THE GEOMETRY AND TOPOLOGY

OF COXETER GROUPS

London, Ontario

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Geometric group theory and topology.

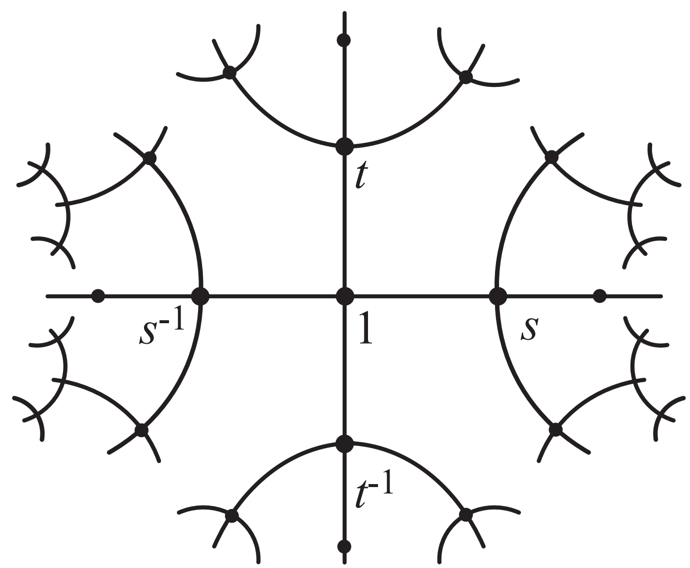
Given a (discrete) gp Γ, find:

1. a connected space on which Γ acts properly

   (i.e., with finite stabilizers)

2. a contractible space on which Γ acts properly (or freely).

The Cayley graph. S a set of generators for Γ. Cay(Γ, S) is graph with vertex set Γ and an edge from g to g' iff $g' = gs^{\pm 1}$. In geometric group theory we often study quasi-isometry invariants of groups, i.e., quasi-isometry invariants of Cay(Γ, S).



Cayley graph of free group

Classifying space of Γ. One can construct a space (in fact, a CW complex) $B\Gamma$ with $\pi_1(B\Gamma) = \Gamma$ and $\pi_i(B\Gamma) = 0$, $\forall i > 1$. ($B\Gamma$ is a "$K(\Gamma, 1)$ complex" or an "aspherical complex.") Its universal cover $E\Gamma$ is contractible and Γ acts via deck transformations. $B\Gamma$ is unique up to homotopy equivalence. Topological invariants of $B\Gamma$, such as its homology groups, are invariants of Γ.

Example. If $\Gamma = \mathbb{Z}$, then $B\Gamma = S^1$.

So, [group theory] $\subset$ [topology].

When Γ has torsion (e.g. if it is finite), $B\Gamma$ must be infinite dimensional.

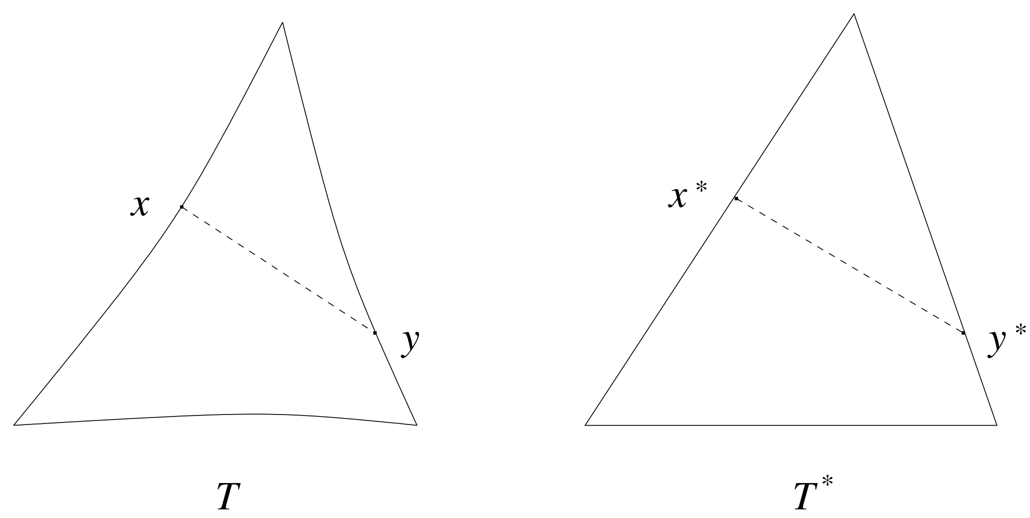
Questions. *When can we choose $B\Gamma$ to be finite dimensional? a finite CW cx (i.e., compact)? a closed mfld?*

When Γ has torsion it is often better to consider $\underline{E}\Gamma$, a cell cx with a proper Γ-action such that $\underline{E}\Gamma^H$ is contractible $\forall$ finite

$H \subset \Gamma$. Such $\underline{E}\Gamma$ always exist. They have universal property that for any space X with proper Γ-action there is an equivariant map $X \to \underline{E}\Gamma$.

Nonpositive curvature. Γ is *word hyperbolic* if the metric space Cay(Γ, S), "in the large," looks like a negatively curved space.

A geodesic metric space X is CAT(0) if any geodesic triangle T in X is thinner than its comparison triangle T^* in $\mathbf{R}^2$.



$$d(x, y) \leq d(x^*, y^*)$$

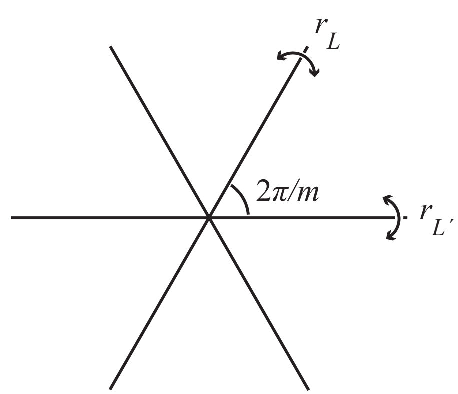
Note: Geodesics are unique in a CAT(0)-space X. This implies X is contractible.

Γ is a CAT(0) *group* if Γ is a discrete gp of isometries acting properly and cocompactly on a CAT(0) space X. (So, $X = \underline{E}\Gamma$.)

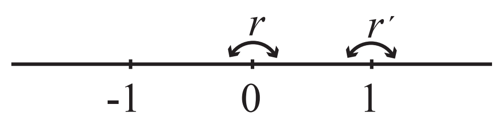
Corollary. *If* Γ *is a torsion-free* CAT(0) *gp, then* $X/\Gamma = B\Gamma$*.*

Reflection groups.

Example. Take two lines in $\mathbf{R}^2$ making an angle of π/m. The gp $\mathbf{D}_m$ generated by the orthogonal reflections across these lines is the *dihedral group* of order $2m$.



Example. The infinite dihedral gp $\mathbf{D}_\infty$ generated by reflections r, r' across 2 points in $\mathbf{R}$.



Example. P a convex polytope in $\mathbf{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$ s.t. all dihedral angles (between codimension one faces) have the form π/m_{ij}, $m_{ij} \in \{2, 3, \ldots\}$. $W =$ the gp generated by

$$S := \{\text{reflections across faces of } P\}$$

Abstract reflection groups. Is there an abstract notion of reflection gp?

First attempt: any gp generated by involutions: a pair (W, S) with $W = \langle S \rangle$, each $s \in S$ of order 2.

Tits proposed two different refinements of the above. The first was that Cay(W, S) had certain separation properties. The second was that W had a presentation of a certain form. Amazingly, these 2 definitions turn out to be equivalent. Details:

(1) Put $\Omega :=$ Cay(W, S). $\forall s \in S$, the fixed set, Ω^s, separates Ω.

(2) For each pair $(s, t) \in S \times S$, let $m_{st} :=$ order(st). (W, S) is a *Coxeter system* if it has a presentation of the form:

$$\langle S \mid (st)^{m_{st}} \rangle_{(s,t) \in S \times S}$$

The equivalence of these two definitions is not obvious. The meaning of (2) is that if we start with Cay(W, S) and fill in orbits of 2-cells corresponding to distinct pairs $\{s, t\}$ with $m_{st} \neq \infty$, then the resulting 2-dim cell cx is simply connected.

Representing an abstract refl gp by a geometric object.

There are two ways to do this. Tits: $\exists$ a faithful representation $\rho : W \hookrightarrow GL(N, \mathbf{R})$ s.t.

- $\forall s \in S, \rho(s)$ is a (not necessarily orthogonal) linear reflection.

- W acts properly on the interior I of a convex cone in $\mathbf{R}^N$.

- Hyperplanes corresponding to S bound a "chamber" $K \subset I$.

For many purposes this representation is completely satisfactory.

Major disadvantage: fundamental domain K is not compact.

The cell complex Σ. $\exists$ a cell cx Σ with a proper W-action s.t.

- $\exists$ a compact fundamental chamber K with $\Sigma/W \cong K$.

- $S = \{$"reflections across faces" of $K\}$.

- Σ is contractible (in fact, CAT(0)).

Right-angled Coxeter groups. (W, S) is *right-angled* if all $m_{st} = 2$ or ∞. For simplicity let's stick to these. Note that $m_{st} = 2$ means $(st)^2 = 1$, i.e., $st = t^{-1}s^{-1} = ts$, i.e., s and t commute. The data for a right-angled Coxeter system is encapsulated in a finite simplicial graph L^1, as follows:

$\{\text{generators}\} = S = \text{Vert}(L^1)$. Relations: $s^2 = 1$, $\forall s \in S$ and

$$(st)^2 = 1 \quad \text{iff} \quad \{s, t\} \in \text{Edge}(L^1).$$

Conversely, given L^1, this presentation defines a right-angled Coxeter system.

One associates to L^1 a simplicial cx (a "flag cx") L as follows: a subset $T \subset S$ spans a simplex σ_T iff any 2 elements of T are connected by an edge. (Remarks: $\dim \sigma_T = \text{Card}(T) - 1$. L keeps track of subsets of pairwise commuting generators. Graph theorists call L the "clique cx" of L^1.)

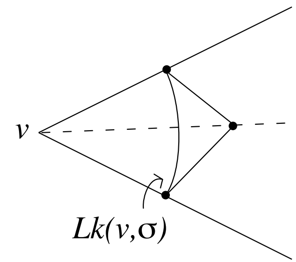
Definition. A simpl cx L is a *flag complex* if any subset of vertices, which are pairwise joined by edges, spans a simplex of L.

(L has no "missing simplices" of dim > 1.)

Remark. This imposes no condition on the topology of L.

Indeed, the barycentric subdivision of any cell cx is a flag cx. So, L can be any polyhedron.

Construction of Σ. Start by declaring the 1-skeleton of Σ to be the Cayley graph: $\Sigma^1 := \text{Cay}(W, S)$. Attach a square to each circuit in $\text{Cay}(W, S)$ labeled $stst$ for each $\{s, t\} \in \text{Edge}(L)$. This is Σ^2. Continue. Add a W-orbit of n-cubes to Σ^{n-1} for each $(n-1)$-simplex in L to get Σ^n. Σ is a cubical cell cx. W acts freely and transitively on $\text{Vert}(\Sigma)$ and the "link" of each vertex is L. Σ has a natural piecewise Euclidean metric in which each cube is identified with a unit cube in Euclidean space.



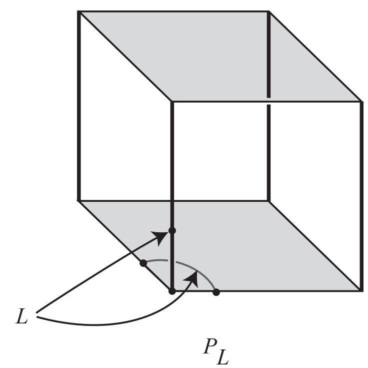
Theorem. (Gromov, Moussong). Σ *is* CAT(0).

Proof. Gromov showed that a cubical cx was locally CAT(0) iff the link of each vertex is a flag cx. Also,

(locally CAT(0) + 1-connected) $\implies$ CAT(0). $\square$

Corollary. Σ *is contractible.*

An alternative construction of Σ. L an arbitrary flag simplicial cx. Put $\square^S := [-1,1]^S$. Define $P_L \subset \square^S$ to be the union of all faces which are parallel to $\square^T$ for some $\sigma_T \subset L$.



The group $(\mathbb{Z}/2)^S$ acts as a reflection group on $\square^S$. A fund chamber for $(\mathbb{Z}/2)^S$ on $\square^S$ is $[0,1]^S$. P_L is $(\mathbb{Z}/2)^S$-stable and a fund chamber is $K := P_L \cap [0,1]^S$. $K \cong \text{Cone}(L)$. $\text{Vert}\, P_L = \text{Vert}\, \square^S = \{\pm 1\}^S$. The link of each vertex of P_L is $\cong L$.

Let $p : \widetilde{P}_L \to P_L$ be the universal cover. Let W be the group of all lifts of elements of $(\mathbb{Z}/2)^S$ to $\widetilde{P}_L$. Then W acts as a reflection group on $\widetilde{P}_L$. Identify an element of S with the appropriate lift of the corresponding reflection in $(\mathbb{Z}/2)^S$. Check that (W, S) is the right-angled Coxeter system associated to L and $\widetilde{P}_L = \Sigma$.

Moreover, $\Gamma := \pi_1(P_L)$ is a torsion-free subgp of W (it is the commutator subgp). So, we have a machine for a converting flag cx L into a finite aspherical cx P_L and gp W acting nicely on its universal cover.

Coxeter groups as a source of examples. A nbhd of a vertex in Σ is $\cong$ to Cone(L). So, Σ is locally $\cong$ to Cone(L). For example, if $L \cong S^{n-1}$, then Σ is an n-mfld. The reason Coxeter groups provide such a potent source of examples is that the topology of L is essentially arbitrary.

Example. $\exists$ closed n-mfld M^n with same homology as S^n and $\pi_1(M^n) \neq 1$. (So-called *homology spheres*.) Take L to be a flag triangulation of a $(n-1)$-dim homology sphere. A slight modification of Σ gives a contractible n-mfld which is not simply connected at ∞. So, $\exists$ aspherical mflds with univ cover $\not\cong \mathbf{R}^n$.

A modification of L shows we can replace "aspherical" by "locally CAT(0)."

Example. Take $L = \mathbf{R}P^2$. Calculation gives:

$$H^i(W; \mathbb{Z}W) = H^i_c(\Sigma) = \begin{cases} 0, & \text{for } i = 0, 1, \\ \bigoplus \mathbb{Z}, & \text{for } i = 2, \\ \mathbb{Z}/2, & \text{for } i = 3 \end{cases}$$

$\Gamma \subset W$ a torsion-free subgp of finite index. Then $\mathrm{cd}_{\mathbb{Z}}(\Gamma) = 3$, $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = 2$. So, $\exists$ torsion-free gps having different cohomological dimension over $\mathbb{Z}$ than over $\mathbb{Q}$.

Example. (*Dranishnikov*) Let L_1 be a flag triangulation of $\mathbf{R}P^2$ as above. L_2 a flag cx $\cong$ space formed by gluing D^2 onto S^1 via a map of degree 3.

$$H^i(L_2) = \begin{cases} \mathbb{Z}/3, & \text{for } i = 2, \\ 0, & \text{for } i \neq 0, 2. \end{cases}$$

We get gps W_1, W_2 and spaces Σ_1, Σ_2. As before,

$$H_c^i(\Sigma_2) = \begin{cases} 0, & \text{for } i = 0, 1, \\ \oplus \mathbb{Z}, & \text{for } i = 2, \\ \mathbb{Z}/3, & \text{for } i = 3 \end{cases}$$

From the Künneth formula:

$$H_c^6(\Sigma_1 \times \Sigma_2) = \mathbb{Z}/2 \otimes \mathbb{Z}/3 = 0.$$

So, $\mathrm{cd}(\Gamma_1 \times \Gamma_2) \neq 6 = \mathrm{cd}(\Gamma_1) + \mathrm{cd}(\Gamma_2)$. cd is not additive.

The reflection group trick. There are plenty of examples of gps π s.t.

a) $B\pi$ is a finite cx (e.g., 2-dimensional) &

b) π has exotic properties, e.g., is not residually finite, has undecidable word problem, etc.

On the other hand, 30 years ago the only known examples of closed aspherical mflds basically had the form $\Gamma\backslash G/K$, for G a Lie gp, K a maximal compact and Γ a torsion-free discrete subgp.

The refl gp trick does the following: given π with $B\pi$ a finite cx, it produces a closed aspherical mfld M which retracts onto $B\pi$. So, $\pi_1(M)$ retracts onto π. Hence, $\pi_1(M)$ will be at least as exotic as π. It also shows that if the Novikov and Borel Conjectures hold for all aspherical mflds, then they hold $\forall$ π with $B\pi$ a finite cx. Here is the construction:

1) Thicken $B\pi$ to X, a compact mfld (e.g., embed $B\pi$ in $\mathbf{R}^n$ and take a regular nbhd of it).

2) Triangulate ∂X as a flag cx L.

3) $W :=$ the right-angled Coxeter gp associated to L; $\Gamma \subset W$ a torsion-free subgp.

4) $\widetilde{M} := (W \times X)/\sim$, the result of pasting together copies of X, one for each element of W. (i.e., take Σ, remove interior of each chamber ($\cong$ Cone(L)), replace with copy of X.)

5) M:=$\widetilde{M}/\Gamma$.

M is obviously a closed mfld and it retracts onto X. (The retraction is induced by $W \times X \to X$.)

Theorem. *$\widetilde{M}$ is aspherical (and so is M).*

Corollary. *$\exists$ closed aspherical mflds M s.t.*

a) $\pi_1(M)$ is not residually finite.

b) $\pi_1(M)$ has undecidable word problem.

Similarly for other properties inherited by gps which retract onto a gp with that property.