Cohomology of relatives to Coxeter groups

Mike Davis

(joint work with Boris Okun)

Oberwolfach
April 30, 2010

1. Introduction
   - Coxeter groups
   - Some previous results

2. Computations
   - Graph products
   - A spectral sequence
   - Artin groups and Bestvina-Brady groups

We want to compute
$H^*(G; \mathbf{Z}G)$ or $H^*(G; L^2(G))$ or possibly
$H^*_G(X; M)$, where X is a G-space and M a G-module.

For $G =$

- Coxeter group
- Artin group
- Bestvina-Brady group
- graph product of groups, all of which are infinite or else all are finite.

Coxeter groups

(W, S) a Coxeter system

$$\begin{aligned}\mathcal{S} &:= \{T \subset S \mid |W_T| < \infty\} \\ &= \text{the poset of spherical subsets}\end{aligned}$$

$L = L(W, S)$ is the *nerve* of (W, S), ie, the simplicial complex with vertex set S and simplices the nonempty elements of $\mathcal{S}$.
K = geometric realization of $\mathcal{S}$ = the cone on L.
$K_s = \text{Cone}(L_s)$, where L_s denotes the link of s in L.

$$K^{S-T} := \bigcup_{s \in S-T} K_s, \quad \partial K := K^S, \quad K_T := \bigcap_{s \in T} K_s$$

Theorem (D)

$H^*(W;\mathbf{Z}W) = \bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes \widehat{A}^T$, *for a certain free abelian gp* $\widehat{A}^T$.

Remarks

- (DDJMO) A similar formula holds for any locally finite bldg of type (W, S).
- In particular since a graph product of finite groups is a locally finite *RAB*, a similar formula holds for such graph products.

Theorem (D - Leary)

A the Artin gp associated to (W, S) *and* X *its Salvetti cx. Then*

$$H^*(X; \mathbf{Z}A) = H^*(K, \partial K) \otimes L^2(A)$$

If $K(A, 1)$ *Conjecture holds, we can replace the left hand side by* $H^*(A; \mathbf{Z}A)$.

I should be saying “reduced” L^2-cohomology.

Theorem (Jensen-Meier)

If A is a RAAG, then

$$H^*(A; \mathbf{Z}A) = \bigoplus_{T \in \mathcal{S}} H^{*-|T|}(K_T, \partial K_T) \otimes \textit{free abelian gp}$$

Remark

$\partial K_T = \mathrm{Lk}(T)$, the link of the simplex corresponding to T in L. On the other hand, K^{S-T} is the complement of this simplex in L.

This theorem was originally proved by using the first theorem and result of DJ that any *RAAG* is commensurable with *RACG*.

The setup

Γ a graph with $\mathrm{Vert}(\Gamma) = S$; L the flag cx determined by the graph and (W, S) the *RACS* with nerve L.
Let $\{G_s\}_{s \in S}$ be a family of groups. For each $T \in \mathcal{S}$ put $G_T := \prod_{s \in T} G_s$ and let

$$G = \prod_{\Gamma} G_s$$

denote their graph product.

Theorem

Suppose $G = \prod_{\Gamma} G_s$, where each G_s is infinite. Then

$$H^n(G; \mathbf{Z}G) = \bigoplus_{T \in \mathcal{S}} \bigoplus_{p+q=n} H^p(K_T, \partial K_T; H^q(G_T; \mathbf{Z}G))$$

- Here G_T is the direct product $\prod_{s\in T} G_s$. So, neglecting torsion
$$H^*(G_T;\mathbf{Z}G_T) = \bigotimes_{\sum i_s = *} H^{i_s}(G_s;\mathbf{Z}G_T)$$
- I should be putting a Gr in front of the LHS for "associated graded".

Idea of proof

- Suppose $\mathcal{P}$ is a poset and $\{X_a\}_{a\in\mathcal{P}}$ is a poset of spaces and
$X = \bigcup_{a\in\mathcal{P}} X_a$
- There is a spectral sequence with

$$E_1^{p,q} = C^p(\mathrm{Flag}(\mathcal{P}); \mathcal{H}^q(\mathcal{V}))$$

converging to $H^*(X)$ where the (nonconstant) coefficient system $\mathcal{H}^q(\mathcal{V})$ associates to a simplex $\sigma \in \mathrm{Flag}(\mathcal{P})$ the abelian gp $H^q(X_{\min a})$
- Want conditions to insure a decomposition:

$$E_2^{p,q} = E_\infty^{p,q} = \bigoplus_{a\in\mathcal{P}} H^p(\mathrm{Flag}(\mathcal{P}_{\le a}), \mathrm{Flag}(\mathcal{P}_{<a}) : H^q(X_a))$$

Put $X_{<a} := \bigcup_{b<a} X_b$.

Main Lemma

The condition we need for this decomposition to hold is that $H^*(X_a) \to H^*(X_{<a})$ *is the* 0*-map,* $\forall a \in \mathcal{P}$

In all situations in which we will apply this lemma, $\mathcal{P} = \mathcal{S}$ so that $\mathrm{Flag}(\mathcal{P}) = K$ and $\forall T \in \mathcal{S}$,

$$(\mathrm{Flag}(\mathcal{P}_{\leq T}), \mathrm{Flag}(\mathcal{P}_{<T})) = (K_T, \partial K_T).$$

The key point

for applying this to graph products is that when each G_s is infinite, $H^0(G_s; \mathbf{Z}G_s) = 0$, so by Künneth Formula, $H^*(G_T; \mathbf{Z}G_T) \to H^*(G_U; \mathbf{Z}G_T)$ is the 0-map whenever $U < T$.

Artin groups

Suppose

- $A = A_L$ is the Artin group associated to (W, S), X_L the associated Salvetti complex.
- For each $T \subset S$, A_T is the subgp generated by T. When T is spherical $H^*(A_T; \mathbf{Z}A_T)$ is free abelian and concentrated in degree $|T|$ (ie A_T is a duality gp)

Theorem

$$H^n(X_L; \mathbf{Z}A_L) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|}(K_T, \partial K_T) \otimes H^{|T|}(A_T; \mathbf{Z}A_L)$$

Bestvina-Brady groups

- Let A_L be the *RAAG* associated to the *RACS* (W, S), where $L =$ nerve of (W, S) (ie A_L is a graph product of $\mathbf{Z}$'s).
- $BB_L =$ kernel of $A_L \to \mathbf{Z}$ which sends each generator to 1.
- If L is acyclic, then BB_L is called a *Bestvina-Brady group*.

Theorem

Suppose BB_L is Bestvina-Brady. Then the cohomology of BB_L with group ring coefficients is isomorphic to that of A_L shifted up in degree by 1:

$$H^n(BB_L; \mathbf{Z}BB_L) = \bigoplus_{T \in \mathcal{S}} H^{n-|T|+1}(K_T, \partial K_T) \otimes \mathbf{Z}(BB_L/BB_L \cap A_T).$$

L^2-cohomology of BB_L

Let $L^2 b^k(BB_L)$ be the k^{th} L^2-Betti number of BB_L.

Theorem

Suppose BB_L is Betvina-Brady. Then

$$L^2 b^k(BB_L) = \sum_{s \in S} b^k(K_s, \partial K_s)$$

where $b^k(K_s, \partial K_s)$ $(= \overline{b}^{k-1}(L_s))$ is the ordinary Betti number.