

# Compactly supported cohomology of buildings

Mike Davis

(joint work with Jan Dymara, Tadeusz Januszkiewicz, John Meier and Boris Okun)

New Brunswick  
December 7, 2010

## Why compactly supported cohomology?

- Suppose a discrete group  $G$  acts properly and cocompactly on contractible CW complex  $\Omega$ . Then

$$H_c^*(\Omega) = H^*(G; \mathbb{Z}G).$$

- The RHS is an invariant of group.  $H^1(G; \mathbb{Z}G)$  tells the number of ends of a gp; higher degrees are related to the gp's cohomological dimension and to its duality properties.

## Combinatorial approach to buildings

The data for a bldg is first of all a Coxeter system  $(W, S)$ . Then

- a set  $\Phi$  of “chambers,”
- a family of equivalence relations on  $\Phi$  (“adjacency relations”) indexed by the set  $S$  (of fundamental generators for the Coxeter gp),
- a “ $W$ -valued distance function”  $\delta : \Phi \times \Phi \rightarrow W$ .

Of course, these objects must satisfy certain axioms.

Classically, one is interested in the case where  $W$  is a finite reflection gp (leading to “spherical bldgs”) or a cocompact Euclidean reflection gp (“affine bldgs”).

## Geometric and topological realizations

We want to associate a topological space to  $\Phi$ . There is a certain freedom of choice in the construction. Choose

- a space  $X$  (as a “model chamber”)
- a family of subspaces  $\{X_s\}_{s \in S}$  indexed by  $S$  (called a “mirror structure”).

Let  $\mathcal{U}(\Phi, X)$  be the result of pasting together copies of  $X$ , one for each element of  $\Phi$ , so that copies corresponding to  $s$ -adjacent elements of  $\Phi$  are pasted together along  $X_s$ .

Let  $X^f$  be the points of  $X$  with a locally finite nbhd in  $\mathcal{U}(W, X)$ .

Two choices for  $X$  stand out:

- $X = \Delta$ , a simplex with its codim 1 faces indexed by  $S$ . (This was Tits' original choice - the resulting space,  $\mathcal{U}(\Phi, \Delta)$ , is called the “classical realization” of  $\Phi$ .)
- $X = K$ , the geometric realization of the poset of “spherical subsets” of  $S$ .  $K \subset \Delta^f$  as its “compact core.” ( $K$  is the so-called “Davis chamber.” The resulting space,  $\mathcal{U}(\Phi, K)$ , is called the “standard realization” of  $\Phi$ .)

All 3 spaces  $\mathcal{U}(\Phi, K)$ ,  $\mathcal{U}(\Phi, \Delta^f)$  and  $\mathcal{U}(\Phi, \Delta)$  are contractible. The first 2 are locally finite, the third usually is not.  $\Delta^f$  is not compact (unless it is  $= \Delta$ ).

The standard realization (when  $X = K$ ) is the most important for geometric group theory. The reason is that if  $\Gamma \subset \text{Aut}(\Phi)$  is a cocompact lattice, then  $H_c^*(\mathcal{U}(\Phi, K)) = H^*(\Gamma; \mathbb{Z}\Gamma)$ . Moreover, the cohomology of  $\Gamma$  with  $\mathbb{Z}\Gamma$  coefficients tells us a good deal about  $\Gamma$ , for example, its vcd (= virtual cohomological dimension), its number of ends and if it is a duality gp.

When  $\Phi$  is an irreducible affine bldg,  $K = \Delta = \Delta^f$  and  $H_c^*(\mathcal{U}(\Phi, \Delta))$  was calculated by Borel-Serre in 1976.

### Theorem of Borel–Serre

$H_c^*(\mathcal{U}(\Phi, \Delta))$  is free abelian and concentrated in the top degree.

From this, they derived cohomological properties of “S-arithmetic gps.”

Our main result will be a computation of  $H_c^*(\mathcal{U}(\Phi, X^f))$  for any space  $X$  with mirror structure. ( $X = K$  is the case of interest.) However, as a first step we will need to calculate  $H_c^*(\mathcal{U}(\Phi, \Delta^f))$ . The result is similar to the Borel–Serre Theorem: it is free abelian and concentrated in the top degree. In contrast,  $H_c^*(\mathcal{U}(\Phi, K))$  usually is not concentrated in a single degree.

## History

- In 1998 I calculated the compactly supported cohomology of (the complex associated to) any Coxeter system  $(W, S)$ .
- In 2002, John Meier and I gave a similar calculation for the compactly supported cohomology of the standard realization of any building of type  $(W, S)$ . However, there was a mistake in the proof.
- In 2006, Dymara, Januszkiewicz, Okun and I gave an equivariant version of the formula for Coxeter gps as well as a proof for right-angled buildings.
- Now we can do the general case using an idea from our 2007 paper on weighted  $L^2$ -cohomology.



- 1 Introduction and history
- 2 Buildings and their geometric realizations
  - Coxeter groups
  - Geometric realizations
- 3 The Main Theorem
  - The free abelian group  $A$
  - Statement and strategy of proof
- 4 The Decomposition Theorem
  - $\mathcal{U}(\Phi, K)$  is CAT(0)
  - References

# Chamber systems

## Definition

A *chamber system* is a set  $\Phi$  (of “chambers”) together with a family of equivalence relations indexed by some set  $S$

Two chambers  $c$  and  $d$  are *s-adjacent* if they are *s-equivalent* and not equal.

A *gallery* in  $\Phi$  is a finite sequence  $(c_0, \dots, c_n)$  of adjacent chambers. It is *type*  $(s_1, \dots, s_n)$  if  $c_i$  and  $c_{i-1}$  are  $s_i$ -adjacent. (So, its type is a word in  $S$ .)

## Definitions

- $\Phi$  is *connected* if any two chambers can be connected by a gallery.
- Given a subset  $T \subset S$ , a gallery of type  $(s_1, \dots, s_n)$  is a *T-gallery* if each  $s_i \in T$ .
- A *residue* of type  $T$  is a *T-gallery* connected component of  $\Phi$ .

## Coxeter groups

A *Coxeter matrix* on a set  $S$  is an  $S \times S$  symmetric matrix  $(m_{st})$  with 1's on the diagonal and off-diagonal entries integers  $\geq 2$  or the symbol  $\infty$ . The associated *Coxeter group* is the group  $W$  with generating set  $S$  and relations:

$$(st)^{m_{st}} = 1$$

(Note: these relations imply that each  $s \in S$  is an involution.)

The pair  $(W, S)$  is a *Coxeter system*.

For each  $T \subset S$ ,  $W_T := \langle T \rangle$ . Also,  $W_s$  is the subgroup  $\langle s \rangle$  of order 2 generated by  $s$ .

## Example

$W$  has the structure of a chamber system over  $S$ .

- $v, w \in W$  are  $s$ -equivalent  $\iff$  they belong to the same coset of  $W_s$ .
- A gallery in  $W$  is an edge path in  $\text{Cay}(W, S)$ .  
(Here a “chamber” is a vertex of  $\text{Cay}(W, S)$ .)
- Its type is a word in  $S$ .
- A  $T$ -residue is a coset of  $W_T$  in  $W$ .

$T$  is a *spherical* subset of  $S$  if  $|W_T| < \infty$ .  
 $S :=$  the poset of spherical subsets.

Suppose  $(W, S)$  is a Coxeter system. A *bldg* of type  $(W, S)$  is

- a set  $\Phi$  (of chambers)
- a family of equivalence relations indexed by  $S$  (“adjacency relations”) st  $\forall s \in S$ , each  $s$ -equivalence class contains at least 2 elements.
- a  $W$ -valued *distance function*  $\delta : \Phi \times \Phi \rightarrow W$  (st  $\varphi, \varphi'$  are connected by a minimal gallery of type  $(s_1, \dots, s_n) \iff \delta(\varphi, \varphi') = s_1 \cdots s_n$ ).

### Example (Thin bldgs)

$W$  is itself a bldg.  $w, w'$  are  $s$ -adjacent if  $w' = ws$ .

$\delta : W \times W \rightarrow W$  is defined by  $\delta(w, w') = w^{-1}w'$ .

A  $T$ -residue is a coset of  $W_T$  (where  $W_T := \langle T \rangle$ .)

### Example (The interesting nonclassical example)

Let  $\{G_s\}_{s \in S}$  be a family of finite gps and  $L^1$  an arbitrary simplicial graph with vertex set  $S$ . Let  $G$  be the *graph product* of the  $G_s$ , ie, it is formed from their free product by imposing the relations  $[g_s, g_t] = 1$  whenever  $\{s, t\} \in \text{Edge}(L^1)$ ,  $g_s \in G_s$ ,  $g_t \in G_t$ . If all  $G_s = \mathbb{Z}/2$ , then  $W$  is a *right-angled Coxeter group* (a *RACG*). Elements of  $G$  have a normal form and there is an obvious notion of  $s$ -adjacency. The elements of  $G$  are the chambers of a *right-angled building* (a *RAB*) of type  $(W, S)$ .



# Geometric realizations

A *mirror structure* on a CW complex  $X$  is a family of subcomplexes  $\{X_s\}_{s \in S}$ . For each  $x \in X$ , put

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Define

$$\mathcal{U}(\Phi, X) = (\Phi \times X) / \sim$$

where  $(\varphi, x) \sim (\varphi', x') \iff x = x'$  and  $\varphi$  and  $\varphi'$  belong to the same  $S(x)$ -residue.

### Example (The classical realization of Tits)

Let  $\Delta$  be a simplex, of dimension  $\text{Card}(S) - 1$ .

$\{\Delta_s\} = \{\text{codim } 1 \text{ faces}\}$ .

Then  $\mathcal{U}(\Phi, \Delta)$  is the *classical realization*.

### Example (The standard realization)

Let  $K$  be the geometric realization of  $\mathcal{S}$ , the poset of spherical subsets of  $S$ .  $K$  is the subcomplex of the barycentric subdivision of  $\Delta$  spanned by the barycenters of spherical faces of  $\Delta$ . Put  $K_s = K \cap \Delta_s$ .  $\mathcal{U}(\Phi, K)$  is the *standard realization*.

Only the spherical faces of  $K$  are nonempty and only spherical residues contribute to  $\mathcal{U}(\Phi, K)$ .

Suppose  $W$  is the *RACG* associated to a graph  $L^1$  and let  $L$  be the associated flag cx. Then  $K$  is the cone on (the barycentric subdivision of)  $L$ . Moreover,  $L$  can be an arbitrary polyhedron.

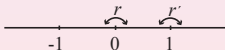
### Example (The edge set of a tree)

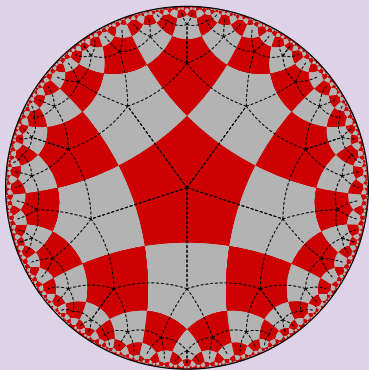
Let  $\mathcal{T}$  be a tree w/o terminal vertices. Any tree is bipartite. Call the colors  $s, t$ . Let  $W$  be the infinite dihedral group on  $\{s, t\}$ .

Then  $\Phi = \text{Edge}(\mathcal{T})$  is a building of type  $(W, \{s, t\})$ .

The  $W$ -distance between  $b, c \in \Phi$  is defined as follows. Take the edge path w/o backtracking from  $b$  to  $c$ . Its type gives a word in  $\{s, t\}$  and hence, an element  $w \in W$  and  $\delta(b, c) = w$ .

$K = \Delta =$  an interval.  $\mathcal{U}(W, K)$  is the real line.  $\mathcal{U}(\Phi, K)$  is the original tree  $\mathcal{T}$ .





$\mathcal{U}(W, K)$ , with  $K$  a pentagon

## Notation

Suppose  $\{X_s\}_{s \in S}$  is a mirror structure. A *face* of  $X$  is a subset

$$X_T := \bigcap_{s \in T} X_s.$$

Given  $U \subset S$ ,

$$X^U := \bigcup_{s \in U} X_s.$$

$X^f$  denotes the complement of the spherical faces of  $X$ .

## The Tits representation

For any Coxeter system  $(W, S)$ , Tits showed  $\exists$  a faithful representation  $W \rightarrow GL(n+1, \mathbf{R})$ ,  $n+1 = |S|$ , as a reflection gp with fundamental domain a simplicial cone. Moreover, if  $W$  is not spherical, we get  $W \hookrightarrow PGL(n+1, \mathbf{R})$  and

$$\bar{C} := \bigcup_{w \in W} w\Delta$$

is a convex cone in  $\mathbf{R}P^n$ . Moreover,  $W$  acts properly on its interior,  $C$ , and  $C = \mathcal{U}(W, \Delta^f)$ . Thus,  $\mathcal{U}(W, \Delta^f)$  can be identified with  $\mathbf{R}^n$ .

# The Main Theorem

Suppose  $\Phi$  is a bldg of type  $(W, S)$ .

Let  $A$  be the set of finitely supported  $\mathbb{Z}$ -valued functions on  $\Phi$ , ie,  $A$  is the free abelian gp on  $\Phi$ . For each  $T \subset S$ , put

$$A^T := \{f \in A \mid f \text{ is constant on each } T\text{-residue}\}.$$

Note that  $A^T = 0$  whenever  $T$  is not spherical. Also,

## Example (The thin bldg)

If  $\Phi = W$ , then  $A = \mathbb{Z}W$  (the group ring) and  $A^T = (\mathbb{Z}W)^{W_T}$  ( $\cong \mathbb{Z}[W/W_T]$ ).



$A^U \subset A^T$  whenever  $U \supset T$ . So, put

$$A^{>T} := \sum_{U \supsetneq T} A^U.$$

**Fact**

$A^T / A^{>T}$  is free abelian.

Let  $\hat{A}^T$  be a complementary summand for  $A^{>T}$  in  $A^T$ .

## Main Theorem

*Suppose  $X$  is a finite CW complex. Then*

$$H_c^*(\mathcal{U}(\Phi, X^f)) = \bigoplus_{T \in \mathcal{S}} H^*(X, X^{S-T}) \otimes \hat{A}^T.$$

$X$  a mirrored CW complex,  $X^{(k)} := \{k\text{-cells}\}$ . Define a cochain complex and “coefficient system”  $\mathcal{I}(A)$  on  $X$  by

$$C^k(X; \mathcal{I}(A)) := \prod_{c \in X^{(k)}} A^{S(c)}.$$

## Proposition

If  $X$  is compact, then  $\mathcal{H}^*(X; \mathcal{I}(A)) = H_c^*(\mathcal{U}(\Phi, X^f))$ .

## Strategy for proving Main Thm

Prove a decomposition result for the coefficient system  $\mathcal{I}(A)$  so that cohomology with coefficients in any given summand can be computed.

## Decomposition Theorem

For any spherical subset  $T$ ,

$$A^T = \bigoplus_{U \subset T} \hat{A}^U.$$

In particular, for  $T = \emptyset$ , we have

$$A = \bigoplus_{U \in \mathcal{S}} \hat{A}^U.$$

To prove this we are led back to classical realizations of bldgs.

## Proposition

*Let  $\Delta$  be the simplex of dimension  $n = \text{Card}(S) - 1$ . Then  $\mathcal{H}^*(\Delta; \mathcal{I}(A))$  (which is  $= H_c^*(\mathcal{U}(\Phi, \Delta^f))$ ) is free abelian concentrated in dimension  $n$ .*

(You also need some similar statements corresponding to certain subcomplexes of  $\Delta$ .)

Why is the proposition plausible?

Because when  $W$  is infinite  $\mathcal{U}(W, \Delta^f)$  can be identified with the image of the interior of a convex cone in real projective space, ie,  $\mathcal{U}(W, \Delta^f) \cong \mathbf{R}^n$ , where  $n = \dim \Delta$ .



## Ingredients for proof of the proposition

- $\mathcal{U}(\Phi, \Delta^f)$  has a piecewise Euclidean CAT(0) metric (extending Moussong's CAT(0) metric on  $\mathcal{U}(\Phi, K)$ ).
- Theorem of Brady-McCammond-Meier on when CAT(0) complexes are highly connected at  $\infty$ . (Sufficient condition: complements of balls of radius  $\frac{\pi}{2}$  in links of cells are sufficiently highly connected).
- Theorem of Dymara-Osajda and independently Schulz: complements of  $\frac{\pi}{2}$ -balls in spherical bldgs are highly connected (homology is concentrated in top degree).

## Outline of argument





- $\mathcal{U}(\Phi, \Delta^f)$  is CAT(0) + BMM + complements of  $\pi/2$ -balls
- $\Rightarrow H^*(\mathcal{U}(\Phi, \Delta^f))$  is free abelian and concentrated in top degree.
- $\Rightarrow$  decomposition of coefficient system  $\mathcal{I}(A)$ .
- $\Rightarrow$  Main Thm (computation of  $H_c^*(\mathcal{U}(\Phi, X^f))$ .)





## Books

-  P. Abramenko and K. Brown, *Approaches to Buildings*, Springer, 2008.
-  M.W. Davis, *The Geometry and Topology of Coxeter Groups*, Princeton Univ. Press, 2007.
-  M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics, vol. 7, Academic Press, San Diego, 1989.



## References

-  A. Borel and J.-P. Serre, *Cohomologie d'immeubles et de groupes  $S$ -arithmétiques*, *Topology* **15** (1976), 211–232.
-  N. Brady, J. McCammond and J. Meier, *Local-to-asymptotic topology for cocompact  $CAT(0)$  complexes*, *Topology and its Applications* **131** (2003), 177–188.
-  M.W. Davis, *The cohomology of a Coxeter group with group ring coefficients*, *Duke Math. J.* **91** (1998), 297–314.
-  M.W. Davis, J. Dymara, T. Januszkiewicz and B. Okun, *Cohomology of Coxeter groups with group ring coefficients: II*, *Algebraic & Geometric Topology* **6** (2006), 1289–1318.

-  \_\_\_\_\_, *Weighted  $L^2$ -cohomology of Coxeter groups*, *Geometry & Topology* **11** (2007), 47–138.
-  M.W. Davis, J. Dymara, T. Januszkiewicz, J.Meier and B. Okun, *The compactly supported cohomology of buildings*, *Comment. Math. Helv.* **85** (2010), 551-582.
-  M.W. Davis and J. Meier, *The topology at infinity of Coxeter groups and buildings*, *Comment. Math. Helv.* **77** (2002), 746–766.
-  J. Dymara and D. Osajda, *Boundaries of right-angled hyperbolic buildings*, *Fund. Math.* **197** (2007), 123–165.