CAT(0)-**spaces**

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“CAT(0)-space” is a term invented by Gromov.

Also, called “Hadamard space.” Roughly, a space which is “non-positively curved” and simply connected.

C = “Comparison” or “Cartan”

A = “Aleksandrov”

T = “Toponogov”

Theorem. *Suppose X is* CAT(0) *and* $\Gamma \subset \mathrm{Isom}(X)$ *is a discrete subgp acting properly on X. Then X is a model for* $\underline{E}\Gamma$ *(i.e., $\forall$ finite subgp $F \subset \Gamma$, X^F is contractible).*

The proof is based on two facts about CAT(0)-spaces.

Fact 1. (Bruhat-Tits Fixed Point Theorem). *A bounded subset of X (e.g. a bounded orbit) has a unique "center of mass."*

Fact 2. $\exists$ *a unique geodesic between any two points of X.*

Proof of Theorem. $F \subset \Gamma$ a finite subgroup. By Fact 1, $X^F \neq \emptyset$. Choose a basepoint $x_0 \in X^F$. Given $x \in X$, $c_x : [0, d] \to X$ is the geodesic from x_0 to x ($d := d(x_0, x)$). If $x \in X^F$, then $\forall \gamma \in F$, $\gamma \cdot c_x$ and c_x are two geodesic segments with the same endpoints. By Fact 2, they are equal. So, $\mathrm{Im}(c_x) \subset X^F$. Therefore, geodesic contraction gives a contraction of X^F to a point. (Geodesic contraction $h : X \times [0, 1] \to X$ is defined by $h(x, t) := c_x(ts)$, where s is the arc length parameter for c_x.) □

In the 1940's and 50's Aleksandrov introduced the notion of a "length space" and the idea of curvature bounds on length space. He was primarily interested in lower curvature bounds (defined by reversing the CAT(κ) inequality). He proved that a length metric on S^2 has nonnegative curvature iff it is isometric to the boundary of a convex body in $\mathbb{E}^3$. First Aleksandrov proved this result for nonnegatively curved piecewise Euclidean metrics on S^2, i.e., any such metric was isometric to the boundary of a convex polytope. By using approximation techniques, he then deduced the general result (including the smooth case) from this.

One of the first papers on nonpositively curved spaces was a 1948 paper of Busemann. The recent surge of interest in nonpositively curved polyhedral metrics was initiated by Gromov's seminal 1987 paper.

Some definitions. Let (X, d) be a metric space. A path $c : [a, b] \to X$ is a *geodesic* (or a *geodesic segment*) if $d(c(s), c(t)) = |s - t|$ for all $s, t \in [a, b]$. (X, d) is a *geodesic space* if any two points can be connected by a geodesic segment.

Given a path $c : [a, b] \to X$, its *length*, $l(c)$, is defined by

$$l(c) := \sup\{\sum_{i=1}^{n} d(c(t_{i-1}, t_i)\},$$

where $a = t_0 < t_1 < \cdots t_n = b$ runs over all possible subdivisions.

The metric space (X, d) is a *length space* if

$$d(x, y) = \inf\{l(c) \mid c \text{ is a path from } x \text{ to } y\}.$$

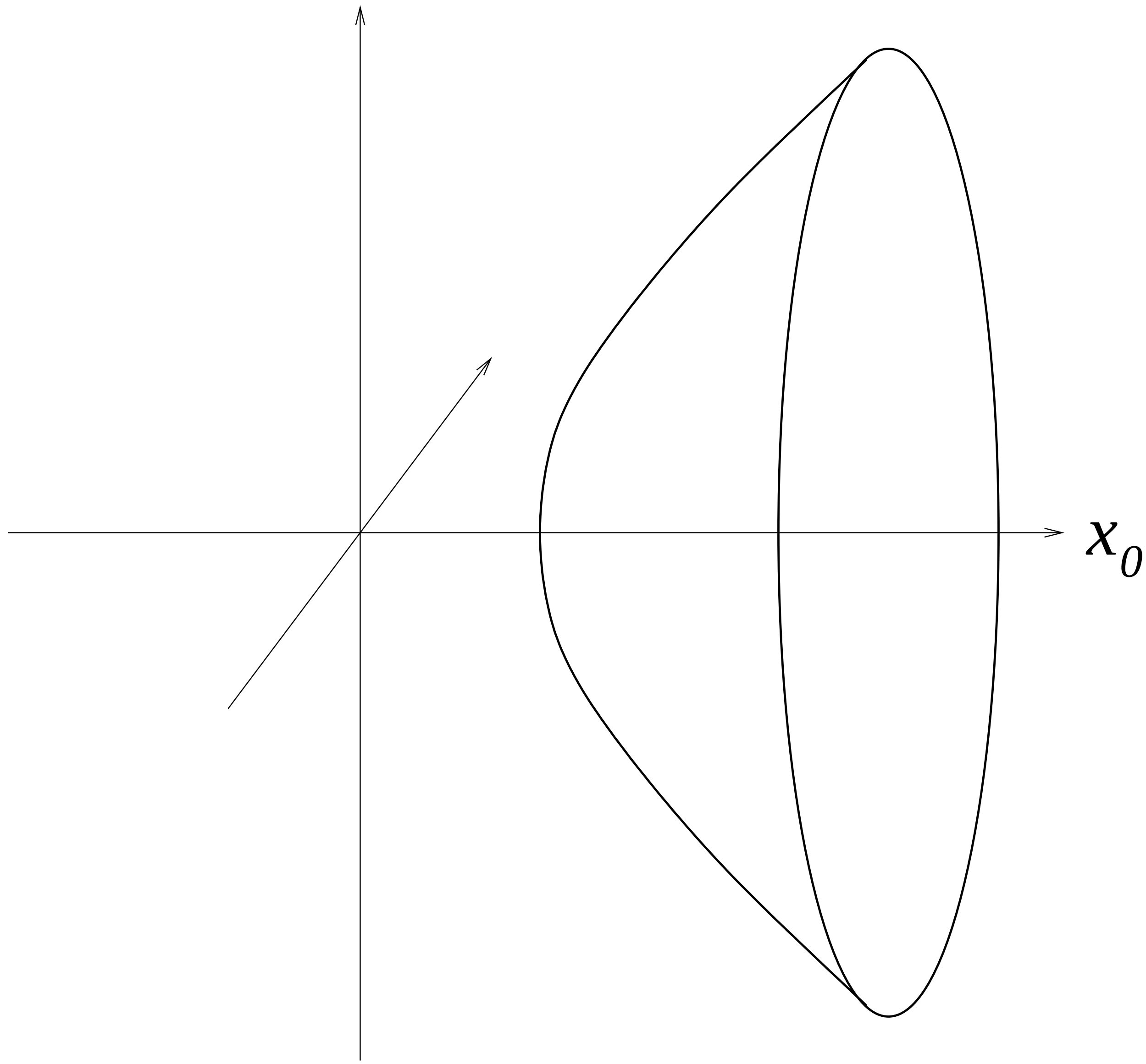
(Here we allow ∞ as a possible value of d.) Thus, a length space is a geodesic space iff the above infimum is always realized and is $\neq \infty$.

The CAT(κ)**-inequality.** For $\kappa \in \mathbb{R}$, $\mathbb{X}^2_\kappa$ is the simply connected, complete, Riemannian 2-manifold of constant curvature κ:

- $\mathbb{X}_0^2$ is the Euclidean plane $\mathbb{E}^2$.

- If $\kappa > 0$, then $\mathbb{X}_\kappa^2 = \mathbb{S}^2$ with its metric rescaled so that its curvature is κ (i.e., it is the sphere of radius $1/\sqrt{\kappa}$).

- If $\kappa < 0$, then $\mathbb{X}_\kappa^2 = \mathbb{H}^2$, the hyperbolic plane, with its metric rescaled.



x_0

A *triangle* T in a metric space X is a configuration of three geodesic segments (the "edges") connecting three points (the "vertices") in pairs. A *comparison triangle* for T is a triangle T^* in $\mathbb{X}^2_\kappa$ with the same edge lengths. When $\kappa \leq 0$, a comparison triangle always exists. When $\kappa > 0$, a comparison triangle exists $\iff l(T) \leq 2\pi/\sqrt{\kappa}$, where $l(T)$ denotes the sum of the lengths of the edges. (The number $2\pi/\sqrt{\kappa}$ is the length of the equator in a 2-sphere of curvature κ.)

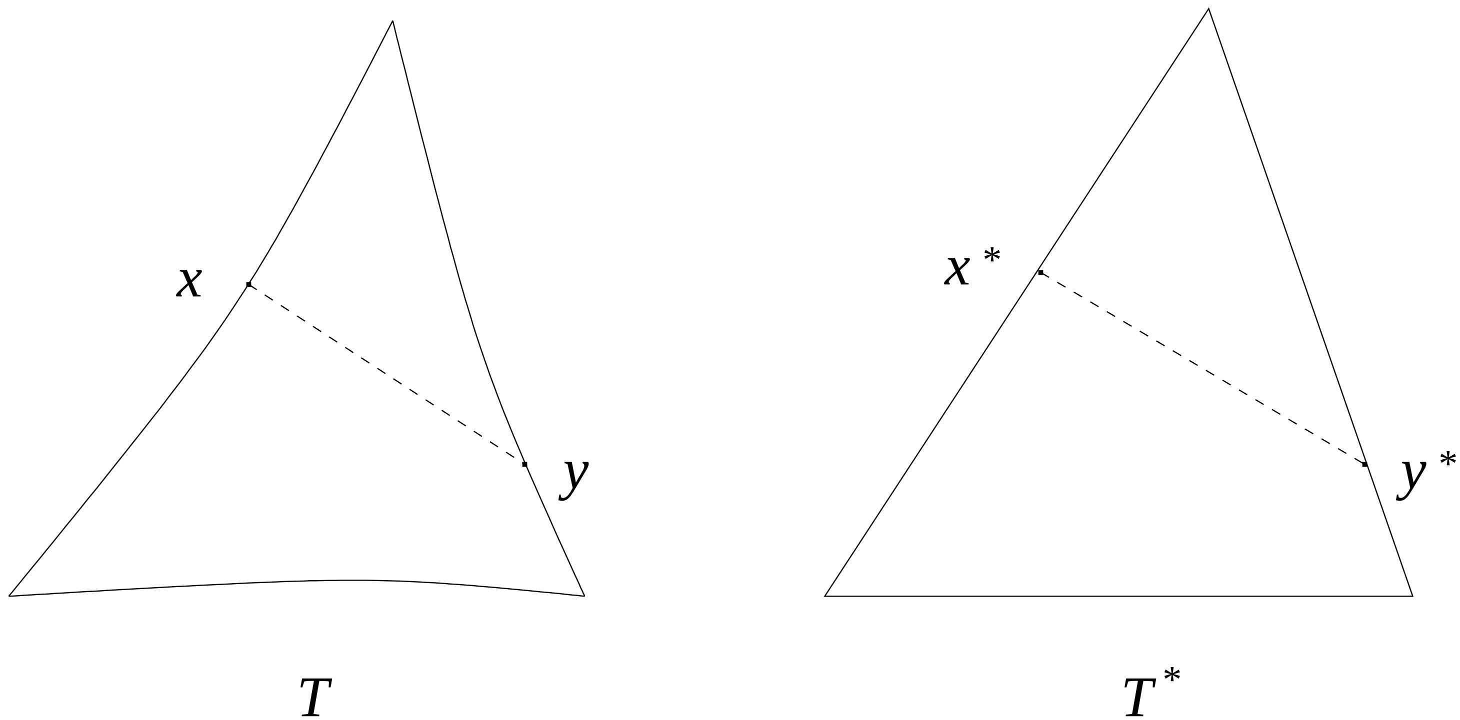
If T^* is a comparison triangle for T, then for each edge of T there is a well-defined isometry, denoted $x \to x^*$, which takes the given edge of T onto the corresponding edge of T^*. A metric space X *satisfies* CAT(κ) (or is a CAT(κ)-*space*) if the following two conditions hold:

- If $\kappa \leq 0$, then X is a geodesic space, while if $\kappa > 0$, it is required there be a geodesic segment between any two points $< \pi/\sqrt{\kappa}$ apart.

- (*The* CAT(κ) *inequality*). For any triangle T (with $l(T) < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and any two points $x, y \in T$, we have

$$d(x, y) \leq d^*(x^*, y^*),$$

where x^*, y^* are the corresponding points in the comparison triangle T^* and d^* is distance in $\mathbb{X}^2_\kappa$.



Definition. A metric space X *has curvature* $\leq \kappa$ if the CAT(κ) inequality holds locally.

Observations. • If $\kappa' < \kappa$, then CAT(κ') $\implies$ CAT(κ).

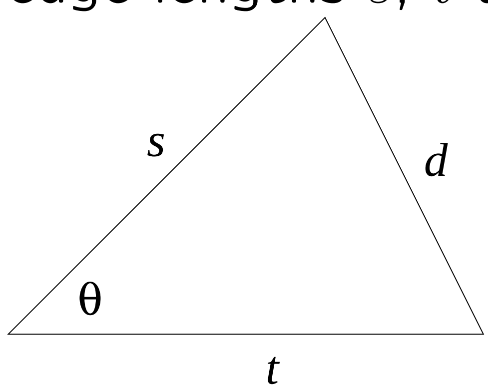
• CAT(0) $\implies$ contractible.

• curvature ≤ 0 $\implies$ aspherical.

Theorem. (Aleksandrov and Toponogov). *A Riemannian mfld has sectional curvature* $\leq \kappa$ *iff* CAT(κ) *holds locally.*

The cone on a CAT(1)**-space.** The *cone on* X, denoted Cone(X), is the quotient space of $X \times [0, \infty)$ by the equivalence relation $\sim$ defined by $(x, s) \sim (y, t)$ if and only if $(x, s) = (y, t)$ or $s = t = 0$. The image of (x, s) in Cone(X) is denoted $[x, s]$. The *cone of radius* r, denoted Cone(X, r), is the image of $X \times [0, r]$.

Given a metric space X and $\kappa \in \mathbb{R}$, we will define a metric d_κ on Cone(X). (When $\kappa > 0$, the definition will only make sense on the open cone of radius $\pi/\sqrt{\kappa}$.) The idea: when $X = \mathbb{S}^{n-1}$, by using "polar coordinates" and the exponential map, Cone($\mathbb{S}^{n-1}$) can be identified with (an open subset of) $\mathbb{X}^n_\kappa$. Transporting the constant curvature metric on $\mathbb{X}^n_\kappa$ to Cone($\mathbb{S}^{n-1}$), gives a formula for d_κ on Cone($\mathbb{S}^{n-1}$). The same formula defines a metric on Cone(X) for any metric space X. To write this formula recall the Law of Cosines in $\mathbb{X}^2_\kappa$. Suppose we have a triangle in $\mathbb{X}^2_\kappa$ with edge lengths s, t and d and angle θ between the first two sides.



The Law of Cosines:

- in $\mathbb{E}^2$:

$$d^2 = s^2 + t^2 - 2st\cos\theta$$

- in $\mathbb{S}^2_\kappa$:

$$\cos\sqrt{\kappa}d = \cos\sqrt{\kappa}s\cos\sqrt{\kappa}t + \sin\sqrt{\kappa}s\sin\sqrt{\kappa}t\cos\theta$$

- in $\mathbb{H}^2_\kappa$:

$$\cosh\sqrt{-\kappa}d = \cosh\sqrt{-\kappa}s\cosh\sqrt{-\kappa}t + \sinh\sqrt{-\kappa}s\sinh\sqrt{-\kappa}t\cos\theta$$

Given $x, y \in X$, put $\theta(x,y) := \min\{\pi, d(x,y)\}$. Define the metric d_0 on Cone(X) by

$$d_0([x,s],[y,t]) := (s^2 + t^2 - 2st\cos\theta(x,y))^{1/2}.$$

Metrics d_κ, $\kappa \neq 0$ are defined similarly. Denote Cone(X) equipped with the metric d_κ by Cone$_\kappa(X)$.

Remark. If X is a $(n-1)$-dimensional spherical polytope, then Cone$_\kappa(X)$ is isometric to a convex polyhedral cone in $\mathbb{X}^n_\kappa$.

Proposition. *Suppose X is a complete and that any two points of distance $\leq \pi$ can be joined by a geodesic. Then*

- $\text{Cone}_\kappa(X)$ *is a complete geodesic space.*

- $\text{Cone}_\kappa(X)$ *is* $\text{CAT}(\kappa)$ *if and only if X is* $\text{CAT}(1)$.

Polyhedra of piecewise constant curvature

We call a convex polytope in $\mathbb{X}_\kappa^n$ an $\mathbb{X}_\kappa$-*polytope* when we don't want to specify n.

Definition. Suppose $\mathcal{F}$ is the poset of faces of a cell complex. An $\mathbb{X}_\kappa$-*cell structure* on $\mathcal{F}$ is a family $(C_F)_{F\in\mathcal{F}}$ of $\mathbb{X}_\kappa$-polytopes s.t. whenever $F' < F$, $C_{F'}$ is isometric to the corresponding face of C_F.

Examples. (i) (*Euclidean cell complexes*). Suppose a collection of convex polytopes in $\mathbb{E}^n$ is a convex cell cx in the classical sense. Then the union Λ of these polytopes is a $\mathbb{X}_0$-polyhedral complex.

(ii) (*Regular cells*). There are three families of regular polytopes which occur in each dimension n: the n-simplex, the n-cube and the n-octahedron. The simplex and the cube have the additional property that each of their faces is of the same type. Requiring each cell of a cell cx to be a simplex (resp. cube) we get the corresponding notion of a simplicial (resp. cubical) cx. In both cases (n-simplex and n-cube) we can be realize the polytope as a regular polytope in $\mathbb{X}^n_\kappa$. This polytope is determined, up to congruence, by its edge length. So, we can define an $\mathbb{X}_\kappa$-structure on a simplicial cx or a cubical cx simply by specifying an edge length.

By now there are many constructions of nonpositively curved PE polyhedral complexes. In the case of PE cubical complexes there are the following:

- products of graphs,
- constructions via reflection groups,
- hyperbolizations (e.g. Gromov's Möbius band procedure),

- certain blowups of $\mathbf{R}P^n$ and

- many other constructions.

Suppose Λ is a $\mathbb{X}_\kappa$-polyhedral complex. A path $c : [a, b] \to \Lambda$ is *piecewise geodesic* if there is a subdivision

$a = t_0 < t_1 \cdots < t_k = b$ s.t. for $1 \le i \le k$, $c([t_{i-1}, t_i])$ is contained in a single (closed) cell of Λ and s.t. $c|_{[t_{i-1}, t_i]}$ is a geodesic. The *length* of the piecewise geodesic c is defined by $l(c) := \sum_{i=1}^{n} d(c(t_i), c(t_{i-1}))$, i.e., $l(c) := b - a$. Λ has a natural length metric:

$$d(x, y) := \inf\{l(c) \mid c \text{ is a piecewise geodesic from } x \text{ to } y\}.$$

(We allow ∞ as a possible value of d.) The length space Λ is a *piecewise constant curvature polyhedron*. As $\kappa = +1, 0, -1$, it is, respectively, *piecewise spherical*, *piecewise Euclidean* or *piece-*

wise hyperbolic.

Piecewise spherical ($= PS$) polyhedra play a distinguished role in this theory. In any $\mathbb{X}_\kappa$-polyhedral cx each "link" naturally has a PS structure.

Geometric links. Suppose P is an n-dimensional $\mathbb{X}_\kappa$-polytope and $x \in P$. The *geometric link*, Lk(x, P), (or "space of directions") of x in P is the set of all inward-pointing unit tangent vectors at x. It is an intersection of a finite number of half-spaces in $\mathbb{S}^{n-1}$. If x lies in the interior of P, then $\mathsf{Lk}(x, P) \cong \mathbb{S}^{n-1}$, while if x is a vertex of P, then $\mathsf{Lk}(x, P)$ is a spherical polytope. Similarly, if $F \subset P$ is a face of P, then $\mathsf{Lk}(F, P)$ is the set of inward-pointing unit vectors in the normal space to F (in the tangent space of P).

If Λ is an $\mathbb{X}_\kappa$-polyhedral complex and $x \in \Lambda$, define

$$\mathsf{Lk}(x,\Lambda) := \bigcup_{x \in P} \mathsf{Lk}(x,P).$$

$\mathsf{Lk}(x,P)$ is a PS length space. Similarly, $\mathsf{Lk}(F,\Lambda) := \bigcup \mathsf{Lk}(F,P)$.

Theorem. *Let* Λ *be an* $\mathbb{X}_\kappa$*-polyhedral complex. TFAE*

- $\mathsf{curv}(\Lambda) \leq \kappa$.
- $\forall x \in \Lambda$, $\mathsf{Lk}(x, \Lambda)$ *is* CAT(1).
- $\forall$*cells* P *of* Λ, $\mathsf{Lk}(P, \Lambda)$ *is* CAT(1).
- $\forall v \in \mathsf{Vert}\,\Lambda$, $\mathsf{Lk}(v, \Lambda)$ *is* CAT(1).

Proof. Any $x \in \Lambda$ has a nbhd isometric to $\mathrm{Cone}_\kappa(\mathrm{Lk}(x, \Lambda), \varepsilon)$. $\square$

Piecewise Euclidean Cubical complexes.

Definition. A simplicial cx L is a *flag complex* if it has the following property: a nonempty finite set of vertices spans a simplex of L iff any two vertices in the set are connected by an edge.

Examples. The boundary of an m-gon is flag iff $m > 3$. The derived cx (= the set of chains) of any poset is a flag cx. Hence, the barycentric subdivision of any cell cx is a flag cx.

Definition. A spherical simplex is *all right* if each of its edge lengths is $\pi/2$. (Equivalently, it is all right if each of its dihedral angles is $\pi/2$.) A PS simplicial complex is *all right* if each of its simplices is all right.

Example. Since each dihedral angle of a Euclidean cube is $\pi/2$, the link of each cell of a Euclidean cube is an all right simplex. Hence, the link of any cell in a PE cubical complex is an all right

PS simplicial cx.

Lemma. (Gromov). *Let L be an all right PS simplicial cx. Then*

$$L \textit{ is } \mathrm{CAT}(1) \iff \textit{flag cx.}$$

Corollary. *A cubical cx is* CAT(0) *iff the link of each vertex is a flag cx.*

A cubical cx with given link. Suppose L is a flag cx and $S := \operatorname{Vert} L$. Put

$$\mathcal{S}(L) := \{T \subseteq S \mid T \text{ is the vertex set of a simplex}\}$$

$$\square^S := [-1, 1]^S.$$

Let $X_L \subseteq \square^S$ be the union of all faces which are parallel to $\square^T$ for some $T \in \mathcal{S}(L)$.

Examples. If $L = S^0$, then $X_L = \partial\square^2 = \partial(\text{4-gon}) \cong S^1$. If

$L = \partial(\text{4-gon})$, then $X_L = T^2 \subset \partial\Box^4$.

Observations.

- $\text{Vert}\, X_L = \{\pm 1\}^S$. So, X_L has $2^{\text{Card}(S)}$ vertices. The link of each vertex is identified with L.

- L a flag cx $\implies$ X_L nonpositively curved.

- If, in addition, $L \cong S^{n-1}$, then X_L is aspherical mfld.

Remark. The group $(\mathbb{Z}/2)^S$ acts as a reflection group on $\square^S$. Let $\widetilde{X}_L$ be the universal cover of X_L and let W_L be the group of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{X}_L$. W_L is a "right-angled" Coxeter group acting as a reflection group on $\widetilde{X}_L$. If L is a flag cx, then $\widetilde{X}_L$ is CAT(0).