WEIGHTED L^2-COHOMOLOGY OF COXETER GROUPS

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Definitions

Coxeter systems, the parameter $\mathbf{q}$,

Hecke algebras,

Hecke -von Neumann algebras,

the growth series $W(\mathbf{q})$, spaces $\mathcal{U}$ and Σ,

$L^2_{\mathbf{q}}$ cellular cochains, $L^2_{\mathbf{q}}$-cohomology,

$L^2_{\mathbf{q}}$-Euler characteristics and

$L^2_{\mathbf{q}}$-Betti numbers.

Results

$\chi_{\mathbf{q}}(\Sigma) = 1/W(\mathbf{q})$,

L^2-cohomology of buildings,

Calculations of $b^k_{\mathbf{q}}(\Sigma)$

A Decomposition Theorem

Interpolating between cohomology and

cohomology with compact supports

A version of the Singer Conjecture

Coxeter groups

S a finite set

(m_{st}) a *Coxeter matrix*, i.e.,

a symmetric $S \times S$ matrix with

entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise.} \end{cases}$$

$$W := \langle S \mid s^2 = 1,\ (st)^{m_{st}} = 1 \rangle.$$

(W, S) is a *Coxeter system.*

$l : W \to \mathbb{N}$ is word length.

For $T \subset S$,

$$W_T := \langle \{s\}_{s \in T} \rangle.$$

If W_T is finite, T is a *spherical subset.*

$$\begin{aligned} \mathcal{S} &:= \{T \subset S \mid W_T \text{ is finite}\} \\ &= \{\text{spherical subsets}\} \end{aligned}$$

The *nerve* of (W, S) is the simplicial complex L, where

$$\text{Vert}(L) = S, \quad \text{and}$$
$$\text{Simp}(L) = \mathcal{S}_{>\emptyset}$$

The parameter $\mathbf{q}$

I an index set

$i : S \to I$ a function s.t.

$i(s) = i(s')$ whenever s and s' are conjugate.

$$\mathbf{q} := (q_i)_{i \in I} \in (0, \infty)^I$$

Write q_s instead of $q_{i(s)}$.

If $w = s_1 \dots s_l$ is a reduced expression, set

$$q_w := q_{s_1} \dots q_{s_l}.$$

It is independent of the reduced expression.

$\mathbf{R}^{(W)} :=$ {finitely supported $W \to \mathbf{R}$}

An inner product on $\mathbf{R}^{(W)}$

$$\langle e_v, e_w \rangle_{\mathbf{q}} := q_w \delta_{vw}.$$

$$L^2_{\mathbf{q}}(W) := \text{completion of } \mathbf{R}^{(W)}.$$

Hecke algebra

$\mathbf{R}_{\mathbf{q}}[W]$ deformed group algebra

$$e_s e_w = \begin{cases} e_{sw}, & l(sw) > l(w); \\ q_s e_{sw} + (q_s - 1) e_w, & l(sw) < l(w). \end{cases}$$

Hecke-von Neumann algebra

$\mathcal{N}_{\mathbf{q}}$ = a completion of $\mathbf{R}_{\mathbf{q}}(W)$

:= $\{\mathbf{R}_{\mathbf{q}}(W)$-equivariant bounded linear operators $L^2_{\mathbf{q}}(W) \to L^2_{\mathbf{q}}(W)\}$

von Neumann dimension

For $\phi \in \mathcal{N}_{\mathbf{q}}$,

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\phi) = \langle \phi(e_1), e_1 \rangle_{\mathbf{q}}.$$

For $\Phi = (\phi_{ij}) \in M_m(\mathcal{N}_{\mathbf{q}})$,

$$\mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\Phi) = \sum \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(\phi_{ii}).$$

Given a $\mathbf{R}_{\mathbf{q}}(W)$-stable, closed subspace

$$V \subset \oplus L^2_{\mathbf{q}}(W), \quad \text{let}$$

$$p_V : \oplus L^2_{\mathbf{q}}(W) \to \oplus L^2_{\mathbf{q}}(W)$$

be orthogonal projection onto V.

$$\dim_{\mathcal{N}_{\mathbf{q}}} V = \mathrm{tr}_{\mathcal{N}_{\mathbf{q}}}(p_V) \in [0, \infty).$$

Growth series

$\mathbf{t} := (t_i)_{i \in I}$ an I-tuple of indeterminates.

$$W(\mathbf{t}) := \sum_{w \in W} t_w \ .$$

It is a rational function in $\mathbf{t}$, e.g.,

$$\frac{1}{W(\mathbf{t})} = \sum_{T \in \mathcal{S}} \frac{(-1)^{\mathrm{Card}(T)}}{W_T(\mathbf{t}^{-1})},$$

where $\mathbf{t}^{-1} := (t_i^{-1})$.

$\mathcal{R} :=$ region of convergence of $W(\mathbf{t})$

$\mathcal{R}^{-1} := \{\mathbf{q} \mid \mathbf{q}^{-1} \in \mathcal{R}\}$

The basic construction.

X a CW complex

$(X_s)_{s\in S}$ a family of subcomplexes.

Define

$$\mathcal{U}(W, X) := (W \times X)/\sim$$

where $\sim$ is the equivalence relation generated by

$$(w, x) \sim (ws, x) \text{ if } x \in X_s.$$

$$K := \text{geometric realization of } \mathcal{S}$$
$$= \text{Cone}(L)$$
$$K_s := \text{geom. realization of } \mathcal{S}_{\geq\{s\}}$$

K is a *chamber*. K_s is a *mirror*.

The complex Σ

$$\Sigma := \mathcal{U}(W, K)$$

Cellular cochains

$$\begin{aligned} \mathcal{E}_k & \quad \text{the set of } k\text{-cells in } \mathcal{U} \\ C^k(\mathcal{U}) & \quad \text{the } k\text{-cochains on } \mathcal{U} \\ & = \text{functions on } \mathcal{E}_k \\ & = \{ \sum_{\text{infinite}} a_\sigma \, \sigma \mid \sigma \in \mathcal{E}_k \} \\ C^k_c(\mathcal{U}) & = \{ \sum_{\text{finite}} a_\sigma \, \sigma \} \end{aligned}$$

How about weighted $L^2_{\mathbf{q}}$-cochains?

Given $\sigma \in \mathcal{E}_k$, let $d(\sigma)$ be the shortest $w \in W$ s.t. $w^{-1}\sigma \subset X$.

Define an inner product on $C_c^k(\mathcal{U})$

$$\langle \sigma, \tau \rangle_{\mathbf{q}} := q_{d(\sigma)} \delta_{\sigma\tau}$$

$L^2_{\mathbf{q}} C^k(\mathcal{U}) :=$ the completion of $C_c^k(\mathcal{U})$

$L^2_{\mathbf{q}} C^*(\mathcal{U})$ is a $\mathcal{N}_{\mathbf{q}}$-module and $\delta : L^2_{\mathbf{q}} C^k(\mathcal{U}) \to L^2_{\mathbf{q}} C^{k+1}(\mathcal{U})$ is a map of $\mathcal{N}_{\mathbf{q}}$-modules.

$L^2_{\mathbf{q}} \mathcal{H}^k(\mathcal{U})$ reduced $L^2_{\mathbf{q}}$-cohomology

$$:= \ker \delta / \overline{\operatorname{im} \delta}$$

$$b^k_{\mathbf{q}}(\mathcal{U}) := \dim L^2_{\mathbf{q}} \mathcal{H}^k(\mathcal{U})$$

$$\chi_{\mathbf{q}}(\mathcal{U}) := \sum (-1)^k b^k_{\mathbf{q}}(\mathcal{U})$$

Theorem. (Dymara)

$$\chi_{\mathbf{q}}(\Sigma) = \frac{1}{W(\mathbf{q})}$$

Theorem. (Dymara) *If* $\mathbf{q} \in \mathcal{R}$*, then* $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$ *is concentrated in dimension* 0*.*

Theorem. (Dymara) *Suppose* Φ *is a building of type* (W, S) *with a chamber transitive automorphism group* G*. Then its* L^2*-Betti number (with respect to* G*),* $b^k(\Phi; G)$*, is equal to* $b^k_{\mathbf{q}}(\Sigma)$*.*

Here $\mathbf{q}$ is the "thickness" of the building.

Remark. For buildings only integral values of $\mathbf{q}$ matter!

Given $w \in W$, set

$$\mathrm{In}(w) := \{s \in S \mid l(ws) < l(w)\}$$

Key fact: $\mathrm{In}(w) \in \mathcal{S}$.

Given $T \subset S$, set

$$W^T := \{w \in W \mid \mathrm{In}(w) = T\}$$

$$X^T := \bigcup_{s \in T} X_s$$

Theorem. (D.)

$$H_*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H_*(X, X^T) \otimes \mathbf{R}^{(W^T)}.$$

Corollary. Σ *is acyclic.*

Theorem. (D.)

$$H_c^*(\mathcal{U}) \cong \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \mathbf{R}^{(W^T)}.$$

Main Theorem.

- *If* $\mathbf{q} \in \mathcal{R}$, *then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^T) \otimes D^T.$$

- *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$L^2_{\mathbf{q}}\mathcal{H}^*(\mathcal{U}) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes D^{S-T}.$$

Here the D^T *are certain* $\mathcal{N}_{\mathbf{q}}$*-submodules of* $L^2_{\mathbf{q}}(W)$.

Decomposition Theorem. *We have direct sum decompositions of $\mathcal{N}_{\mathbf{q}}$-modules.*

- *If* $\mathbf{q} \in \mathcal{R}$, *then*

$$L^2_{\mathbf{q}} = \bigoplus_{T \in \mathcal{S}} D^T$$

- *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$L^2_{\mathbf{q}} = \bigoplus_{T \in \mathcal{S}} D^{S-T}$$

(In the formulas of the previous two theorems, the RHS might only be a dense subspace of the LHS).

Corollary. • *If* $\mathbf{q} \in \mathcal{R}$, *then*

$$H_k(\mathcal{U}; \mathbf{R}) \to L^2_{\mathbf{q}} \mathcal{H}_k(\mathcal{U})$$

is injective with dense image.

• *If* $\mathbf{q} \in \mathcal{R}^{-1}$, *then*

$$H^k_c(\mathcal{U}; \mathbf{R}) \to L^2_{\mathbf{q}} \mathcal{H}^k(\mathcal{U})$$

is injective with dense image.

Example. Suppose Σ is an n-manifold (e.g., L is a triangulation of S^{n-1}).

- If $\mathbf{q} \in \mathcal{R}$, then $L^2_{\mathbf{q}} \mathcal{H}^*(\Sigma)$ is concentrated in dimension 0.
- If $\mathbf{q} \in \mathcal{R}^{-1}$, then $L^2_{\mathbf{q}} \mathcal{H}^*(\Sigma)$ is concentrated in dimension n.

Example. Suppose $n = 2$ and L is a circle. $\mathbf{q} = q \in (0, \infty)$.

ρ, the radius of convergence of $W(t)$.

Then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension

$$\begin{cases} 0, & \text{if } q \leq \rho; \\ 1, & \text{if } \rho < q < 1/\rho; \\ 2, & \text{if } q \geq \rho. \end{cases}$$

Question. *What happens in the intermediate range,* $\mathbf{q} \notin \mathcal{R} \cup \mathcal{R}^{-1}$*?*

Conjecture. (A version of the Singer Conjecture). *Suppose* Σ *is an* n*-manifold . Then for* $\mathbf{q} \leq \mathbf{1}$*,*

$$L^2_{\mathbf{q}}\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k > \frac{n}{2}.$$

Similarly, for $\mathbf{q} \geq \mathbf{1}$*,*

$$L^2_{\mathbf{q}}\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k < \frac{n}{2}.$$

Some idempotents in $\mathcal{N}_{\mathbf{q}}$

For $\mathbf{q} \in \mathcal{R}_T$,

$$a_T := \frac{1}{W_T(\mathbf{q})} \sum_{w \in W_T} e_w.$$

For $\mathbf{q} \in \mathcal{R}_T^{-1}$,

$$h_T := \frac{1}{W_T(\mathbf{q}^{-1})} \sum_{w \in W_T} (-1)^{l(w)} q_w^{-1} e_w.$$

$$A_T := \operatorname{im} a_T \quad \text{and} \quad H_T := \operatorname{im} h_T.$$

$$D_T := A_{S-T} \cap \left(\sum_{U \subsetneq T} A_{S-U} \right)^{\perp}$$