_1), e_1 \rangle_q.$$

For $\Phi = (\varphi_{ij}) \in M_m(\mathcal{N}_q)$, set

$$\mathrm{tr}_{\mathcal{N}_q}(\Phi) = \sum \mathrm{tr}_{\mathcal{N}_q}(\varphi_{ii}).$$

von Neumann dimension. Given a $\mathbf{R}_q W$-stable, closed subspace $V \subset \oplus L_q^2(W)$, let $p_V : \oplus L_q^2(W) \to \oplus L_q^2(W)$ be orthogonal projection onto V. Define

$$\dim_{\mathcal{N}_q} V = \mathrm{tr}_{\mathcal{N}_q}(p_V) \in [0, \infty).$$

Idempotents in $\mathcal{N}_q$: sample calculations. Define

$$\tilde{a}_S := \sum_{w \in W} e_w$$

$$\tilde{a}_S e_s = q\tilde{a}_S \quad \text{therefore,} \quad \tilde{a}_S e_w = q^{l(w)} \tilde{a}_S.$$

So, $(\tilde{a}_S)^2 = W(q)\tilde{a}_S$ and $\tilde{a}_S$ is bounded $\iff q < \rho$.

Lemma. *(i) For $q < \rho$, the following is an idempotent in $\mathcal{N}_q$:*

$$a_S := \frac{1}{W(q)} \sum_{w \in W} e_w. \qquad Similarly,$$

(ii) For $q > \rho^{-1}$, the following is an idempotent in $\mathcal{N}_q$:

$$h_S := \frac{1}{W(q^{-1})} \sum_{w \in W} (-1)^{l(w)} q^{-l(w)} e_w.$$

These have dimensions:

$$\dim_{\mathcal{N}_q}(\operatorname{im} a_S) = \frac{1}{W(q)}, \qquad \dim_{\mathcal{N}_q}(\operatorname{im} h_S) = \frac{1}{W(q^{-1})}$$

Spaces on which *W* acts. X a cell complex and $(X_s)_{s \in S}$ a family of subcomplexes. Put $S(x) := \{s \in S \mid x \in X_s\}$. Define

$$\mathcal{U}(W, X) := (W \times X)/\sim$$

where $\sim$ is the equivalence relation given by

$$(w,x) \sim (v,y) \iff x = y \text{ and } w^{-1}v \in W_{S(x)}.$$

Given $T \subset S$, set

$$X_T := \bigcap_{s \in T} X_s, \qquad X^T := \bigcup_{s \in T} X_s.$$

Always assume: $X_T = \emptyset$ whenever $T \notin \mathcal{S}$ (i.e., whenever $|W_T| = \infty$).

A special case: the complex Σ

$$K := \text{geometric realization of } \mathcal{S} \quad \text{(a cone)}$$

$$K_s := \text{geom. realization of } \mathcal{S}_{\geq\{s\}}$$

$$\Sigma := \mathcal{U}(W, K)$$

K is a *chamber*. K_s is a *mirror*.

Fact. Σ *is contractible.*

Weighted L^2-cohomology. Let $\mathcal{U} = \mathcal{U}(W, X)$. $\forall$ cell $\sigma \in \mathcal{E}_k$, let $e_\sigma \in C_c^k(\mathcal{U})$ be its characteristic function. Define an inner product on $C_c^k(\mathcal{U})$ by

$$\langle e_\sigma, e_\tau \rangle_q := q^{l(w(\sigma))}\delta(\sigma,\tau)$$

where $w(\sigma)$ is the shortest $w \in W$ s.t. $w^{-1}\sigma \in X$.

$$L_q^2 C^k(\mathcal{U}) := \text{ completion of } C_c^k(\mathcal{U})$$

$L_q^2 C^*(\mathcal{U})$ is a $\mathcal{N}_q$-module and $\delta : L_q^2 C^k(\mathcal{U}) \to L_q^2 C^{k+1}(\mathcal{U})$ is a map of $\mathcal{N}_q$-modules.

$$L_q^2\mathcal{H}^k(\mathcal{U}) := \ker\delta/\overline{\operatorname{im}\delta}$$

$$b_q^k(\mathcal{U}) := \dim L_q^2\mathcal{H}^k(\mathcal{U})$$

$$\chi_q(\mathcal{U}) := \sum(-1)^k b_q^k(\mathcal{U})$$

Theorem. (Dymara) $\chi_q(\Sigma) = \frac{1}{W(q)}$

Theorem. $b_q^k(\mathcal{U})$ *is a continuous function of* q.

Theorem. (Dymara) *If* $q < \rho$, *then* $L^2_q\mathcal{H}^*(\Sigma)$ *is concentrated in dimension* 0.

Theorem. (Dymara) *Suppose* Φ *is a building of type* (W, S) *with a chamber transitive automorphism group* G *and its "thickness" is* q. *Then the* L^2*-Betti number (with respect to* G*),* $b^k(\Phi; G)$*, is equal to* $b^k_q(\Sigma)$.

For buildings only integral values of q matter!

$W^T := \{w \in W \mid l(ws) = l(w) - 1, \forall s \in T, \text{and } l(ws) = l(w) + 1, \forall s \notin T\}$

Let $\mathbf{Z}(W^T) =$ the free abelian gp on W^T.

Theorem. (D.)

$$H_*(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H_*(X, X^T) \otimes \mathbf{Z}(W^T)$$

$$H_c^*(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H_*(X, X^{S-T}) \otimes \mathbf{Z}(W^T).$$

Main Theorem. (DDJO)

$$L_q^2\mathcal{H}^*(\mathcal{U}) = \bigoplus_{T\in\mathcal{S}} H^*(X, X^T)\otimes D^T, \qquad \text{if } q < \rho$$

$$L_q^2\mathcal{H}^*(\mathcal{U}) = \bigoplus_{T\in\mathcal{S}} H^*(X, X^{S-T})\otimes D^{S-T}, \quad \text{if } q > \rho^{-1}$$

Here the D^T *are certain specific* $\mathcal{N}_q$*-submodules of* $L_q^2(W)$ *whose dimensions can be explicitly computed:*

$$\dim_{\mathcal{N}_q} D_T = W^T(q)/W(q) \qquad \text{if } q < \rho,$$

$$\dim_{\mathcal{N}_q} D_{S-T} = W^T(q^{-1})/W(q^{-1}) \quad \text{if } q > \rho^{-1}, \text{ where}$$

$$W^T(q) := \sum_{w \in W^T} q^{l(w)}.$$

Corollary. *If $q < \rho$, then $H_k(\mathcal{U}; \mathbf{R}) \to L^2_q\mathcal{H}_k(\mathcal{U})$ is injective with dense image.*

If $q > \rho^{-1}$, then $H^k_c(\mathcal{U}; \mathbf{R}) \to L^2_q\mathcal{H}^k(\mathcal{U})$ is injective with dense image.

Decomposition Theorem. *We have direct sum decompositions of $\mathcal{N}_q$-modules:*

$$L^2_q = \bigoplus_{T \in \mathcal{S}} D^T \qquad \textit{if } q < \rho,$$

$$L^2_q = \bigoplus_{T \in \mathcal{S}} D^{S-T} \quad \textit{if } q > \rho^{-1}.$$

(In the formulas of this theorem and the Main Theorem, the RHS might only be a dense subspace of the LHS.)

Example. Suppose Σ is an n-manifold.

If $q < \rho$, then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension 0.

If $q > \rho^{-1}$, then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension n.

Example. Suppose Σ is an 2-manifold.

Then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension

$$\begin{cases} 0, & \text{if } q \leq \rho; \\ 1, & \text{if } \rho < q < \rho^{-1}; \\ 2, & \text{if } q \geq \rho^{-1}. \end{cases}$$

Question. *In general, what happens in the intermediate range,*

$\rho < q < \rho^{-1}$*?*

Conjecture. (A version of the Singer Conjecture). *Suppose* Σ *is an* n*-manifold . Then for* $q \leq 1$,

$$L^2_q\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k > \frac{n}{2}.$$

Similarly, for $q \geq 1$,

$$L^2_q\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k < \frac{n}{2}.$$

Question. *What about groups other than Coxeter groups?*