The compactly supported cohomology of buildings

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- In 1976 Borel & Serre calculated the compactly supported cohomology of affine buildings. Their motivation was to study the cohomology and finiteness properties of S-arithmetic groups
- In 1998 I calculated the compactly supported cohomology of (the complex associated to) any Coxeter system (W, S).
- In 2002, John Meier and I gave a similar calculation for the compactly supported cohomology of any building of type (W, S). However, there was a mistake in the proof.
- In 2006, Dymara, Januszkiewicz, Okun and I gave an equivariant version of the formula for Coxeter gps as well as a proof for right-angled buildings.
- Here we do the general case using an idea from our 2007 paper on weighted L^2-cohomology.

References

- A. Borel and J.-P. Serre, Cohomologie d'immeubles et de groupes S-arithmétiques. *Topology* **15** (1976), 211–232.
- M.W. Davis *The cohomology of a Coxeter group with group ring coefficents*, Duke Math. J. **91** (1998), 297–314.
- M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Cohomology of Coxeter groups with group ring coefficents: II*, Algebraic & Geometric Topology **6** (2006), 1289–1318.
- ———, *Weighted L^2-cohomology of Coxeter groups*, Geometry & Topology **11** (2007), 47–138.
- M.W. Davis and J. Meier, *The topology at infinity of Coxeter groups and buildings*, Comment. Math. Helv. **77** (2002), 746–766.

Suppose (W, S) is a Coxeter system. A *bldg* of type (W, S) is

- a set Φ (of chambers)
- a family of equivalence relations indexed by S ("adjacency relations") st $\forall s \in S$, each s-equivalence class contains at least 2 elements.
- a *W-valued distance function* $\delta : \Phi \times \Phi \to W$ (st φ, φ' are connected by a minimal gallery of type $(s_1, \ldots, s_n) \iff \delta(\varphi, \varphi') = s_1 \cdots s_n$).

Definition

A *gallery* is a sequence of successively adjacent chambers. Given $T \subset S$, a gallery is a *T-gallery* if any 2 successive chambers are t-adjacent for some $t \in T$. A *T-residue* is a T-gallery connected component.

Example (Thin bldgs)

W is itself a bldg. w, w' are *s-adjacent* if $w' = ws$.
$\delta : W \times W \to W$ is defined by $\delta(w, w') = w^{-1}w'$.
A T-residue is a coset of W_T (where $W_T := \langle T \rangle$.)

There are different ways to associate a topological space to a bldg Φ.

Mirror structures

A *mirror structure* on a CW complex X is a family of subcomplexes $\{X_s\}_{s\in S}$. For each $x \in X$, put

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Define

$$\mathcal{U}(\Phi, X) = (\Phi \times X)/\sim$$

where $(\varphi, x) \sim (\varphi', x') \iff x = x'$ and φ and φ' belong to the same $S(x)$-residue.

Example (The classical realization of Tits)

Let Δ be a simplex, of dimension $\text{Card}(S) - 1$.
$\{\Delta_s\} = \{\text{codim 1 faces}\}$.
Then $\mathcal{U}(\Phi, \Delta)$ is the *classical realization*.

Example (The standard realization)

A subset $T \subset S$ is *spherical* if W_T is finite. Let $\mathcal{S}$ be the poset of spherical subsets (including $\emptyset$). Let K be the geometric realization of $\mathcal{S}$. K is a subset of the barycentic subdivision of Δ. Put $K_s = K \cap \Delta_s$. $\mathcal{U}(\Phi, K)$ is the *standard realization*. Only the spherical faces of K are nonempty and only spherical residues contribute to $\mathcal{U}(\Phi, K)$.

Suppose Φ is a bldg of type (W, S).
Let A be the set of finitely supported $\mathbb{Z}$-valued functions on Φ, ie, A is the free abelian gp on Φ. For each $T \subset S$, put

$$A^T := \{f \in A \mid f \text{ is constant on each } T\text{-residue}\}.$$

Note that $A^T = 0$ whenever T is not spherical. Also,

Example (The thin bldg)

If $\Phi = W$, then $A = \mathbb{Z}W$ (the group ring) and $A^T = (\mathbb{Z}W)^{W_T}$ ($\cong \mathbb{Z}[W/W_T]$ when T is spherical).

$A^U \subset A^T$ whenever $U \supset T$. So, put

$$A^{>T} := \sum_{U \supsetneq T} A^U.$$

Fact

$A^T / A^{>T}$ is free abelian.

Let $\hat{A}^T$ be a complementary summand for $A^{>T}$ in A^T.

Main Theorem

$$H_c^*(\mathcal{U}(\Phi, K) = \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes \hat{A}^T, \text{ where}$$

$$K^{S-T} = \bigcup_{s\in S-T} K_s.$$

X a mirrored CW complex, $X^{(k)} := \{k\text{-cells}\}$. Define a cochain complex and "coefficient system" $\mathcal{I}(A)$ on X by

$$\mathcal{C}^k(X;\mathcal{I}(A)) := \prod_{c\in X^{(k)}} A^{S(c)}.$$

$$X^f := \{x \in X \mid S(x) \text{ is spherical}\}$$

Proposition

If X is compact, then $\mathcal{H}^*(X;\mathcal{I}(A)) = H_c^*(\mathcal{U}(\Phi, X^f))$.

Strategy for proving Main Thm

Prove a decomposition result for for the coefficient system $\mathcal{I}(A)$ so that cohomology with coefficients in any given summand can be computed.

Decomposition Theorem

For any spherical subset T,

$$A^T = \bigoplus_{U \subset T} \hat{A}^U.$$

In particular, for $T = \emptyset$, we have

$$A = \bigoplus_{U \in \mathcal{S}} \hat{A}^U.$$

To prove this we are led back to classical realizations of bldgs.

Proposition

Let Δ *be the simplex of dimension* $n = \mathrm{Card}(S) - 1$*. Then* $\mathcal{H}^*(\Delta; \mathcal{I}(A))$ *(which is* $= H_c^*(\mathcal{U}(\Phi, \Delta^f)))$ *is free abelian concentrated in dimension n.*

You also need some similar statements corresponding to certain subcomplexes of Δ.