Introduction

$L^2$-Betti numbers

Hyperplane arrangements

$L^2$-cohomology of hyperplane complements

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1 Introduction

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A a collection of affine hyperplanes in \( \mathbb{C}^n \)

**Definition**

\( \text{rk}(A) \), the *rank*, of \( A \) is the maximum codimension of a nonempty intersection of hyperplanes in \( A \). Usually we denote \( \text{rk}(A) \) by \( l \).

\( A \) is *essential* if \( \text{rk}(A) = n \).

\[ Σ(A) := ∪_{H ∈ A} H \]

\[ M(A) := \mathbb{C}^n - Σ(A) \]
Fact

$H^*(\mathbb{C}^n, \Sigma)$ vanishes except in dimension $l (= \text{rk}(\mathcal{A}))$. In fact, $\Sigma \sim \bigvee S^{l-1}$. 

The number $\alpha(\mathcal{A})$

\[
\alpha(\mathcal{A}) := \dim H_l(\mathbb{C}^n, \Sigma) := b_l(\mathbb{C}^n, \Sigma)
\]

= the number of spheres in the wedge
Example

Suppose $\mathcal{A}_R$ is an essential hyperplane arrangement in $\mathbb{R}^n$. It divides $\mathbb{R}^n$ into convex regions.

- $\dim H_n(\mathbb{R}^n, \Sigma(\mathcal{A}_R))$ is the number of bounded components of $\mathbb{R}^n - \Sigma(\mathcal{A}_R)$.

- If $\mathcal{A}$ is complexification of $\mathcal{A}_R$, then

$$\left(\mathbb{C}^n, \Sigma(\mathcal{A})\right) \sim \left(\mathbb{R}^n, \Sigma(\mathcal{A}_R)\right).$$

- So $\alpha(\mathcal{A})$ is the number of bounded components of $\mathbb{R}^n - \mathcal{A}_R$. 
Main Theorem

Suppose $\mathcal{A}$ is an arrangement of a finite number of affine hyperplanes in $\mathbb{C}^n$ with $\text{rk}(\mathcal{A}) = l$. Then the $L^2$-Betti numbers of $M(\mathcal{A})$ (the complement of the hyperplanes) are all 0, except in dimension $l$, where

$$\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A}).$$

Here $\beta_i(\ )$ denotes the $i^{th}$ $L^2$-Betti number (to be defined later).
Theorem

Suppose $L$ is a “generic” flat complex line bundle over $M(\mathcal{A})$. Then $H^*(M(\mathcal{A}); L)$ vanishes except in dimension $l$ and

$$\dim_{\mathbb{C}} H^l(M(\mathcal{A}); L) = \alpha(\mathcal{A}).$$

Basic idea

If $L$ is a flat line bundle over $S^1$ giving a nonconstant local coefficient system, then $H^*(S^1; L) = 0$ for $* = 0, 1$.

Similarly, the basic idea for the Main Theorem is that the $L^2$-Betti numbers of $S^1$ vanish.
$L^2$-Betti numbers

The regular representation

$\pi$ is a countable discrete gp.

\[ L^2_\pi := \{ f : \pi \to \mathbb{C} \mid \sum |f(x)|^2 < \infty \}, \]

where the sum is over all $x \in \pi$.

$L^2_\pi$ is a Hilbert space with Hermitian inner product:

\[ f \cdot f' := \sum_{x \in \pi} f(x)f'(x). \]

There are unitary $\pi$-actions on $L^2_\pi$ by left or right translation.
**Introduction**

L₂-Betti numbers

Hyperplane arrangements

**The regular representation**

L₂-(co)homology

Idea of the proof

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**von Neumann dimension**

Using the von Neumann algebra \( \mathcal{N}_\pi \) is of \( \pi \)-equivariant bounded linear operators on \( L^2_\pi \), it is possible to attach a “dimension” to any \( \pi \)-stable closed subspace of \( \bigoplus L^2_\pi \).

- \( \dim_\pi V \) is a nonnegative real number.
- It is \( = 0 \) iff \( V = 0 \).
- Also, \( \dim_\pi L^2_\pi = 1 \).
(Co)homology with local coefficients

- $X$ a CW complex
- $\tilde{X}$ its universal cover
- $C_i(\tilde{X})$ the cellular $i$-chains on $\tilde{X}$
- $\pi = \pi_1(X)$. Suppose $M$ is a $\pi$-module.

$$C_i(X; M) := C_i(\tilde{X}) \otimes_{\pi} M$$
$$C^i(X; M) := \text{Hom}_\pi(C_i(\tilde{X}), M)$$

are the (co)chains with local coefficients in $M$,

$H_*(X; M)$ and $H^*(X; M)$ are the corresponding (co)homology groups.
To fix ideas, let’s stick to cohomology.

At first approximation $L^2$-cohomology means local coefficients in $L^2\pi$, i.e., $H^*(X; L^2\pi)$.

$C^*(X; L^2\pi)$ is a Hilbert space but $H^*(X; L^2\pi)$ need not be. $\text{Ker} \, \delta$ is a closed subspace but $\text{Im} \, \delta$ need not be.

Define

$$\mathcal{H}^*(X; L^2\pi) := \text{Ker} \, \delta / \text{Im} \, \delta.$$ 

$\mathcal{H}^*(X; L^2\pi)$ is a closed, $\pi$-stable subspace of $C^*(X; L^2\pi)$. (It is $= \text{Ker} \, \delta \cap (\text{Im} \, \delta)^\perp$.)
If $X$ is a finite complex, then $C^i(X; L^2\pi)$ is a direct sum of finitely many copies of $L^2\pi$ (one for each $i$-cell of $X$).

So the closed, $\pi$-stable subspace $H^i(X; L^2\pi)$ has a well-defined von Neumann dimension called the $i^{th}$ $L^2$-Betti number

$$\beta_i(X) := \dim_\pi H^i(X; L^2\pi).$$

If $X$ is a finite complex then $C^*(X; L^2\pi)$ can be identified with the square summable cochains on $\tilde{X}$ (denoted by $L^2 C^*(\tilde{X})$). The corresponding (reduced) cohomology groups are denoted $L^2 H^*(\tilde{X})$. 
**Lemma**

The $L^2$-Betti numbers of $S^1$ vanish.

**Corollary**

All $L^2$-Betti numbers of $S^1 \times B$ vanish.

**Proof.**

Künneth Formula.
Rough idea of proof of theorems

Suppose $\mathcal{U} = \{U_i\}$ is a cover of $X$ by connected open subsets and $\mathcal{V}$ is a subcover s.t.

- $\mathcal{V} = \{U_i \in \mathcal{U} \mid \pi_1(U_i) \neq 1\}$.
- $\forall \sigma \in N(\mathcal{U}), \pi_1(U_\sigma) \rightarrow \pi_1(X) (= \pi)$ is injective. (Here $N(\mathcal{U})$ denotes the nerve of $\mathcal{U}$ and $U_\sigma = U_{i_1} \cap \cdots \cap U_{i_k}$, where $\sigma = \{i_1, \ldots, i_k\}$.)
- $\forall \sigma \in N(\mathcal{U}) - N(\mathcal{V}), U_\sigma$ is contractible.
- $\forall \sigma \in N(\mathcal{V}), U_\sigma = S^1 \times (\text{something})$. 

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$L^2$-cohomology of hyperplane complements
There is a Mayer-Vietoris spectral sequence converging to $H^*(X; L^2_\pi)$ with $E_2$-term

$$E_2^{p,q} = H^p(N(U); H^q(U_\sigma; L^2(\pi_1(U_\sigma))),$$

where the coefficient system is the functor $\sigma \rightarrow H^q(U_\sigma; L^2(\pi_1(U_\sigma))).$

Hypotheses $\implies E_2^{p,q}$ is concentrated on the bottom row $q = 0$ and

$$E_2^{p,0} = H^p(N(U), N(V)) \otimes L^2_\pi.$$

(We are ignoring terms with vanishing $L^2$-Betti numbers.) So, $\beta_p(X) = b_p(N(U), N(V)).$
More on hyperplane arrangements

- $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{C}^n$ and $\Sigma$ is the union of hyperplanes.
- A *subspace of* $\mathcal{A}$ is a nonempty intersection of hyperplanes in $\mathcal{A}$.
- $L(\mathcal{A})$ is the poset of subspaces of $\mathcal{A}$.
- $\mathcal{A}$ is *central* if the intersection of all hyperplanes in $\mathcal{A}$ is nonempty.
- Given $G \in L(\mathcal{A})$, we have a central subarrangement
  \[ \mathcal{A}_G := \{ H \in \mathcal{A} \mid G \subset H \}. \]
Definition

An open convex subset $U \subset \mathbb{C}^n$ is small (wrt $A$) if

- $\{ G \in L(A) \mid G \cap U \neq \emptyset \}$ has a minimum element $U_{\min}$.
- Given $H \in A$, $H \cap U \neq \emptyset \iff H \supset U_{\min}$.

- Given a small $U$, put $\hat{U} := U - \Sigma$.
- So, $\hat{U} \cong M(A_{U_{\min}})$, the complement of a central arrangement.
Choose a cover $\mathcal{U}$ of $\mathbb{C}^n$ by small open sets. For each $\sigma \in N(\mathcal{U})$, $A_\sigma$ is the corresponding central arrangement.

$\mathcal{V} := \{ U \in \mathcal{U} \mid A_\sigma \text{ is not trivial} \}$. $\mathcal{V}$ is an open cover of a nbhd of $\Sigma$ homotopy equiv to $\Sigma$.

Since each element of $\mathcal{U}$ is convex,

$$H^*(N(\mathcal{U}), N(\mathcal{V})) = H^*(\mathbb{C}^n, \Sigma).$$
Similarly, we have open covers $\hat{\mathcal{U}} := \{\hat{U}\}_{U \in \mathcal{U}}$ and $\hat{\mathcal{V}} := \{\hat{U}\}_{U \in \mathcal{V}}$. ($\hat{\mathcal{U}}$ is an open cover of $M(\mathcal{A})$.)

**Key point**

$N(\hat{\mathcal{V}}) = N(\mathcal{V})$ and $N(\hat{\mathcal{U}}) = N(\mathcal{U})$.

**Lemma**

*Suppose $\mathcal{A}$ is a nonempty central arrangement. Then*

$$M(\mathcal{A}) = S^1 \times (\text{something}).$$
Main Theorem

The $L^2$-Betti numbers of $M(\mathcal{A})$ are all 0, except in dimension $l$, where $\beta_l(M(\mathcal{A})) = \alpha(\mathcal{A})$.

Proof.

- $\hat{\mathcal{U}}$ is open cover of $M(\mathcal{A})$.
- $\forall \sigma \subset N(\hat{\mathcal{U}}) - N(\hat{\mathcal{V}})$, $U_\sigma \sim \ast$.
- $\forall \sigma \subset N(\hat{\mathcal{V}})$, $U_\sigma = S^1 \times \text{(something)}$.
- Use spectral sequence and fact that $H^*(N(\hat{\mathcal{U}}), N(\hat{\mathcal{V}})) = H^*(\mathbb{C}^n, \Sigma)$ to complete the proof.