ANDREEV'S THEOREM
&
EULER
CHARACTERISTICS OF
4-MANIFOLDS

(with Boris Okun)

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The Euler Characteristic Conjecture. *Suppose M^{2n} is a closed aspherical manifold. Then*

$$(-1)^n \chi(M^{2n}) \geq 0.$$

Hopf and Chern conjectured this in the case M^{2n} is Riemannian and nonpositively curved.

Theorem. (Milnor – Chern) *True, for Riemannian M^4 with $K \leq 0$.*

Theorem. (D. – Okun) *True, for M^4 a cubical cx with PE metric, with $K \leq 0$. (So, $\chi(M^4) \geq 0$.)*

Definition. A simplicial cx L is a *flag complex* iff it is "determined by its 1-skeleton," i.e.,

$$\{v_0, \dots, v_k\} \subset \mathrm{Vert}(L)$$

spans a k-simplex iff $\{v_i, v_j\}$ spans an edge $\forall \{i, j\}$, $0 \le i < j \le k$.

Example. Suppose L is an m-gon ($L \cong S^1$). Then L is flag iff $L \neq \partial\Delta^2$ (i.e., $m \neq 3$).

Example. Suppose $L \cong S^2$. Then L is flag iff

- $L \neq \partial\Delta^3$
- L has no "empty triangles."

Gromov's Lemma for nonpositive curvature

X a cubical cx

$x \in \mathrm{Vert}(X)$

$L_x = \mathrm{Link}(x, X)$

Lemma. (Gromov). *X has $K \leq 0$ iff L_x is a flag cx $\forall x \in \mathrm{Vert}(X)$.*

Example. Suppose X is a 2-manifold. Then $K \leq 0$ iff at least 4 squares meet at each vertex.

The Combinatorial Gauss–Bonnet Theorem

Theorem. *X a cell cx with PE metric. Then*

$$\chi(X) = \sum_{x \in \mathrm{Vert}(X)} \lambda(L_x),$$

where $\lambda(L)$ is a local contribution depending on the (spherical) cells in L.

Cubical case: If X is cubical, then simplices of L_x are "all right" and formula for λ is:

$$\lambda(L) = \sum_{i=-1}^{\dim L} (-\frac{1}{2})^{i+1} f_i \ ,$$

where

$$f_{-1} = 1 \qquad \text{and}$$

$$f_i = \# \ (i\text{-simplices}).$$

Example. If L is an m-gon, then

$$\lambda(L) = 1 - \frac{m}{2} + \frac{m}{4} \ .$$

Flag Complex Conjecture. (Charney – D.) *Suppose L is a flag triangulation of S^{2n-1}. Then*

$$(-1)^n \lambda(L) \geq 0 \ .$$

Remark. Flag Cx Conj ⇒ Euler Char Conj for cubical manifolds.

Theorem. (D. – Okun). *True for flag triangulations of S^3, i.e.,*

$$1 - \frac{1}{2}f_0 + \frac{1}{4}f_1 - \frac{1}{8}f_2 + \frac{1}{16}f_3 \geq 0 \ .$$

Andreev's Theorem.

This gives necessary and sufficient conditions for realizing a polytope P as a convex polytope (compact or at least of finite volume) in $\mathbb{H}^3$ with prescribed dihedral angles in $(0, \pi/2]$.

First, in the compact case, P should be simple, i.e., ∂P should be dual to a triangulation L of S^3. In the right–angled case, Andreev's conditions are the following:

Conditions on L when P is compact:

- L is a flag cx.
- L has no "empty" 4 circuits.

Further conditions on L when P has ideal vertices (the ideal vertices are square cusps):

- valence 4 vertices are allowed, but there are no other empty 4 circuits.
- L is not the suspension of a 4- or 5-gon.

Theorem. (Andreev). *P can be realized as a right–angled convex polytope in $\mathbb{H}^3$ iff L satisfies the above conditions.*

Example. Let W be the group generated by reflections across the faces of P. So, $W \subset \mathrm{Isom}(\mathbb{H}^3)$ and $\mathbb{H}^3$ is tiled by copies of P.

Construction of cubical complexes.

L a simplicial cx

$I = \mathrm{Vert}(L)$

Define a subcomplex X_L of $[-1,1]^I$

$$X_L = \bigcup_{\sigma \subset L} \text{faces parallel to } [-1,1]^\sigma$$

Main property: $L_x = L, \forall x \in I$.

So,

- $L \cong S^{n-1} \Rightarrow X_L$ is an n-manifold
- L a flag cx $\Rightarrow K \leq 0$.

The group $\{\pm 1\}^I$ acts on $[-1,1]^I$ and X_L is $\{\pm 1\}^I$-stable.

$$\Sigma_L = \text{universal cover of } X_L \ .$$

$$W_L = \text{group of lifts of}$$

$$\{\pm 1\}^I - \text{action to } \Sigma_L \ .$$

W_L is a right-angled Coxeter group.

generators: $\{s_i\}_{i \in I}$

relations: $s_i^2 = 1, \quad (s_i s_j)^2 = 1$

if $\{i, j\}$ is an edge.

$$\chi(X_L) = 2^{|I|} \lambda(L)$$

$$\chi^{\text{orb}}(\Sigma_L / W_L) = \lambda(L)$$

ℓ^2-homology.

G a countable discrete group.

$\ell^2(G) = \{f : G \to \mathbb{R} \mid \sum f(g)^2 < \infty\}$

Σ a proper, cocompact G CW–complex

$$C_i^{(2)}(\Sigma) = C_i(\Sigma) \otimes \ell^2(G)$$
$$H_i^{(2)}(\Sigma) = H_i(C_*^{(2)}(\Sigma))$$

Reduced ℓ^2-homology:

$$\mathcal{H}_i(\Sigma) = \ker \partial \;/\; \overline{\operatorname{im} \partial}$$

- a Hilbert space with orthogonal G-action
- a G homotopy invariant

von Neumann dimension:

Given a G-stable, closed subspace

$$V \subset \ell^2(G)^m, \quad \text{let}$$

$$p_V : \ell^2(G)^m \to V$$

be orthogonal projection.

$$\dim_G \ell^2(G) = \mathrm{tr}_G(p_V) \in [0, \infty)$$

ℓ^2-Betti numbers:

$$b_i^{(2)}(\Sigma; G) = \dim_G \mathcal{H}_i(\Sigma)$$
$$\chi_G^{(2)}(\Sigma) = \sum (-1)^i b_i^{(2)}(\Sigma; G)$$

Atiyah's Formula:

$$\chi_G^{(2)}(\Sigma) = \sum_{\text{orbits}} \frac{(-1)^{\dim \sigma}}{|G_\sigma|}$$
$$= \chi^{\text{orb}}(\Sigma/G) \ .$$

Σ a contractible n-manifold.

G a discrete group acting properly, cocompactly on Σ.

Singer Conjecture.

$$b_i^{(2)}(\Sigma; G) = 0, \quad \forall i \neq \frac{n}{2} \ .$$

Proof of Flag Cx Conj for S^3 :

- Singer Conj $\Rightarrow$ χ Conj.
- Given L a flag triangulation of S^3, try to prove Singer Conj for Σ_L. (This $\Rightarrow$ Flag Cx Conj for L.)
- Singer Conj is true for hyperbolic manifolds (or orbifolds).
- Andreev's Thm + M–V sequence $\Rightarrow$ Singer Conj for $L \cong S^2$.
- Inductive argument $\Rightarrow$ Singer Conj for $L \cong S^3$.