Cohomology of relatives to Coxeter groups

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1 Introduction
   - Coxeter groups
   - Some previous results

2 Computations
   - Graph products
   - A spectral sequence
   - Artin groups and Bestvina-Brady groups
We want to compute
\[ H^* (G; \mathbb{Z}_G) \] or \[ H^* (G; L^2(G)) \] or possibly \[ H^*_G(X; M), \] where \( X \) is a \( G \)-space and \( M \) a \( G \)-module.

For \( G = \)
- Coxeter group
- Artin group
- Bestvina-Brady group
- graph product of groups, all of which are infinite or else all are finite.
Coxeter groups

$(W, S)$ a Coxeter system

$$S := \{ T \subset S \mid |W_T| < \infty \}$$

$= \text{the poset of spherical subsets}$

$L = L(W, S)$ is the \textit{nerve} of $(W, S)$, ie, the simplicial complex with vertex set $S$ and simplices the nonempty elements of $S$. $K = \text{geometric realization of } S = \text{the cone on } L$. $K_s = \text{Cone}(L_s)$, where $L_s$ denotes the link of $s$ in $L$.

$$K^{S-T} := \bigcup_{s \in S-T} K_s, \quad \partial K := K^S, \quad K_T := \bigcap_{s \in T} K_s$$
Theorem (D)

\[ H^* (W; \mathbb{Z}W) = \bigoplus_{T \in S} H^* (K, K^{S-T}) \otimes \hat{A}^T, \text{ for a certain free abelian gp } \hat{A}^T. \]

Remarks

- (DDJMO) A similar formula holds for any locally finite bldg of type \((W, S)\).
- In particular since a graph product of finite groups is a locally finite \(RAB\), a similar formula holds for such graph products.
Theorem (D - Leary)

A the Artin gp associated to \((W, S)\) and \(X\) its Salvetti cx. Then

\[
H^*(X; \mathbb{Z} A) = H^*(K, \partial K) \otimes L^2(A)
\]

If \(K(A, 1)\) Conjecture holds, we can replace the left hand side by \(H^*(A; \mathbb{Z} A)\).

I should be saying “reduced” \(L^2\)-cohomology.
**Theorem (Jensen-Meier)**

*If A is a RAAG, then*

\[ H^*(A; \mathbb{Z}A) = \bigoplus_{T \in S} H^{|T|}(K_T, \partial K_T) \otimes \text{free abelian gp} \]

**Remark**

\( \partial K_T = \text{Lk}(T) \), the link of the simplex corresponding to \( T \) in \( L \). On the other hand, \( K^{S-T} \) is the complement of this simplex in \( L \).

This theorem was originally proved by using the first theorem and result of DJ that any RAAG is commensurable with RACG.
The setup

Γ a graph with Vert(Γ) = S; L the flag cx determined by the graph and (W, S) the RACS with nerve L.
Let \{G_s\}_{s \in S} be a family of groups. For each \( T \in S \) put \( G_T := \prod_{s \in T} G_s \) and let

\[
G = \prod_{\Gamma} G_s
\]

denote their graph product.

Theorem

Suppose \( G = \prod_{\Gamma} G_s \), where each \( G_s \) is infinite. Then

\[
H^n(G; \mathbb{Z}G) = \bigoplus_{T \in S} \bigoplus_{p+q=n} H^p(K_T, \partial K_T; H^q(G_T; \mathbb{Z}G))
\]
Here $G_T$ is the direct product $\prod_{s \in T} G_s$. So, neglecting torsion

$$H^*(G_T; \mathbb{Z}G_T) = \bigotimes \sum_{i_s = *} H^{i_s}(G_s; \mathbb{Z}G_T)$$

I should be putting a Gr in front of the LHS for “associated graded”.
Idea of proof

Suppose $\mathcal{P}$ is a poset and $\{X_a\}_{a \in \mathcal{P}}$ is a poset of spaces and
$$X = \bigcup_{a \in \mathcal{P}} X_a$$

There is a spectral sequence with
$$E_1^{p,q} = C^p(\text{Flag}(\mathcal{P}); \mathcal{H}^q(\mathcal{V}))$$

converging to $H^*(X)$ where the (nonconstant) coefficient system $\mathcal{H}^q(\mathcal{V})$ associates to a simplex $\sigma \in \text{Flag}(\mathcal{P})$ the abelian gp $H^q(X_{\text{min} a})$

Want conditions to insure a decomposition:
$$E_2^{p,q} = E_\infty^{p,q} = \bigoplus_{a \in \mathcal{P}} H^p(\text{Flag}(\mathcal{P}_{\leq a}), \text{Flag}(\mathcal{P}_{< a}) : H^q(X_a))$$
Put $X_{<a} := \bigcup_{b < a} X_b$.

**Main Lemma**

The condition we need for this decomposition to hold is that $H^*(X_a) \rightarrow H^*(X_{<a})$ is the 0-map, $\forall a \in P$

In all situations in which we will apply this lemma, $P = S$ so that $\text{Flag}(P) = K$ and $\forall T \in S$,

$$(\text{Flag}(P_{\leq T}), \text{Flag}(P_{< T})) = (K_T, \partial K_T).$$

**The key point**

for applying this to graph products is that when each $G_s$ is infinite, $H^0(G_s; \mathbb{Z}G_s) = 0$, so by K"unneth Formula,

$H^*(G_T; \mathbb{Z}G_T) \rightarrow H^*(G_U; \mathbb{Z}G_T)$ is the 0-map whenever $U < T$. 

Suppose

- \( A = A_L \) is the Artin group associated to \((W, S)\), \( X_L \) the associated Salvetti complex.
- For each \( T \subset S \), \( A_T \) is the subgp generated by \( T \). When \( T \) is spherical \( H^*(A_T; \mathbb{Z} A_T) \) is free abelian and concentrated in degree \( |T| \) (ie \( A_T \) is a duality gp)

**Theorem**

\[
H^n(X_L; \mathbb{Z} A_L) = \bigoplus_{T \in S} H^{n-|T|}(K_T, \partial K_T) \otimes H^{|T|}(A_T; \mathbb{Z} A_L)
\]
Let $A_L$ be the RAAG associated to the RACS $(W, S)$, where $L = \text{nerve of } (W, S)$ (ie $A_L$ is a graph product of $\mathbb{Z}$’s).

$BB_L = \text{kernel of } A_L \to \mathbb{Z}$ which sends each generator to 1.

If $L$ is acyclic, then $BB_L$ is called a Bestvina-Brady group.

**Theorem**

Suppose $BB_L$ is Bestvina-Brady. Then the cohomology of $BB_L$ with group ring coefficients is isomorphic to that of $A_L$ shifted up in degree by 1:

$$H^n(BB_L; \mathbb{Z}BB_L) = \bigoplus_{T \in S} H^{n-|T|+1}(K_T, \partial K_T) \otimes \mathbb{Z}(BB_L/BB_L \cap A_T).$$
Let $L^2 b^k(BB_L)$ be the $k^{th}$ $L^2$-Betti number of $BB_L$.

**Theorem**

Suppose $BB_L$ is Betvina-Brady. Then

$$L^2 b^k(BB_L) = \sum_{s \in S} b^k(K_s, \partial K_s)$$

where $b^k(K_s, \partial K_s) = \overline{b}^{k-1}(L_s)$ is the ordinary Betti number.