WEIGHTED L^2-COHOMOLOGY

OF COXETER GROUPS

Happy Birthday Wu-chung!

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work with

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Given a Coxeter group W, $\exists$ a contractible, simplicial cx Σ with a proper, cocompact W-action.

Goal: Define groups $L^2_q\mathcal{H}^*(\Sigma)$, depending on a parameter $q \in (0, \infty)$, which interpolate between $H^*(\Sigma)$ and $H^*_c(\Sigma)$ (coefficients in $\mathbf{R}$).

The $L^2_q\mathcal{H}^*(\Sigma)$ are Hilbert spaces with a "dimension;"

so we can define "Betti numbers:"

$$b^i_q(\Sigma) := \dim L^2_q\mathcal{H}^*(\Sigma).$$

Properties: • The b_q^i vary continuously with q.

• Explicit formulas for the b_q^i in certain range of q.

• For $q = 1$, get the standard L^2-cohomology of Σ.

• For q an integer, the $b_q^i(\Sigma)$ compute standard L^2-cohomology of buildings of type W and "thickness" q.

• When Σ is a n-mfld, we have Poincaré duality: $b_q^i = b_{1/q}^{n-i}$.

Cellular cochains. Let Y be a cell complex, usually not compact, often contractible.

$$\mathcal{E}_n := \{n\text{-cells in } Y\}$$

$$C^n(Y) = \{\mathbf{R}\text{-valued cochains on } Y\} := \{f : \mathcal{E}_n \to \mathbf{R}\}$$

$$C^n_c(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid f \text{ is finitely supported}\}$$

$$L^2C^n(Y) := \{f : \mathcal{E}_n \to \mathbf{R} \mid \sum f(\sigma)^2 < \infty\}$$

Cohomology. $\delta : C^n(Y) \to C^{n+1}(Y)$ is the coboundary map.

$L^2C^n(Y)$ is a Hilbert space. In favorable situations δ is a bounded linear operator. So, $\ker\delta$ is a closed subspace of $L^2C^n(Y)$.

However, $\operatorname{im}\delta$ might not be. Define the *reduced* L^2*-cohomology*:

$$\mathcal{H}^n(Y) := \ker\delta/\overline{\operatorname{im}\delta}.$$

Example. Suppose Y^n is a contractible n-mfld (e.g. $Y^n = \mathbf{R}^n$).

$$H^i(Y^n) = \begin{cases} \mathbf{R}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$H_c^i(Y^n) = \begin{cases} \mathbf{R}, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

What about $\mathcal{H}^*(Y^n)$? In many situations, $\mathcal{H}^i(Y^n) = 0$ for $i \neq \frac{n}{2}$

and can be nonzero for $i = \frac{n}{2}$.

Bringing a group into the picture. Γ is a countable discrete group.

$$\mathbf{R}\Gamma := \{f : \Gamma \to \mathbf{R} \mid f \text{ is finitely supported}\}$$

$$= \{\text{finite sums, } \sum f(\gamma)e_\gamma\} = \text{the gp algebra}$$

$$L^2(\Gamma) := \{f : \Gamma \to \mathbf{R} \mid \sum f(\gamma)^2 < \infty\}$$

$L^2(\Gamma)$ is the Hilbert space completion of $\mathbf{R}\Gamma$ with inner product:

$$e_\gamma \cdot e_{\gamma'} := \delta(\gamma, \gamma').$$

There are two orthogonal Γ-actions on $L^2(\Gamma)$ (by left or right translation).

Γ-dimension. Let $\mathcal{N}(\Gamma)$ be the set of Γ-equivariant bounded linear endomorphisms of $L^2(\Gamma)$ (the *von Neumann algebra*). Given $\varphi \in \mathcal{N}(\Gamma)$, define $\mathrm{tr}_\Gamma(\varphi) := \varphi(e_1) \cdot e_1$. Suppose

$$V \subset \bigoplus_{\text{finite}} L^2(\Gamma)$$

is a closed, Γ-stable subspace (a *Hilbert Γ-module*).

Let $p_V : \oplus L^2(\Gamma) \to \oplus L^2(\Gamma)$ be orthogonal projection. Represent p_V as a matrix (p_{ij}) with entries in $\mathcal{N}(\Gamma)$ and define

$$\dim_\Gamma V = \mathrm{tr}_\Gamma(p_V) := \sum \mathrm{tr}_\Gamma(p_{ii}) \in [0, \infty).$$

Properties:

- $\dim_\Gamma V = 0 \iff V = 0$,
- $\dim_\Gamma L^2(\Gamma) = 1$,
- $\dim_\Gamma (V_1 \oplus V_2) = \dim_\Gamma V_1 + \dim_\Gamma V_2$,
- If F is a finite gp, then $\dim_F V = \frac{1}{|F|} \dim_{\mathbf{R}} V$,
- If $F \subset \Gamma$ is finite subgp, then $\dim_\Gamma L^2(\Gamma)^F = \frac{1}{|F|}$.

Suppose now that Γ acts properly and cocompactly on a CW cx Y by permuting the cells. Note

$$C_c^i(Y) = C_\Gamma^i(Y; \mathbf{R}\Gamma) \quad \text{and} \quad L^2C^i(Y) = C_\Gamma^i(Y; L^2(\Gamma)).$$

Since there are only finitely many orbits of cells in Y,

$$L^2C^i(Y) = \bigoplus_{\text{orbits of cells}} L^2(\Gamma)^{\Gamma_\sigma} \quad \subset \bigoplus L^2(\Gamma).$$

is a Hilbert Γ-module. So, $L^2C^i(Y)$ and $\mathcal{H}^i(Y)$ have Γ-dimensions.

The Γ-dimension of the second is called the i^{th} L^2-*Betti number*:

$$b^i(Y;\Gamma) := \dim_\Gamma \mathcal{H}^i(Y)$$

$$L^2\chi(Y;\Gamma) := \sum (-1)^i b^i(Y;\Gamma)$$

$$\dim_\Gamma L^2C^i(Y) \ = \bigoplus_{\text{orbits of } i\text{-cells}} \frac{1}{|\Gamma_\sigma|}$$

Atiyah's Formula.

$$L^2\chi(Y;\Gamma) = \sum_{\textit{orbits of cells}} \frac{(-1)^{\dim\sigma}}{|\Gamma_\sigma|} := \chi^{\text{orb}}(Y/\Gamma)$$

Coxeter groups. S a finite set. (m_{st}) a *Coxeter matrix*, i.e.,

a symmetric $S \times S$ matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that

$$m_{st} = \begin{cases} 1, & \text{if } s = t; \\ \geq 2, & \text{otherwise.} \end{cases}$$

$$W := \langle S \mid (st)^{m_{st}} = 1,\ (s,t) \in S \times S \rangle.$$

(W, S) is a *Coxeter system*. For $T \subset S$, $W_T := \langle \{s\}_{s \in T} \rangle$.

If W_T is finite, T is a *spherical subset*.

$$\mathcal{S} := \{T \subset S \mid W_T \text{ is finite}\} = \{\text{spherical subsets}\}$$

The complex Σ. Fast definition: geometric realization of poset

$$W\mathcal{S} := \coprod_{T \in \mathcal{S}} W/W_T \ .$$

Slow definition:

$$K := \text{geom realization of } \mathcal{S} \quad \text{(a cone)}$$

$$K_s := \text{geom. realization of } \mathcal{S}_{\geq \{s\}}$$

$$S(x) := \{s \in S \mid x \in K_s\}$$

$$\Sigma := (W \times K)/\sim \quad ,$$

where $\sim$ is the equivalence relation given by

$$(w,x) \sim (v,y) \iff x = y \text{ and } w^{-1}v \in W_{S(x)}.$$

K is a *chamber*. K_s is a *mirror*.

W acts properly and cocompactly on Σ.

Fact. Σ *is contractible.*

Growth series. Define a power series:

$$W(t) := \sum_{w \in W} t^{l(w)}, \quad \text{where } l(w) \text{ denotes word length.}$$

It is a rational function in t, e.g.,

$$\frac{1}{W(t)} = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(t^{-1})},$$

$$\rho := \text{ radius of convergence of } W(t)$$

$$= \text{smallest root of } 1/W(t).$$

An inner product. Let $q \in (0, \infty)$.

$$\mathbf{R}^{(W)} := \{\text{finitely supported functions } W \to \mathbf{R}\}$$

$$\langle e_v, e_w \rangle_q := q^{l(w)} \delta(e_v, e_w)$$

where $\delta(\ ,\ )$ is the Kronecker delta.

$$L^2_q(W) := \text{completion of } \mathbf{R}^{(W)}.$$

The Hecke algebra. $\mathbf{R}_q W$ is a deformation of the group algebra.

Multiplication:

$$e_s e_w = \begin{cases} e_{sw}, & l(sw) > l(w); \\ q e_{sw} + (q-1) e_w, & l(sw) < l(w). \end{cases}$$

Define an anti-involution $*$ on $\mathbf{R}_q W$ by $(e_w)^* := e_{w^{-1}}$.

Key properties:

$$(xy)^* = y^* x^* \quad \text{and} \quad \langle xy, z \rangle_q = \langle y, x^* z \rangle_q.$$

$L^2_q(W)$ is an $\mathbf{R}_q W$-bimodule.

The Hecke-von Neumann algebra

$$\mathcal{N}_q = \text{a completion of } \mathbf{R}_q W$$

$$:= \{\mathbf{R}_q W\text{-equivariant bounded linear operators on } L^2_q(W)\}$$

von Neumann trace. For $\varphi \in \mathcal{N}_q$, set

$$\mathrm{tr}_{\mathcal{N}_q}(\varphi) = \langle \varphi(e_1), e_1 \rangle_q.$$

For $\Phi = (\varphi_{ij}) \in M_m(\mathcal{N}_q)$, set

$$\mathrm{tr}_{\mathcal{N}_q}(\Phi) = \sum \mathrm{tr}_{\mathcal{N}_q}(\varphi_{ii}).$$

von Neumann dimension. Given a $\mathbf{R}_q W$-stable, closed subspace $V \subset \oplus L_q^2(W)$, let $p_V : \oplus L_q^2(W) \to \oplus L_q^2(W)$ be orthogonal projection onto V. Define

$$\dim_{\mathcal{N}_q} V = \mathrm{tr}_{\mathcal{N}_q}(p_V) \in [0, \infty).$$

Idempotents in $\mathcal{N}_q$: sample calculations. Define

$$\tilde{a}_S := \sum_{w \in W} e_w$$

$$\tilde{a}_S e_s = q \tilde{a}_S \quad \text{therefore,} \quad \tilde{a}_S e_w = q^{l(w)} \tilde{a}_S.$$

So, $(\tilde{a}_S)^2 = W(q)\tilde{a}_S$ and $\tilde{a}_S$ is bounded $\iff q < \rho$

(where $\rho =$ radius of conv.)

Lemma. *(i) For $q < \rho$, the following is an idempotent in $\mathcal{N}_q$:*

$$a_S := \frac{1}{W(q)} \sum_{w \in W} e_w. \qquad \textit{Similarly},$$

(ii) For $q > \rho^{-1}$, the following is an idempotent in $\mathcal{N}_q$:

$$h_S := \frac{1}{W(q^{-1})} \sum_{w \in W} (-1)^{l(w)} q^{-l(w)} e_w.$$

These have dimensions:

$$\dim_{\mathcal{N}_q}(\operatorname{im} a_S) = \frac{1}{W(q)}, \qquad \dim_{\mathcal{N}_q}(\operatorname{im} h_S) = \frac{1}{W(q^{-1})}$$

Weighted L^2-cohomology. $\forall$ cell $\sigma \subset \Sigma$, let $e_\sigma \in C_c^k(\Sigma)$ be its characteristic function. Define an inner product on $C_c^k(\Sigma)$ by

$$\langle e_\sigma, e_\tau \rangle_q := q^{l(w(\sigma))} \delta(\sigma, \tau)$$

where $w(\sigma)$ is the shortest $w \in W$ s.t. $w^{-1}\sigma \in K$.

$$L_q^2 C^k(\Sigma) := \text{ completion of } C_c^k(\Sigma)$$

$L^2_q C^*(\Sigma)$ is a $\mathcal{N}_q$-module and $\delta : L^2_q C^k(\Sigma) \to L^2_q C^{k+1}(\Sigma)$ is a map of $\mathcal{N}_q$-modules.

$$L^2_q \mathcal{H}^k(\Sigma) := \ker \delta / \overline{\operatorname{im} \delta}$$

$$b^k_q(\Sigma) := \dim L^2_q \mathcal{H}^k(\Sigma)$$

$$\chi_q(\Sigma) := \sum (-1)^k b^k_q(\Sigma)$$

Theorem. (Dymara) $\chi_q(\Sigma) = \frac{1}{W(q)}$

Sketch of Proof. Similar to Atiyah's Formula. Σ can be cellulated with one orbit of cells for each $T \in \mathcal{S}$.

Moreover, dim(cell) $= |T|$.

$$\dim_{\mathcal{N}_q}(L^2_q C^{|T|}(\text{orbit of cell}) = \dim(\operatorname{im} h_T) = \frac{1}{W_T(q^{-1})}$$

So, $$\chi_q(\sigma) = \sum_{T \in \mathcal{S}} \frac{(-1)^{|T|}}{W_T(q^{-1})} = \frac{1}{W(q)}.$$

$\square$

Theorem. (DDJO). $b_q^k(\Sigma)$ *is a continuous function of* q.

Theorem. (Dymara) *Suppose* $q < \rho$. *Then* $L_q^2\mathcal{H}^*(\Sigma)$ *is concentrated in dimension* 0, *i.e.,* $b_q^i(\Sigma) = 0$, $\forall i > 0$.

Theorem. (Dymara) *Suppose* Φ *is a building of type* (W, S) *with a chamber transitive automorphism group* G *and "thickness"* q*. Then the* L^2*-Betti number (with respect to* G*),* $b^k(\Phi; G)$*, is equal to* $b_q^k(\Sigma)$*.*

For buildings only integral values of q matter!

Notation: Given $T \subset S$, set

$$K^T := \bigcup_{s \in T} K_s$$

$$W^T := \{w \in W \mid l(ws) = l(w) - 1, \forall s \in T, l(ws) = l(w) + 1, \forall s \notin T\}$$

$\mathbf{Z}(W^T) =$ the free abelian gp on W^T.

Theorem. (D.)

$$H_c^*(\Sigma) = \bigoplus_{T \in \mathcal{S}} H_*(K, K^{S-T}) \otimes \mathbf{Z}(W^T).$$

Main Theorem. (DDJO). *Suppose* $q > \rho^{-1}$. *Then*

$$L^2_q\mathcal{H}^*(\Sigma) = \overline{\bigoplus_{T\in\mathcal{S}} H^*(K, K^{S-T}) \otimes D^{S-T}}.$$

Here the D^{S-T} are certain specific $\mathcal{N}_q$-submodules of $L^2_q(W)$

whose dimensions can be explicitly computed:

$$\dim_{\mathcal{N}_q} D_{S-T} = \sum_{U\in\mathcal{S}_{\geq T}} \frac{(-1)^{|U-T|}}{W_U(q)} = \frac{W^T(q^{-1})}{W(q^{-1})}$$

where $W^T(q) := \sum_{w\in W^T} q^{l(w)}$

Corollary. *If* $q > \rho^{-1}$, *then* $H_c^k(\Sigma; \mathbf{R}) \to L_q^2\mathcal{H}^k(\Sigma)$ *is injective with dense image.*

Decomposition Theorem. *We have direct sum decompositions of* $\mathcal{N}_q$*-modules:*

$$L_q^2(W) = \overline{\bigoplus_{T \in \mathcal{S}} D^T}, \qquad \text{if } q < \rho,$$

$$L_q^2(W) = \overline{\bigoplus_{T \in \mathcal{S}} D^{S-T}}, \qquad \text{if } q > \rho^{-1}.$$

Example. Suppose Σ is an n-manifold.

If $q < \rho$, then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension 0.

If $q > \rho^{-1}$, then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension n.

Remark. If W is an affine Coxeter group, then $\rho = 1$.

Example. Suppose Σ is an 2-manifold. Then $L^2_q\mathcal{H}^*(\Sigma)$ is concentrated in dimension

$$\begin{cases} 0, & \text{if } q \leq \rho; \\ 1, & \text{if } \rho < q < \rho^{-1}; \\ 2, & \text{if } q \geq \rho^{-1}. \end{cases}$$

(Σ a 2-mfld, continued). Suppose K is a right-angled k-gon.

$$\chi_q(\Sigma) = \frac{1}{W(q)} = \frac{q^2 + (2-k)q + 1}{(1+q)^2}$$

So,

$$\rho^{\pm 1} = \frac{(k-2) \pm \sqrt{k^2 - 4k}}{2},$$

e.g. when $k = 5$, $\rho^{-1} = \frac{3+\sqrt{5}}{2}$, $\quad 2 < \rho^{-1} < 3$.

Question. *In general, what happens in the intermediate range,*

$\rho < q < \rho^{-1}$*?*

Conjecture. (A version of the Singer Conjecture). *Suppose* Σ *is an* n*-manifold . Then for* $q \leq 1$,

$$L_q^2\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k > \frac{n}{2}.$$

Similarly, for $q \geq 1$,

$$L_q^2\mathcal{H}^k(\Sigma) = 0 \quad \textit{for } k < \frac{n}{2}.$$

Question. *What about groups other than Coxeter groups acting on CW complexes?*