L^2-cohomology of hyperplane complements

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$\mathcal{A}$ a collection of affine hyperplanes in $\mathbf{C}^n$

Definition

rk$(\mathcal{A})$, the *rank*, of $\mathcal{A}$ is the maximum codimension of a nonempty intersection of hyperplanes in $\mathcal{A}$. Usually we denote rk$(\mathcal{A})$ by l.

$\mathcal{A}$ is *essential* if rk$(\mathcal{A}) = n$.

$$\Sigma(\mathcal{A}) := \bigcup_{H \in \mathcal{A}} H$$

$$M(\mathcal{A}) := \mathbf{C}^n - \Sigma(\mathcal{A})$$

Fact

$H^*(\mathbf{C}^n, \Sigma)$ vanishes except in dimension l $(= \mathrm{rk}(\mathcal{A}))$. In fact, $\Sigma \sim \bigvee S^{l-1}$.

The number $\alpha(\mathcal{A})$

$$\begin{aligned} \alpha(\mathcal{A}) &:= \dim H_l(\mathbf{C}^n, \Sigma) \\ &:= b_l(\mathbf{C}^n, \Sigma) \\ &= \text{the number of spheres in the wedge} \end{aligned}$$

Example

Suppose $\mathcal{A}_{\mathbf{R}}$ is an essential hyperplane arrangement in $\mathbf{R}^n$. It divides $\mathbf{R}^n$ into convex regions.

- $\dim H_n(\mathbf{R}^n, \Sigma(\mathcal{A}_{\mathbf{R}}))$ is the number of bounded components of $\mathbf{R}^n - \Sigma(\mathcal{A}_{\mathbf{R}})$.
- If $\mathcal{A}$ is complexification of $\mathcal{A}_{\mathbf{R}}$, then

$$(\mathbf{C}^n, \Sigma(\mathcal{A})) \sim (\mathbf{R}^n, \Sigma(\mathcal{A}_{\mathbf{R}})).$$

- So $\alpha(\mathcal{A})$ is the number of bounded components of $\mathbf{R}^n - \mathcal{A}_{\mathbf{R}}$.

Main Theorem

Suppose $\mathcal{A}$ is an arrangement of a finite number of affine hyperplanes in $\mathbf{C}^n$ with $\text{rk}(\mathcal{A}) = l$. Then the L^2-Betti numbers of $M(\mathcal{A})$ (the complement of the hyperplanes) are all 0, except in dimension l, where

$$\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A}).$$

Here $\beta_i(\ )$ denotes the i^{th} L^2-Betti number (to be defined later).

Theorem

Suppose L is a "generic" flat complex line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension l and

$$\dim_{\mathbf{C}} H^l(M(\mathcal{A}); L) = \alpha(\mathcal{A}).$$

Basic idea

If L is a flat line bundle over S^1 giving a nonconstant local coefficient system, then $H^*(S^1; L) = 0$ for $* = 0, 1$.

Similarly, the basic idea for the Main Theorem is that the L^2-Betti numbers of S^1 vanish.

L^2-Betti numbers

The regular representation

π is a countable discrete gp.

- $$L^2\pi := \{f : \pi \to \mathbf{C} \mid \sum |f(x)|^2 < \infty\},$$
  where the sum is over all $x \in \pi$.
- $L^2\pi$ is a Hilbert space with Hermitian inner product:
  $$f \cdot f' := \sum_{x \in \pi} f(x)\overline{f'(x)}.$$
- There are unitary π-actions on $L^2\pi$ by left or right translation.

von Neumann dimension

Using the *von Neumann algebra* $\mathcal{N}\pi$ is of π-equivariant bounded linear operators on $L^2\pi$, it is possible to attach a "dimension" to any π-stable closed subspace of $\bigoplus L^2\pi$.

- $\dim_\pi V$ is a nonnegative real number.
- It is $= 0$ iff $V = 0$.
- Also, $\dim_\pi L^2\pi = 1$.

(Co)homology with local coefficients

- X a CW complex $\widetilde{X}$ its universal cover
- $C_i(\widetilde{X})$ the cellular i-chains on $\widetilde{X}$
- $\pi = \pi_1(X)$. Suppose M is a π-module.
- $$C_i(X; M) := C_i(\widetilde{X}) \otimes_\pi M$$
  $$C^i(X; M) := \mathrm{Hom}_\pi(C_i(\widetilde{X}), M)$$
- are the (co)chains with *local coefficients in M*,
- $H_*(X; M)$ and $H^*(X; M)$ are the corresponding (co)homology groups.

L^2-(co)homology

- To fix ideas, let's stick to cohomology.
- At first approximation L^2-cohomology means local coefficients in $L^2\pi$, i.e., $H^*(X; L^2\pi)$.
- $C^*(X; L^2\pi)$ is a Hilbert space but $H^*(X; L^2\pi)$ need not be. $\mathrm{Ker}\,\delta$ is a closed subspace but $\mathrm{Im}\,\delta$ need not be.
- Define
$$\mathcal{H}^*(X; L^2\pi) := \mathrm{Ker}\,\delta / \overline{\mathrm{Im}\,\delta}.$$
- $\mathcal{H}^*(X; L^2\pi)$ is a closed, π-stable subspace of $C^*(X; L^2\pi)$. (It is $= \mathrm{Ker}\,\delta \cap (\mathrm{Im}\,\delta)^\perp$.)

- If X is a finite complex, then $C^i(X; L^2\pi)$ is a direct sum of finitely many copies of $L^2\pi$ (one for each i-cell of X).
- So the closed, π-stable subspace $\mathcal{H}^i(X; L^2\pi)$ has a well-defined von Neumann dimension called the i^{th} L^2-*Betti number*

$$\beta_i(X) := \dim_\pi \mathcal{H}^i(X; L^2\pi).$$

If X is a finite complex then $C^*(X; L^2\pi)$ can be identified with the square summable cochains on $\widetilde{X}$ (denoted by $L^2C^*(\widetilde{X})$). The corresponding (reduced) cohomology groups are denoted $L^2\mathcal{H}^*(\widetilde{X})$.

Lemma

The L^2-Betti numbers of S^1 vanish.

Corollary

All L^2-Betti numbers of $S^1 \times B$ vanish.

Proof.

Künneth Formula. □

Rough idea of proof of theorems

Suppose $\mathcal{U} = \{U_i\}$ is a cover of X by connected open subsets and $\mathcal{V}$ is a subcover s.t.

- $\mathcal{V} = \{U_i \in \mathcal{U} \mid \pi_1(U_i) \neq 1\}$.
- $\forall \sigma \in N(\mathcal{U})$, $\pi_1(U_\sigma) \to \pi_1(X)$ $(= \pi)$ is injective.
  (Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$ and
  $U_\sigma = U_{i_1} \cap \cdots \cap U_{i_k}$, where $\sigma = \{i_1, \dots i_k\}$.)
- $\forall \sigma \in N(\mathcal{U}) - N(\mathcal{V})$, U_σ is contractible.
- $\forall \sigma \in N(\mathcal{V})$, $U_\sigma = S^1 \times$ (something).

There is a Mayer-Vietoris spectral sequence converging to $H^*(X; L^2\pi)$ with E_2-term

$$E_2^{p,q} = H^p(N(\mathcal{U}); H^q(U_\sigma; L^2(\pi_1(U_\sigma)),$$

where the coefficient system is the functor $\sigma \longrightarrow H^q(U_\sigma; L^2(\pi_1(U_\sigma))$.

Hypotheses $\implies$ $E_2^{p,q}$ is concentrated on the bottom row $q = 0$ and

$$E_2^{p,0} = H^p(N(\mathcal{U}), N(\mathcal{V})) \otimes L^2\pi.$$

(We are ignoring terms with vanishing L^2-Betti numbers.) So, $\beta_p(X) = b_p(N(\mathcal{U}), N(\mathcal{V}))$.

More on hyperplane arrangements

- $\mathcal{A}$ is a hyperplane arrangement in $\mathbf{C}^n$ and Σ is the union of hyperplanes.
- A *subspace of* $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$.
- $L(\mathcal{A})$ is the poset of subspaces of $\mathcal{A}$.
- $\mathcal{A}$ is *central* if the intersection of all hyperplanes in $\mathcal{A}$ is nonempty.
- Given $G \in L(\mathcal{A})$, we have a central subarrangement

$$\mathcal{A}_G := \{H \in \mathcal{A} \mid G \subset H\}.$$

Definition

An open convex subset $U \subset \mathbf{C}^n$ is *small* (wrt $\mathcal{A}$) if

- $\{G \in L(\mathcal{A}) \mid G \cap U \neq \emptyset\}$ has a minimum element U_{min}.
- Given $H \in \mathcal{A}$, $H \cap U \neq \emptyset \iff H \supset U_{min}$.

- Given a small U, put $\widehat{U} := U - \Sigma$.
- So, $\widehat{U} \cong M(\mathcal{A}_{U_{min}})$, the complement of a central arrangement.

- Choose a cover $\mathcal{U}$ of $\mathbf{C}^n$ by small open sets. For each $\sigma \in N(\mathcal{U})$, $\mathcal{A}_\sigma$ is the corresponding central arrangement.
- $\mathcal{V} := \{U \in \mathcal{U} \mid \mathcal{A}_\sigma \text{ is not trivial}\}$. $\mathcal{V}$ is an open cover of a nbhd of Σ homotopy equiv to Σ.
- Since each element of $\mathcal{U}$ is convex,

$$H^*(N(\mathcal{U}), N(\mathcal{V})) = H^*(\mathbf{C}^n, \Sigma).$$

Similarly, we have open covers $\widehat{\mathcal{U}} := \{\widehat{U}\}_{U \in \mathcal{U}}$ and $\widehat{\mathcal{V}} := \{\widehat{U}\}_{U \in \mathcal{V}}$. ($\widehat{\mathcal{U}}$ is an open cover of $M(\mathcal{A})$.)

Key point

$N(\widehat{\mathcal{V}}) = N(\mathcal{V})$ and $N(\widehat{\mathcal{U}}) = N(\mathcal{U})$.

Lemma

Suppose $\mathcal{A}$ is a nonempty central arrangement. Then

$$M(\mathcal{A}) = S^1 \times (\mathit{something}).$$

Main Theorem

The L^2-Betti numbers of $M(\mathcal{A})$ are all 0, except in dimension l, where $\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A})$.

Proof.

- $\widehat{\mathcal{U}}$ is open cover of $M(\mathcal{A})$.
- $\forall \sigma \subset N(\widehat{\mathcal{U}}) - N(\widehat{\mathcal{V}})$, $U_\sigma \sim *$.
- $\forall \sigma \subset N(\widehat{\mathcal{V}})$, $U_\sigma = S^1 \times$ (something).
- Use spectral sequence and fact that $H^*(N(\widehat{\mathcal{U}}), N(\widehat{\mathcal{V}})) = H^*(\mathbf{C}^n, \Sigma)$ to complete the proof.

□