COXETER GROUPS AS A SOURCE OF EXAMPLES IN GEOMETRIC GROUP THEORY

Spring Topology and Dynamical Systems Conference

Greensboro, NC

March 24, 2006

I. CAT(0)**-spaces**

II. Reflection groups & the complex Σ

III. Examples

IV. The reflection group trick

I. CAT(0)-spaces and polyhedra

Roughly, a space which is "nonpositively curved" and simply connected.

C = "Comparison" or "Cartan"

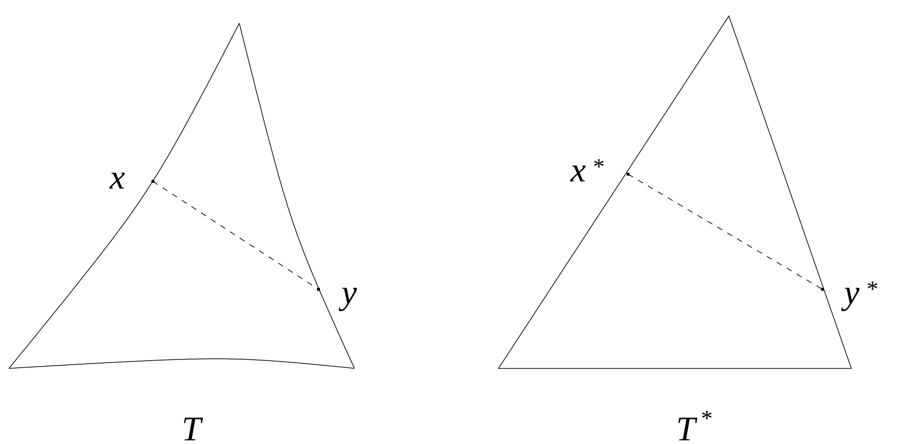
A = "Aleksandrov"

T = "Toponogov"

Some definitions. Let (X, d) be a metric space. A path $c : [a, b] \to X$ is a *geodesic* (or a *geodesic segment*) if $d(c(s), c(t)) = |s - t|$ for all $s, t \in [a, b]$.

(X, d) is a *geodesic space* if any two points can be connected by a geodesic segment.

The CAT(0)-**inequality.** A *triangle* T in X is a configuration of three geodesic segments (the "edges") connecting three points (the "vertices") in pairs. A *comparison triangle* for T is a triangle T^* in $\mathbf{R}^2$ with the same edge lengths. A geodesic space X is a CAT(0)-*space* if for any triangle T and any two points $x, y \in T$, we have $d(x,y) \leq d^*(x^*, y^*)$.



where x^*, y^* are the corresponding points in the comparison triangle T^* and d^* is distance in $\mathbf{R}^2$.

Observation. CAT(0) $\implies$ contractible.

Definition. X *is nonpositively curved* if the CAT(0) inequality holds locally.

Fact. nonpositive curv. $+$ 1-connected $\implies$ CAT(0).

Definition. X is *aspherical* if its univ cover is contractible.

So, nonpositive curv. $\Longrightarrow$ asphericity.

CAT(0)-polyhedra. Suppose X is a finite dimensional cell complex. Give it a "piecewise Euclidean metric" by declaring each cell to be a convex cell in Euclidean space and then measure the length of paths using Euclidean arc length. For example, X might be a cubical cell cx with each n-cell a regular Euclidean n-cube of edge length 1.

Definition. A simplicial cx L is a *flag cx* iff every finite set of vertices which are pairwise connected by edges spans a simplex of L.

Remark. The barycentric subdivision of any cell cx is a flag cx. So, the condition that a polyhedron L be a flag cx places no restriction on its topology.

Theorem. (Gromov). *A piecewise Euclidean cubical cell cx is nonpositively curved iff the link of each vertex is a flag cx..*

The visual boundary of a $\mathrm{CAT}(0)$-space

Adjoin a space ∂X of "ideal points" to a complete CAT(0) space X obtaining $\overline{X} = X \cup \partial X$. When X is locally compact, $\overline{X}$ will be a compactification of X. Fix a base point $x_0 \in X$. Rough idea: $\overline{X}$ is formed by adding an "endpoint" $c(\infty)$ to each geodesic ray $c : [0, \infty) \to X$, which begins at x_0. ∂X is the set of such endpoints. $\overline{X}$ has the "inverse limit topology." Consider the system of closed balls, $\{\overline{B}(x_0, r)\}_{r \in [0,\infty)}$. For each $s > r$, there is a retraction $p_{s,r} : \overline{B}(x_0, s) \to \overline{B}(x_0, r)$.

$$\overline{X} := \varprojlim \overline{B}(x_0, r).$$

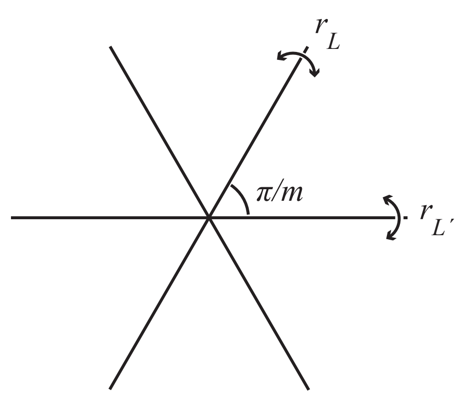
$$X \subset \overline{X} \quad \text{and} \quad \partial X := \overline{X} - X.$$

Example. If $X = \mathbf{R}^n$ or $\mathbb{H}^n$, then $(\overline{X}, \partial X) = (D^n, S^{n-1})$.

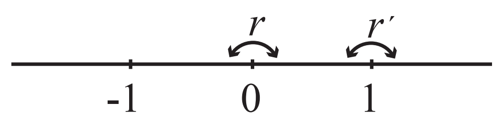
Example. If X is a regular tree (valence > 2), then ∂X is a Cantor set.

II. Reflection groups.

Example. Take two lines in $\mathbf{R}^2$ making an angle of π/m. The gp $\mathbf{D}_m$ generated by the orthogonal reflections across these lines is the *dihedral group* of order $2m$.



Example. The infinite dihedral gp $\mathbf{D}_\infty$ generated by reflections r, r' across 2 points in $\mathbf{R}$.



Example. P a convex polytope in $\mathbf{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$ s.t. all dihedral angles (between codimension one faces) have the form π/m_{ij}, $m_{ij} \in \{2, 3, \ldots\}$. W = the gp generated by

$$S := \{\text{reflections across faces of } P\}$$

Abstract reflection groups. Is there an abstract notion of reflection gp?

First attempt: any gp generated by involutions: a pair (W, S) with $W = \langle S \rangle$, each $s \in S$ of order 2.

Tits proposed two different refinements of the above. The first was that Cay(W, S) had certain separation properties. The second was that W had a presentation of a certain form. Amazingly, these 2 definitions turn out to be equivalent. Details:

(1) Put $\Omega :=$ Cay(W, S). $\forall s \in S$, the fixed set, Ω^s, separates Ω.

(2) For each pair $(s,t) \in S \times S$, let $m_{st} :=$ order(st). (W, S) is a *Coxeter system* if it has a presentation of the form:

$$\langle S \mid (st)^{m_{st}} \rangle_{(s,t) \in S \times S}$$

The equivalence of these two definitions is not obvious. The meaning of (2) is that if we start with Cay(W, S) and fill in orbits of 2-cells corresponding to distinct pairs $\{s, t\}$ with $m_{st} \neq \infty$, then the resulting 2-dim cell cx is simply connected.

Representing an abstract refl gp by a geometric object.

There are two ways to do this. Tits: $\exists$ a faithful representation $\rho : W \hookrightarrow GL(N, \mathbf{R})$ s.t.

- $\forall s \in S, \rho(s)$ is a (not necessarily orthogonal) linear reflection.

- W acts properly on the interior I of a convex cone in $\mathbf{R}^N$.

- Hyperplanes corresponding to S bound a "chamber" $K \subset I$.

For many purposes this representation is completely satisfactory.

Major disadvantage: fundamental domain K is not compact.

The cell complex Σ. $\exists$ a cell cx Σ with a proper W-action s.t.

- $\exists$ a compact fundamental chamber K with $\Sigma/W \cong K$.
- $S = \{$"reflections across faces" of $K\}$.
- Σ is contractible (in fact, CAT(0)).

Right-angled Coxeter groups. (W, S) is *right-angled* if all $m_{st} = 2$ or ∞. Let's stick to this case. Note that $m_{st} = 2$ means $(st)^2 = 1$, i.e., $st = t^{-1}s^{-1} = ts$, i.e., s and t commute.

The data for a right-angled Coxeter system is encapsulated in a finite simplicial graph L^1, as follows:

$\{$generators$\} = S = \text{Vert}(L^1)$. Relations: $s^2 = 1$, $\forall s \in S$ and

$$(st)^2 = 1 \quad \text{iff} \quad \{s, t\} \in \text{Edge}(L^1).$$

Conversely, given L^1, this presentation defines a right-angled Coxeter system.

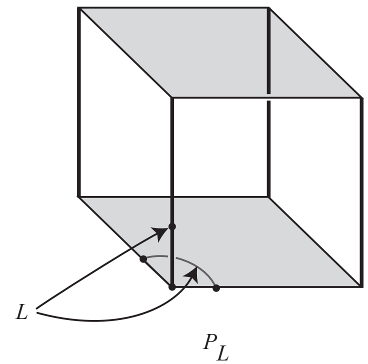
One associates to L^1 a simplicial cx (a "flag cx") L as follows: a subset $T \subset S$ spans a simplex σ_T iff any 2 elements of T are connected by an edge.

Construction of Σ. The 1-skeleton of Σ to be the Cayley graph: $\Sigma^1 :=$ Cay(W, S). Attach a square to each circuit in Cay(W, S) labeled $stst$ for each $\{s,t\} \in$ Edge(L). This is Σ^2. Continue. Add a W-orbit of n-cubes to Σ^{n-1} for each $(n-1)$-simplex in L to get Σ^n. Σ is a cubical cell cx. W acts freely and transitively on Vert(Σ). The "link" of each vertex is L. Σ has a natural piecewise Euclidean metric in which each cube is identified with a unit cube in Euclidean space.

Theorem. (Gromov, Moussong). Σ *is* CAT(0).

Corollary. Σ *is contractible.*

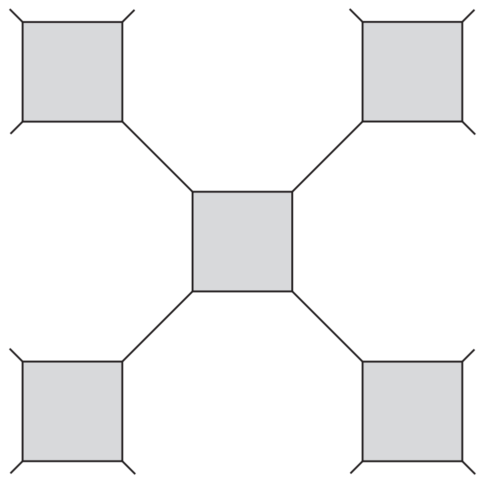
An alternative construction of Σ. L an arbitrary flag simplicial cx. Put $\square^S := [-1,1]^S$. Define $P_L \subset \square^S$ to be the union of all faces which are parallel to $\square^T$ for some $\sigma_T \subset L$.



$$L = \text{point} \cup \text{interval}$$

The group $(\mathbb{Z}/2)^S$ acts as a reflection group on $\square^S$. A fund chamber for $(\mathbb{Z}/2)^S$ on $\square^S$ is $[0,1]^S$. P_L is $(\mathbb{Z}/2)^S$-stable and a fund chamber is $K := P_L \cap [0,1]^S$. $K \cong \text{Cone}(L)$.

Vert P_L = Vert $\square^S = \{\pm 1\}^S$. The link of each vertex of P_L is $\cong L$. Let $p : \widetilde{P}_L \to P_L$ be the universal cover.



$\tilde{P}_L$

Let W be the group of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{P}_L$.

W acts as a reflection group on $\widetilde{P}_L$. Identify an element of S with the appropriate lift of the corresponding reflection in $(\mathbb{Z}/2)^S$. Check that (W, S) is the right-angled Coxeter system associated to L and $\widetilde{P}_L = \Sigma$. Moreover, $\Gamma := \pi_1(P_L)$ is a torsion-free subgp of W (it is the commutator subgp). So, we have a machine for a converting flag cx L into a finite aspherical cx P_L and gp W acting nicely on its universal cover.

III. Coxeter groups as a source of examples. A nbhd of a vertex in Σ is $\cong$ to Cone(L). So, Σ is locally $\cong$ to Cone(L). For example, if $L \cong S^{n-1}$, then Σ is an n-mfld. The reason Coxeter groups provide such a potent source of examples is that the topology of L is essentially arbitrary.

If L is a PL mfld, then $\partial\Sigma$ is the inverse limit of increasing number of connected sums of L.

Example. Take $L = \mathbf{R}P^2$. $\partial\Sigma$ is the inverse limit of connected sums of $\mathbf{R}P^2$, i.e., $\partial\Sigma$ is a Pontrjagin surface. We have:

$$H^i(W;\mathbb{Z}W) = H^i_c(\Sigma) = \check{H}^{i-1}(\partial\Sigma) \quad \text{so}$$

$$H^i(W;\mathbb{Z}W) = \begin{cases} 0, & \text{for } i = 0,1, \\ \oplus\,\mathbb{Z}, & \text{for } i = 2, \\ \mathbb{Z}/2, & \text{for } i = 3 \end{cases}$$

$\Gamma \subset W$ a torsion-free subgp of finite index. Then $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 3$, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = 2$. So, $\exists$ torsion-free gps having different cohomological dimension over $\mathbb{Z}$ than over $\mathbb{Q}$.

Example. (*Dranishnikov*) Let L_1 be a flag triangulation of $\mathbf{R}P^2$ as above. L_2 a flag cx $\cong$ space formed by gluing D^2 onto S^1 via a map of degree 3.

$$H^i(L_2) = \begin{cases} \mathbb{Z}/3, & \text{for } i = 2, \\ 0, & \text{for } i \neq 0, 2. \end{cases}$$

We get gps W_1, W_2 and spaces Σ_1, Σ_2. As before,

$$H^i_c(\Sigma_2) = \begin{cases} 0, & \text{for } i = 0, 1, \\ \oplus \mathbb{Z}, & \text{for } i = 2, \\ \mathbb{Z}/3, & \text{for } i = 3 \end{cases}$$

From the Künneth formula:

$$H_c^6(\Sigma_1 \times \Sigma_2) = \mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0.$$

So, $\mathrm{cd}(\Gamma_1 \times \Gamma_2) \neq 6 = \mathrm{cd}(\Gamma_1) + \mathrm{cd}(\Gamma_2)$. cd( ) is not additive.

Example. For $n \geq 4$, $\exists$ closed $(n-1)$-mflds M^{n-1} with same homology as S^{n-1} and $\pi_1(M^{n-1}) \neq 1$ (*homology spheres*). Take L to be a flag triangulation of a $(n-1)$-dim homology sphere. $\partial\Sigma$ is inverse limit of connected sums of L. $\partial\Sigma$ is a homology mfld with same homology as S^{n-1}. It is not an ANR (not locally 1-connected). $\pi_1^\infty(\Sigma)$ is inverse limit of free products of $\pi_1(L)$. In particular, $\pi_1^\infty(\Sigma) \neq 1$. A slight modification of Σ makes it into a contractible n-mfld which is not simply connected at ∞. So, for $n \geq 4$, $\exists$ aspherical mflds with univ cover $\ncong \mathbf{R}^n$. Another variation gives

Theorem. *For $n \geq 5$, $\exists$ nonpositively curved closed mflds with univ cover $\ncong \mathbf{R}^n$.*

IV. The reflection group trick. Given a gp π, $B\pi$ means a $K(\pi, 1)$ cx. There are plenty of examples of gps π s.t.

a) $B\pi$ is a finite cx (e.g., 2-dimensional) &

b) π has exotic properties, e.g., is not residually finite, has undecidable word problem, etc.

On the other hand, 30 years ago the only known examples of closed aspherical mflds basically had the form $\Gamma \backslash G/K$, for G a Lie gp, K a maximal compact and Γ a torsion-free discrete subgp.

The refl gp trick does the following: given π with $B\pi$ a finite cx, it produces a closed aspherical mfld M which retracts onto $B\pi$. So, $\pi_1(M)$ retracts onto π. Hence, $\pi_1(M)$ will be at least as exotic as π. It also shows that if the Novikov and Borel Conjectures hold for all aspherical mflds, then they hold $\forall$ π with $B\pi$ a finite cx. Here is the construction:

1) Thicken $B\pi$ to X, a compact mfld (e.g., embed $B\pi$ in $\mathbf{R}^n$ and take a regular nbhd of it).

2) Triangulate ∂X as a flag cx L.

3) $W :=$ the right-angled Coxeter gp associated to L; $\Gamma \subset W$ a torsion-free subgp of finite index

4) $\widetilde{M} := (W \times X)/\sim$, the result of pasting together copies of X, one for each element of W. (i.e., take Σ, remove interior of each chamber ($\cong$ Cone(L)), replace with copy of X.)

5) M:=$\widetilde{M}/\Gamma$.

M is obviously a closed mfld and retracts onto X. (The retraction is induced by $W \times X \to X$.)

Theorem. *$\widetilde{M}$ is aspherical (and so is M).*

Corollary. *$\exists$ closed aspherical mflds M s.t.*

a) $\pi_1(M)$ is not residually finite.

b) $\pi_1(M)$ has undecidable word problem.

Similarly for other properties inherited by gps which retract onto a gp with that property.

Theorem. (D. - Hausmann). *For n large, $\exists$ closed aspherical PL n-mflds not h.e. to smooth mflds.*

Sketch of proof. Choose X not h.e. to a smooth mfld with ∂.

$\square$

Remark. Using different technique Januszkiewicz and I showed for $n \geq 4$, $\exists$ closed aspherical mflds not h.e. to PL mflds.

Theorem. *For* $n \geq 4$, $\exists$ PD^n*-gps* π *which cannot be finitely presented. (Such a* π *cannot be the fundamental gp of a closed mfld).*

Sketch. Apply reflection gp trick to Bestvina–Brady examples (of finite 2-cx Z and non finitely presented quotient $\pi_1(Z) \to \pi$ s.t. induced cover $\widetilde{Z} \to Z$ is acyclic). $\square$