Explicit descriptions of all 1-Meixner random vectors

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Abstract. The 1-Meixner random vectors consist of d real-valued random variables, having finite moments of all orders, for which the commutators between their joint semi-quantum operators and the multiplication operators that they generate are linear combinations of these multiplication operators and the identity operator. This property of random variables imposes a system of first-order partial differential equations on their joint Laplace transform on a neighborhood of 0 in a Euclidean d-space. We provide explicit descriptions of all solutions of the system just mentioned in every dimension d.

1. Introduction

The goal of this paper is to find all solutions of the equation

$$[\nabla\varphi](x) \,=\, \alpha(x,[\nabla\varphi](x)) \,+\, \varphi(x)x \tag{1.1}$$

imposed on the unknown C^2 function $x \mapsto \varphi(x)$ defined on a sufficiently small connected neighborhood of 0 in V, where the following objects are involved:

(1.2) a Euclidean space V of dimension $d \geq 1$ with the inner product $\langle\,,\rangle$ and a bilinear mapping $\alpha : V \times V \to V$ such that $\langle \alpha(x,y), z\rangle$ is totally symmetric in $x, y, z \in V$, where we also write $\alpha_x = \alpha(x,\,\cdot\,) : V \to V$, for any $x \in V$.

As shown in [**12**] — see Proposition 9.1 below – *the Laplace transform* φ *of every nondegenerate* 1*-Meixner random vector* $Y = (Y_1 \ldots, Y_d)$*, normalized so that*

$$Y_1, \ldots, Y_d \text{ are } L^2\text{-orthonormal and } L^2\text{-orthogonal to constants,} \tag{1.3}$$

is a solution of equation (1.1)*, and* $\varphi(0) = 1$.

The solutions φ of (1.1) with $\varphi(0) = 1$ naturally form two classes, described in Theorem A. The present paper also establishes a partial converse of the above italicized statement, by exhibiting the joint probability distributions of the components of such normalized 1-Meixner random vectors corresponding to some suitably rescaled versions of solutions belonging to one the two classes just mentioned.

It is worth pointing out, which we do in Proposition 3.1, that

$$\text{any solution } \varphi \text{ to (1.1) is uniquely determined by } \varphi(0), \tag{1.4}$$

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as well as real-analytic, while, unless φ vanishes identically,

$$\varphi \text{ uniquely determines both } \alpha \text{ and the inner product } \langle\,,\rangle. \tag{1.5}$$

Whenever φ is not identically zero, (1.1) with $\sigma = \log|\varphi|$ reads

$$[\nabla\sigma](x) \,=\, \alpha(x,[\nabla\sigma](x)) \,+\, x \text{ for } x \text{ near } 0 \text{ in } V. \tag{1.6}$$

Section 3 describes several trivial ways of modifying given examples of φ with (1.1), for suitable V, d, α and $\langle\,,\rangle$, to obtain new examples with the same property. The first two are the $\mathrm{O}(d)$-*equivalence* and *orthogonal direct-sum construction*. Then, there is *rescaling*: with V, d and $\langle\,,\rangle$ in (1.2) unchanged, we use any fixed $c \in \mathbb{R} \smallsetminus \{0\}$ to replace α, and a solution $\sigma = \log|\varphi|$ to (1.6), by

$$\widetilde{\alpha} = c\alpha \text{ and the function } x \mapsto \widetilde{\sigma}(x) = c^{-2}\sigma(cx). \tag{1.7}$$

Here is our first main result.

Theorem A. *All solutions φ to* (1.1)*, with their associated data $V, \alpha, \langle\,,\rangle$, arise via the trivial modifications just mentioned from two classes of examples. In both classes we have*

$$\varphi = ae^{\sigma} \text{ for an arbitrary } a \in \mathbb{R} \text{ and some specific function } \sigma. \tag{1.8}$$

In the first class, $V = \mathbb{R}^d$, with any $d \geq 1$, and $\langle\,,\rangle$ is the standard inner product, while the components $\alpha_{ijk} = \langle \alpha(e_i, e_j), e_k\rangle$ of α in the standard orthonormal basis $e_1, \ldots, e_d$ and the function σ are given by

$$\begin{aligned} &\alpha_{iid} = \alpha_{idi} = \alpha_{dii} = 1 \text{ for all } i, \text{ and } \alpha_{ijk} = 0 \text{ otherwise,} \\ &2\sigma(x_1, \ldots, x_d) \,=\, -2x_d \,-\, \log|(1-x_d)^2 - \textstyle\sum_{i<d} x_i^2|. \end{aligned} \tag{1.9}$$

The second class of examples uses a vector subspace V, closed under the anticommutator operation $(y,z) \mapsto yz + zy$, of the vector space of all self-adjoint endomorphisms of some given Euclidean space E, with $\langle\,,\rangle$, α and σ defined, for $y, z \in V$ and x near 0 in V, by

$$\begin{aligned} &2\alpha(y,z) \,=\, yz + zy, \quad 2\langle y,z\rangle \,=\, \mathrm{tr}\, yz, \\ &2\sigma(x) \,=\, -\mathrm{tr}\, x \,-\, \log|\det(\mathrm{Id} - x)|. \end{aligned} \tag{1.10}$$

Note that all φ in (1.8) – (1.9) are $\mathrm{O}(d-1)$-invariant:

$$\varphi = e^{-\xi}[(1-\xi)^2 - s^2]^{-1/2}, \text{ for } \xi = x_d \text{ and } s \geq 0 \text{ with } s^2 = \textstyle\sum_{i<d} x_i^2. \tag{1.11}$$

Remark B. Important special cases of the second class of examples arise when

(i) V is the full space of all self-adjoint endomorphisms of E, or
(ii) E is the underlying real Euclidean space of a complex or quaternionic inner-product space, and V consists of those self-adjoint endomorphisms which are also $\mathbb{C}$-linear or, respectively, $\mathbb{H}$-linear, with tr and det still standing for the *real* trace and determinant; or,
(iii) V is the space of 3×3 octonion matrices, self-adjoint in a suitable sense, and viewed as 24×24 real symmetric matrices, so that $E = \mathbb{R}^{24}$.

See Section 5 for more details.

Let α be as in (1.2). For $x \in V$ we set $K[x] = \mathrm{span}\{\alpha_x^n x : n \in \mathbb{N} \cup \{0\}\}$, where $\mathbb{N}$ is the set of positive integers. Thus, since $\alpha_x^0 = \mathrm{Id}$ even for $\alpha_x = 0$,

$$K[x] \,=\, \mathrm{span}\,[\{x\} \cup \{\alpha_x^n x : n \in \mathbb{N}\}]. \tag{1.12}$$

Our two further results characterize the case where (1.1) is nontrivially solvable.

Theorem C. *Given* α *with* (1.2)*, three conditions are mutually equivalent*:

(i) *some* φ*, not identically zero, satisfies* (1.1)*,*
(ii) $\alpha_x = \alpha(x, \cdot\,)$ *and* $\alpha(\alpha_x^n x, \cdot\,)$ *commute for every* $x \in V$ *and* $n \in \mathbb{N}$*,*
(iii) *whenever* $x \in V$*, all* α_y*, where* $y \in K[x]$*, commute with one another.*

Corollary D. *For* α *as in* (1.2)*, equation* (1.1) *has a nontrivial solution* φ *if and only if* V *with the multiplication* α *constitutes a Jordan algebra.*

We use the two-sided, or bilateral, Laplace transform (no restriction of integration to a "positive" quadrant or half-line), with the same sign convention as Rothaus [**11**]. Thus, a Borel measure μ on a Euclidean d-space V has the Laplace transform φ defined on a suitable (necessarily convex) subset of V by

$$\varphi(x) = \int_V e^{\langle x,y\rangle}\, d\mu_y. \tag{1.13}$$

When μ is absolutely continuous with respect to the Lebesgue measure, we treat it as a function, by identifying it with its Radon–Nikodym derivative.

In the next theorem, the $\mathrm{O}(d-1)$-invariance of f reflects the same property of its Laplace transform, in the form of a rescaled version of (1.11).

The symbol u always denotes the unit-step (Heaviside) function.

Theorem E. *In the first class of examples of Theorem* A*, the rescaled version of* φ *with* (1.8) – (1.9)*, for every dimension* $d \geq 2$ *and any* c *in* (1.7) *such that* $(d+2)c^2 \in (0,1]$*, has the inverse Laplace transform* f *which depends on*

$$y = (y_1, \ldots, y_d) \text{ only through } \eta = y_d \text{ and } r \geq 0 \text{ given by } r^2 = \textstyle\sum_{i<d} y_i^2. \tag{1.14}$$

Specifically, with a constant κ *depending on* d *and* c*,*

$$f \,=\, \kappa e^{-\eta/c}[(1+c\eta)^2 - c^2r^2]^{p/2}\, u(1+c\eta-|c|r), \text{ where } p = -d + 1/c^2. \tag{1.15}$$

To prove Theorem E, in Sect. 6, we use the uniqueness of a solution φ to (1.1) with $\varphi(0) = 1$, mentioned in (1.4), which relieves us of a rather tedious task: evaluating the Laplace transform of f directly from (1.13).

For $d \in \{1, 2, 3\}$, the inverse Laplace transforms of the original functions φ with (1.8) – (1.9) and $\varphi(0) = 1$, not subjected to any rescaling, were described in [**12**]. We provide concise versions of those three descriptions in the Appendix.

The paper is organized as follows. We devote the final three sections to reviewing briefly the joint quantum and semi-quantum operators generated by a finite family of random variables having finite moments of all orders, and stating the result from [**12**] which shows that equation (1.1) holds for the joint Laplace transforms of the components of all suitably normalized nondegenerate 1-Meixner random vectors. That material is preceded by comments concerning intrinsic aspects of (1.1), unrelated to its quantum-probability interpretation, and proofs of the results listed above. To be specific, Sections 2 through 4 contain a quick verification of (1.1) for the examples described in Theorem A, a description of the trivial modifications of solutions to (1.1), and a proof of Theorem C. To prove the uniqueness assertion of Theorem A, in Section 5, we first trivially derive Corollary D from Theorem C. We then point out that, due to (1.2), any Jordan algebra arising in Corollary D is of the Euclidean type, and hence formally real. In their seminal 1934 paper [**4**], Pascual Jordan, John von Neumann and Eugene Wigner classified all formally real Jordan

algebras, and the uniqueness claim in Theorem A is an immediate consequence of their classification. Finally, in Section 6, we prove Theorem E.

The present paper leaves unanswered two questions, to be addressed in the future. One concerns a possible converse of [**12**, Lemma 6], that is, of Proposition 9.1, also stated in the lines surrounding (1.3). More precisely, let an $\mathbb{R}^d$-valued random variable $Y = (Y_1 \dots, Y_d)$ on a probability space have finite moments of all orders and satisfy the normalizing condition (1.3). For the Laplace transform φ of the resulting push-forward Borel measure on $\mathbb{R}^d$, *does equation* (1.1) *with any suitable data* (1.2) *guarantee that* Y *is a* 1*-Meixner random vector*? The other question is: *Can the* 1*-Meixner condition be characterized by certain group invariance, for instance, in the spirit of* [**6**] *and* [**7**]?

2. The examples described in Theorem A

For α appearing in (1.9), equation (1.6) obviously amounts to

$$(1-x_d)\sigma_i = (1+\sigma_d)x_i \ \text{ if } \ i<d \ \text{ and } \ (1-x_d)\sigma_d = x_d + \textstyle\sum_{i<d} x_i\sigma_i, \tag{2.1}$$

with subscripts applied to σ denoting partial differentiations. The relations (2.1) are easily verified to hold if σ is defined by (1.9).

When α is given by (1.10), equation (1.6) takes the form

$$2[d_y\sigma](x) = [d_{xy+yx}\sigma](x) + \operatorname{tr} xy \ \text{ for any } \ y\in V \ \text{ and } \ x\in V \ \text{ close to } 0. \tag{2.2}$$

In the case of a real vector space E of any finite dimension, $\theta : \mathrm{GL}(E) \to \mathbb{R}$ defined by $\theta(x) = \log|\det x|$ satisfies, whenever $x \in \mathrm{GL}(E)$ and $y \in \mathfrak{gl}(E)$, the following well-known relation (see Remark 2.1 below):

$$[d_y\theta](x) \ = \ \operatorname{tr} x^{-1}y. \tag{2.3}$$

With σ as in (1.10), using (2.3) we get $2[d_z\sigma](x) = -\operatorname{tr} z + \operatorname{tr}(\mathrm{Id}-x)^{-1}z$ for x near 0 and any $z\in V$. Applied to $z=y$ and $z = xy+yx$, this gives

$$\begin{aligned} 2[d_y\sigma](x) - [d_{xy+yx}\sigma](x) - \operatorname{tr} xy = {} & [-\operatorname{tr} y + \operatorname{tr}(\mathrm{Id}-x)^{-1}y] \\ & + [\operatorname{tr} xy - \operatorname{tr}(\mathrm{Id}-x)^{-1}xy] - \operatorname{tr} xy. \end{aligned}$$

Canceling, on the right-hand side, the two occurrences of $\operatorname{tr} xy$ with opposite signs, we see that the right-hand side equals the inner product of $2y$ and

$$-\mathrm{Id} \ + \ (\mathrm{Id}-x)^{-1} \ - \ (\mathrm{Id}-x)^{-1}x \ = \ -\mathrm{Id} \ + \ (\mathrm{Id}-x)^{-1}(\mathrm{Id}-x) \ = \ 0,$$

proving (2.2) and, consequently, (1.6).

Remark 2.1. For any $z \in \mathfrak{gl}(E)$, applying d/dt at $t=0$ to $\det\exp tz$ we get $\operatorname{tr} z$, since $\operatorname{tr} : \mathfrak{gl}(E) \to \mathbb{K}$ is the Lie-algebra homomorphism induced by the Lie-group homomorphism $\det : \mathrm{GL}(E) \to \mathbb{K}\smallsetminus\{0\}$. (Equivalently, tr is the differential of det at Id, since $-z$ has, in dimension m, the characteristic polynomial $t \mapsto (-1)^m[t^m + (\operatorname{tr} z)t^{m-1}]$ plus lower-order terms, and so $\det(\mathrm{Id}+tz)$ equals $1+(\operatorname{tr} z)t$ plus higher-order terms.) Using the curve $t \mapsto x\exp tx^{-1}y$, which for $t=0$ arrives at x with the velocity y, one easily obtains (2.3).

3. Trivial modifications

We begin by justifying the claims (1.4) and (1.5) made in the Introduction. First, concerning the operator $\mathrm{Id} - \alpha_x : V \to V$, depending as in (1.2) on $x \in V$,

$$\mathrm{Id} - \alpha_x \text{ is invertible for } x = 0, \text{ and hence for all } x \text{ near } 0, \tag{3.1}$$

so that (1.1) is in normal form (solved for the derivatives), and (1.4) follows.

Proposition 3.1. *If* (1.6) *has a* C^2 *solution* σ *on a neighborhood of* 0 *in* V*, then* σ *is real-analytic, uniquely determined by* α *up to an additive constant, and, explicitly,* $\sigma(x) = \langle x, \gamma_x x\rangle$*, where, with the notation of* (1.2)*,* γ_x *is the linear operator* $V \to V$ *given by*

$$\gamma_x = \int_0^1 t\,[\mathrm{Id} - t\alpha_x]^{-1} dt \text{ or, equivalently, } \gamma_x = \sum_{n=0}^{\infty} (n+2)^{-1} \alpha_x^n. \tag{3.2}$$

Furthermore, σ *uniquely determines both* α *and the inner product* $\langle\,,\rangle$*, while* σ *has at* 0 *a strict local minimum, with the Hessian equal to* $\langle\,,\rangle$*.*

Proof. We have (3.1), which turns (1.6) into a formula for the gradient of σ. Thus, if a solution σ exists, it may be assumed to vanish at 0, and then $\sigma(x)$ equals, for x close to 0, the line integral of the gradient along the curve $[0,1] \ni t \mapsto tx$, which proves the first claim.

The final clause is in turn obvious since, for all $x, y, z \in V$,

$$[d_x\sigma](0) = 0, \quad [d_x d_y \sigma](0) = \langle x, y\rangle, \quad [d_x d_y d_z \sigma](0) = 2\langle \alpha(x,y), z\rangle,$$

as one sees differentiating (1.6) twice. □

There are at least three trivial ways of constructing new examples of α with (1.2) out of old ones, while keeping equation (1.6) solvable.

The first one may be termed *rescaling*. With V, d and $\langle\,,\rangle$ in (1.2) unchanged, we use any fixed $c \in \mathbb{R} \smallsetminus \{0\}$ to replace α, and a solution σ to (1.6), by (1.7). Secondly, we say that two examples of α with (1.2), in the same dimension d, are $\mathrm{O}(d)$*-equivalent* if one has the same components as the other in suitably chosen orthonormal bases of the two underlying Euclidean spaces.

Finally, we have the *orthogonal direct-sum* construction, where V and α in (1.2) are the orthogonal direct sums of V', α' and V'', α'' satisfying the analogs of (1.2) in dimensions d' and d'', so that

$$\alpha(x', x'') = 0, \quad \alpha(x', y') = \alpha'(x', y'), \quad \alpha(x'', y'') = \alpha''(x'', y'')$$

whenever $x', y' \in V'$ and $x'', y'' \in V''$. The solvability of (1.6) for σ amounts to the same for both summands, and then it follows that $\sigma(x' + x'') = \sigma'(x') + \sigma''(x'')$, where σ' and σ'' are the solutions for V', α' and V'', α''. To see this, one just fixes x' or x'' in $\sigma(x' + x'')$.

Following [**4**, p. 37], we call α *irreducible* when it is not such a direct sum, and *reducible* otherwise. Thus, when α is reducible, our problem is reduced to understanding what happens in lower dimensions.

In addition, (1.6) remains unaffected, since so does the gradient of σ, when σ and the inner product $\langle\,,\rangle$ are replaced by $r\sigma$ and $r\langle\,,\rangle$, with any real $r > 0$.

4. Proof of Theorem C

Lemma 4.1. *For α as in* (1.2)*, the existence of a solution σ to* (1.6) *is equivalent to the condition* (ii) *in Theorem* C.

Proof. A solution σ to (1.6), defined near 0 in V, exists if and only if

$$\begin{aligned}&\text{for } x \text{ near } 0 \text{ in } V \text{ the operator } \alpha(x,\,\cdot\,)\\ &\text{commutes with } \alpha([\mathrm{Id}-\alpha(x,\,\cdot\,)]^{-1}x,\,\cdot\,).\end{aligned} \tag{4.1}$$

In fact, let us set $w(x) = [\mathrm{Id} - \alpha(x,\,\cdot\,)]^{-1}x$, so that the requirement in question is for the vector field w to be, locally, a gradient or, in other words, for the directional derivative operator sending $y \in V$ to $d_y w \in V$ to be self-adjoint at each point. Let $A(x) = \mathrm{Id} - \alpha(x,\,\cdot\,)$ and $B(x) = [A(x)]^{-1}$, so that $w(x) = [B(x)]x$ and, if we suppress the dependence on x in the notation, $d_y B = B[\alpha(y,\,\cdot\,)]B$. Thus, $[d_y w](x) = [B(x)][\alpha(y,\,\cdot\,)][B(x)]x + [B(x)]y$ which, as $w(x) = [B(x)]x$, reads

$$[d_y w](x) = [B(x)][\alpha(y, w(x)) + y].$$

On the right-hand side we have, evaluated on y, a composition of two self-adjoint operators, $[B(x)]$ and $\alpha(w(x),\,\cdot\,) + \mathrm{Id}$, which is itself self-adjoint if and only if they commute. Ignoring the occurrences of Id, we see that the condition is the commutation of $\alpha(w(x),\,\cdot\,)$ with $B(x)$, that is, with its inverse $A(x) = \mathrm{Id} - \alpha(x,\,\cdot\,)$, and consequently with $\alpha(x,\,\cdot\,)$. This is precisely (4.1), as required.

With $\alpha_x = \alpha(x,\,\cdot\,)$, we obviously have, for x near 0 in V and $t \in [0,1]$,

$$[\mathrm{Id} - t\alpha(x,\,\cdot\,)]^{-1} = \sum_{n=0}^{\infty} t^n \alpha_x^n. \tag{4.2}$$

Due to (4.1) which, in view of (1.7), remains unaffected when one multiplies α by any $t \neq 0$, it now follows that (i) in Theorem C is equivalent to (ii). □

Lemma 4.2. *Suppose that α satisfies* (1.2) *and* (1.6) *has a solution σ.*

(i) *There exists a dense open set $Z \subseteq V$ formed by x such that $\alpha_x = \alpha(x,\,\cdot\,)$ has the maximum possible number of distinct eigenvalues and, at the same time, $K[x]$ in* (1.12) *is of dimension* $\max\{\dim K[y] : y \in V\}$.

(ii) *Given $x \in Z$, any orthonormal basis $e_1, \dots, e_d$ of V, diagonalizing α_x, must also diagonalize all α_y with $y \in K[x]$.*

Proof. The assertion (i) is immediate, since Z is defined to be the intersection of two specific dense open sets.

Any fixed orthonormal basis diagonalizing α_x diagonalizes α_y for $y \in K[x]$ sufficiently close to x. In fact, as $x \in K[x]$, such α_y commute with α_x due to Lemma 4.1, so that they leave the eigenspaces of α_x invariant, and have to act on each eigenspace as a multiple of the identity, or else α_y would have more distinct eigenvalues than α_x does. An obvious real-analyticity argument yields the same conclusion for all $y \in K[x]$, whether or not close to x, proving (ii). □

Proof of Theorem C. Assume (i). Choosing $x \in Z$ and a basis $e_1, \dots, e_d$ as in part (ii) of Lemma 4.2, we see that $\alpha(\alpha_x^n x,\,\cdot\,)$, for all integers $n \geq 0$, being simultaneously diagonalizable, commute with one another. Since $Z \subseteq V$ is open and dense, the same holds, in view of real-analyticity, for all $x \in V$, so that (iii) follows. As (iii) trivially implies (ii), Lemma 4.1 completes the proof. □

5. Proof of uniqueness in Theorem A

By a *Jordan algebra* [**3**, p. 24] one means a real or complex vector space V with a symmetric bilinear multiplication $\alpha : V \times V \to V$ such that, for every $x \in V$, the α-multiplications by x and by the α-square of x commute:

$$[\alpha(x, \cdot\,),\, \alpha(\alpha(x,x), \cdot\,)] \,=\, 0.$$

Corollary D is thus a trivial consequence of (ii) in Theorem C for $n = 1$ and the well-known fact [**3**, p. 27] that (ii) in Theorem C, as a property of α, holds whenever α is a Jordan-algebra multiplication.

One refers to a real Jordan algebra V as *Euclidean* [**3**, p. 42] if it satisfies (1.2) for some inner product $\langle\,,\rangle$ in V. This is known [**3**, pp. 42, 154] to be the case if and only if the algebra is *formally real* in the sense that $\alpha(x,x) + \alpha(y,y) \neq 0$, unless $x = y = 0$. Thus, being obviously Euclidean,

(5.1) the Jordan algebras arising in Corollary D are all formally real.

A Jordan algebra is said to be *irreducible* [**4**, p. 37] if it is not a direct sum of lower-dimensional ones. Irreducible formally real Jordan algebras were classified by Jordan, von Neumann and Wigner [**4**, p. 63, Fundamental Theorem 2], and their list of types consists of the first class of examples of α in Theorem A (which they list under $\gamma = 1$ and $\gamma = 2$) and the three cases of Remark B (referred to as $\gamma = 3$ and $\gamma \geq 4$). See also [**3**, Chapter V].

Due to (5.1), this proves the uniqueness clause of Theorem A.

As pointed out by Jordan, von Neumann and Wigner [**4**, p. 63], in our first class of examples of α, the cases $d = 3$ and $d = 4$ coincide with the case (i) of Remark B and, respectively, (ii) for the scalar field $\mathbb{C}$ in Remark B where, in both (i) and (ii), E is of real/complex dimension two.

The examples in our Theorem A were designed with the purpose of maximally simplifying the presentation, and result in abundant choices of Jordan algebras which are reducible. For instance, V in the second class of examples could be the algebra generated by $\{\mathrm{Id}, x\}$, with a single self-adjoint endomorphism x of E.

6. Proof of Theorem E

For any $c \in \mathbb{R} \smallsetminus \{0\}$ and $\lambda, \varepsilon \in (0, \infty)$,

$$\int_0^\infty \Pi(r)\, dr \,<\, \infty, \text{ where } \Pi(r) = \int_{r/|c|}^\infty e^{-\lambda t} t^\varepsilon dt. \tag{6.1}$$

In fact, $\Pi(r) = \lambda^{-\varepsilon-1}\Gamma(\varepsilon + 1, \lambda r/|c|)$, where $\Gamma(\delta, q) = \int_q^\infty e^{-t} t^{\delta-1} dt$ is the upper incomplete gamma function. Now (6.1) follows since $\Gamma(\delta, 0) = \Gamma(\delta)$ is finite while, according to [**9**, p. 70], $\Gamma(\delta, q) \leq e^{-q} q^\delta/(q - \delta + 1)$ whenever $q > \delta - 1 > 0$.

If $c \in \mathbb{R} \smallsetminus \{0\}$ and $x = (x_1, \ldots, x_d)$ lies in the open half-cone given by

$$1 - c\xi > |c|s, \text{ for } \xi = x_d \text{ and } s \geq 0 \text{ such that } s^2 = \textstyle\sum_{i<d} x_i^2, \tag{6.2}$$

then, for $\eta = y_d$, any $q \in (0, \infty)$, and $r \geq 0$ with $r^2 = \sum_{i<d} y_i^2$, as in (1.14),

$$\int_{\mathbb{R}^d} e^{\langle x,y\rangle - \eta/c} (1 + c\eta)^q u(1 + c\eta - |c|r)\, dy \,<\, \infty. \tag{6.3}$$

To prove (6.3), we begin by setting $t = (1 + c\eta)/c^2$. We thus have

$$\langle x, y\rangle \,-\, \eta/c \,\leq\, -\lambda t \,+\, (1 - c\xi)/c^2 \text{ with } \lambda = 1 - c\xi - |c|s. \tag{6.4}$$

Namely, the Schwarz inequality gives $\langle x,y\rangle \le sr+\xi\eta$, and so $c^2(\langle x,y\rangle-\eta/c)\le |c|s|c|r+(c\xi-1)c\eta$. As $|c|r\le 1+c\eta=c^2t$ in view of the Heaviside factor in (6.3), replacing $c\eta$ in the term $(c\xi-1)c\eta$ with c^2t-1, we get (6.4).

Let, again, $t=(1+c\eta)/c^2$, and $(y_1,\dots,y_{d-1})=rz$, where $r\in[0,\infty)$ and z ranges over the unit sphere S^{d-2} around 0 in $\mathbb{R}^{d-1}$. Now (6.3) reads

$$|c|^{2q}\int_0^\infty\int_{r/|c|}^\infty\int_{S^{d-2}} e^{\langle x,y\rangle-\eta/c}t^q r^{d-2}dz\,dt\,dr\ <\ \infty.$$

Since $r^{d-2}\le |c|^{d-2}t^{d-2}$ wherever $t\ge r/|c|$, with V_{d-2} denoting the $(d-2)$-dimensional volume of S^{d-2}, we see that, by (6.4), the last integral does not exceed

$$|c|^{2q+d-2}e^{1/c^2}e^{-\xi/c}\,\mathrm{V}_{d-2}\int_0^\infty\int_{r/|c|}^\infty e^{-\lambda t}t^{q+d-2}dt\,dr,$$

and (6.3) follows from (6.1) for $\varepsilon=q+d-2$, as (6.2) and (6.4) give $\lambda>0$.

For f defined by (1.15), and $i=1,\dots,d$, we have

$$\text{a)}\ \ \partial_i\widehat{f}=\widehat{y_if},\qquad \text{b)}\ \ x_i\widehat{f}=-\widehat{\partial_if}. \tag{6.5}$$

Here $\widehat{\ }$ stands for the Laplace transform, the transforms of functions of $(y_1,\dots,y_d)$ depend on $(x_1,\dots,x_d)$, while ∂_i equals $\partial/\partial x_i$ in (6.5-a) and $\partial/\partial y_i$ in (6.5-b). Also, the domains of all the transforms appearing in (6.5) contain the open half-cone (6.2).

The last claim, about the domains of the transforms of f, y_if and, respectively, ∂_if, follows from (6.3) and the obvious inequality $p\ge 2$ in (1.15) since,

$$\text{setting}\ \ \psi\,=\,(1+c\eta)^2-c^2r^2,\ \text{one has}\ \ 0\le\psi^{\delta/2}\le(1+c\eta)^\delta \tag{6.6}$$

whenever $\delta\in[0,\infty)$, due to the Heaviside factor in (6.3). More precisely, for f we use (6.6) and (6.3) with $q=p\ge 2$. If $i<d$, the definition of r preceding (6.3) and the Heaviside factor in (6.3) give $|c||y_i|\le|c|r\le 1+c\eta$. Thus, as $|\partial_i\psi^{p/2}|=pc^2|y_i|\psi^{(p-2)/2}$ and $|y_i\psi^{p/2}|=|y_i|\,|\psi^{p/2}|$, we get, again from (6.6), $|\partial_i\psi^{p/2}|\le p|c|(1+c\eta)^{p-1}$ and $|c||y_i\psi^{p/2}|\le(1+c\eta)^{p+1}$. Now (6.3) with q equal to $p-1\ge 1$ or, respectively, to $p+1$, yields the required conclusion about the domains. When $i=d$, similarly, $|c||\partial_d(e^{-\eta/c}\psi^{p/2})|\le e^{-\eta/c}[\psi^{p/2}+pc^2\psi^{(p-2)/2}(1+c\eta)]\le e^{-\eta/c}[(1+c\eta)^p+pc^2(1+c\eta)^{p+1}]$ and, with $|c||y_d|=|c\eta|\le(1+c\eta)+1$, one sees that $|c||y_d\psi^{p/2}|\le|c|[(1+c\eta)^{p+1}+(1+c\eta)^p]$.

We obtain (6.5-a) from (6.3) combined with the Lebesgue dominated convergence theorem. The integration by parts leading to (6.5-b) is also permitted: the boundary terms in question all vanish. For $i<d$ this is clear as ψ given by (6.6) equals 0 on the boundary half-cone $1+c\eta=|c|r$, and hence so does f due to positivity of the exponent $p\ge 2$ in (1.15). Finally, when $i=d$, every line parallel to the dth coordinate direction intersects the solid half-cone $1+c\eta\ge|c|r$ along a half-line L, and $f=0$ at both "endpoints" of L (one of which is infinite).

On the other hand, (1.15) easily implies that, with $\partial_i=\partial/\partial y_i$,

$$\begin{aligned}&(1+cy_d)\partial_if+cy_i\partial_df\,=\,-y_if\ \ \text{for all}\ \ i<d,\\&(1+cy_d)\partial_df\,+\,c\textstyle\sum_{i<d}y_i\partial_if\,=\,-(cd\,+\,y_d)f\end{aligned} \tag{6.7}$$

which, as a consequence of (6.5), amounts to

$$\begin{aligned}&(1-cx_d)\partial_i\widehat{f}=(\widehat{f}+c\partial_d\widehat{f})x_i\ \ \text{if}\ \ i<d,\\&(1-cx_d)\partial_d\widehat{f}\,=\,x_d\widehat{f}\,+\,c\textstyle\sum_{i<d}x_i\partial_i\widehat{f}.\end{aligned} \tag{6.8}$$

The system (6.8) is nothing else than (2.1) with α replaced by $\widetilde{\alpha} = c\alpha$, and treated as a condition imposed on $\widehat{f} = e^{\sigma}$, rather than σ. The uniqueness – mentioned in (1.4) – of a solution φ to (1.1) with $\varphi(0) = 1$, applied to $\widetilde{\alpha} = c\alpha$ rather than α, now proves Theorem E, for a suitably chosen constant κ.

7. Background and summary

In dimension one, the classic Meixner class consists of the random variables having Szegő-Jacobi parameters α_n and ω_n that are, respectively, linear in n and quadratic in n with zero constant term. Up to rescaling and translations, this class can be split into six types: Gaussian, Poisson, Gamma, negative binomial, two -parameter hyperbolic, and binomial random variables. Attempts to extend this definition to higher dimensions used to run into a difficulty, due to the absence of a reasonable replacement of the Szegő-Jacobi parameters. However, in the past two decades, such a replacement has been proposed, in the form of joint quantum operators: creation, preservation, and annihilation, generated by the random variables $Y_1, \ldots, Y_d$, which are the components of a random vector. Splitting each preservation operator into two halves, one attached to the corresponding creation operator, the other to the annihilation operator, one obtains the semi-quantum operators: semi-creation and semi-annihilation operators. As pointed out in [**10**], the Meixner random variables can be characterized by the fact that a nested double commutator involving the semi-annihilation operator and the multiplication operator generated by the random variable Y is a linear combination of the semi-quantum operators. Moreover, within this class, the Meixner subclass consisting, up to a rescaling and translation, of the Gaussian and Gamma random variables, have the property that the (simple) commutator of the semi-annihilation operator U and the multiplication operator Y equals a linear combination of Y and the identity operator. These observations allow us to both recover the classic Meixner class and define a hierarchy of Meixner random variables, in an operatorial manner, involving the number of nested commutators, the joint semi-annihilation operator U_i, and the multiplication operators Y_i, $i = 1, \ldots, d$, all of which are used in defining them. Those using only one commutator in their definition are called 1-Meixner random vectors, those with two nested commutators – referred to as 2-Meixner, etc. It was shown in [**10**] that every nondegenerate two-dimensional 1-Meixner random vector can be reduced, via an invertible affine transformation, to a random vector whose components are independent Gamma or Gaussian random variables. An analogous result of [**12**] states that a nondegenerate three-dimensional 1-Meixner random vector can be similarly reduced either to a random vector with components as above, or to one having components which are three random variables with a joint probability distribution supported in a solid half-cone or on its boundary surface.

According to yet another result of [**12**], stated below as Proposition 9.1, equation (1.1) is satisfied by the Laplace transforms φ of all suitably normalized nondegenerate 1-Meixner random vectors.

8. Some details

Let us fix an integer $d \geq 1$, a probability space $(\Omega, \mathcal{F}, P)$, and an $\mathbb{R}^d$-valued random variable $Y = (Y_1 \ldots, Y_d)$, assumed to have *finite moments of all orders*, that is, to satisfy the condition $Y_i \in L^q(\Omega, \mathcal{F}, P)$ for every $q \in (0, \infty)$ and all $i =$

$1, \ldots, d$. We are thus requiring $|Y_i|^q$, for each q and each component $Y_i : \Omega \to \mathbb{R}$, to be integrable. Two vector subspaces of $L^2(\Omega, \mathcal{F}, P)$ are of interest to us:

$$\begin{array}{l} \text{the subalgebra } F \text{ generated by } Y_1, \ldots, Y_d, \\ \text{and the closure } H \text{ of } F \text{ in } L^2(\Omega, \mathcal{F}, P), \end{array} \tag{8.1}$$

the word 'subalgebra' referring to the associative algebra of measurable functions $\Omega \to \mathbb{R}$, and 'closure' to the L^2 topology. The space F consists of all $f(Y_1, \ldots, Y_d)$, with f ranging over polynomial functions of d real variables. For such f having degrees at most n, the resulting functions $f(Y_1, \ldots, Y_d)$ form a vector subspace F_n of F, and we denote by G_n the L^2-orthogonal complement of F_{n-1} in F_n, where we set $F_{-1} = G_{-1} = \{0\}$. Thus, $G_0 = F_0 = \mathbb{R}$.

Clearly, H is the Hilbert-space direct sum of the family $\{G_n\}_{n=0}^{\infty}$.

Some terminology: functions from F are called *polynomial random variables*, those among them lying in G_n are said to be *degree n homogeneous*, while each G_n, and H, is referred to as the *n-th homogeneous chaos space* (and, respectively, the *chaos space*) *generated by* $Y_1, \ldots, Y_d$.

We identify each Y_i with the operator $F \to F$ of multiplication by Y_i, that is,

$$f(Y_1, \ldots, Y_d) \mapsto Y_i f(Y_1, \ldots, Y_d). \tag{8.2}$$

As shown in [**2**] and [**1**], each operator (8.2) sends each G_n into $G_{n-1} \oplus G_n \oplus G_{n+1}$.

For any integer $n \geq 0$, we denote by $P_n : F \to G_n$ the L^2-orthogonal projection operator, and by $I : F \to F$ the identity operator. Thus, $I = \sum_{n \geq 0} P_n$, the sum being finite when applied to any vector in F. The same finiteness property is obviously shared by the operators $a_i^0, a_i^- : F \to F$, where $i = 1, \ldots, d$, defined by $a_i^0 = \sum_{n \geq 0} P_n Y_i P_n$, and $a_i^- = \sum_{n \geq 0} P_{n-1} Y_i P_n$. One calls

$$U_i = a_i^- + \frac{1}{2} a_i^0 \tag{8.3}$$

the ith *semi-annihilation* operator of (or, generated by) $Y_1 \ldots, Y_d$.

9. 1-Meixner random vectors and equation (1.1)

For $(\Omega, \mathcal{F}, P)$ and $Y_1 \ldots, Y_d$ satisfying the assumptions listed in the initial paragraph of Section 8, and F given by (8.1), let $U_1, \ldots, U_d$ be the corresponding joint semi-annihilation operators, defined by (8.3). Following [**12**, Definition 2], we say that $Y = (Y_1, \ldots, Y_d)$ is a 1-*Meixner random vector* if, for all $i, j \in \{1, \ldots, d\}$, the commutator $[U_i, Y_j] : F \to F$, with Y_j as in (8.2), is a linear combination of $Y_1, \ldots, Y_d$ and I, where $I : F \to F$ denotes the identity operator. In other words,

$$[U_i, Y_j] = \sum_{k=1}^{d} \alpha_{ijk} Y_j + \beta_{ij} I \text{ for all } i, j \in \{1, \ldots, d\} \text{ and some } \alpha_{ijk}, \beta_{ij} \in \mathbb{R}. \tag{9.1}$$

A 1-Meixner random vector $Y = (Y_1, \ldots, Y_d)$ is called *nondegenerate* if the multiplication operators $Y_1, \ldots, Y_d$ and I are linearly independent. In this case, as shown in [**12**, Proposition 2], the coefficients α_{ijk} in (9.1) are not only (obviously) unique, buy also *totally symmetric in* i, j, k. Moreover, using an invertible affine transformation, we are then always free to assume the normalizing condition (1.3). The following conclusion was established in [**12**, Lemma 6].

Proposition 9.1. *Let $Y = (Y_1 \ldots, Y_d)$ be a 1-Meixner random vector with* (1.3). *Then equation* (1.1) *is satisfied by the Laplace transform φ of $Y : \Omega \to \mathbb{R}^d$, explicitly given by $\varphi(x_1, \ldots, x_d) = \int e^{\langle x, Y\rangle} dP$, where $x = (x_1, \ldots, x_d)$ and $\langle x, Y\rangle = x_1 Y_1 + \ldots + x_d Y_d$, for the data $V, \alpha, \langle\,,\rangle$ in* (1.2) *consisting of $V = \mathbb{R}^d$, its standard inner product $\langle\,,\rangle$, and α with the components $\alpha_{ijk} = \langle \alpha(e_i, e_j), e_k\rangle$, in the standard orthonormal basis $e_1, \ldots, e_d$, equal to the coefficients α_{ijk} in* (9.1).

The assumption that Y is nondegenerate, made in [**12**, Lemma 6], may be dropped, as it follows from (1.3). Namely, if $I : F \to F$ were a linear combination of the multiplication operators in (8.2), the equality $I = s_1 Y_1 + \ldots + s_d Y_d$ applied to each Y_i would, via (1.3) combined with integration, give $s_i = 0$.

Appendix. The inverse Laplace transforms for (1.9) with $d \le 3$

For the reader's convenience, we describe here the inverse Laplace transforms, appearing in [**12**], of the functions φ with (1.8) – (1.9) and $\varphi(0) = 1$.

If $d = 1$, as $\varphi : (-\infty, 1) \to \mathbb{R}$ is given by $\varphi(x) = e^{-x}/(1-x)$, from (1.13),

$$\varphi \text{ is, obviously, the Laplace transform of } y \mapsto e^{-y-1} u(y+1), \tag{10.1}$$

u being the unit-step (Heaviside) function. Next, as one sees using the new variable $\sqrt{1-x}\sqrt{y+1/2}$ and the Gaussian integral $\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}$, if $x \in (-\infty, 1)$,

$$\int_{-1/2}^{\infty} \frac{e^{(x-1)y}}{\sqrt{y+1/2}}\, dy \;=\; \sqrt{\pi}\, \frac{e^{(1-x)/2}}{\sqrt{1-x}}\,. \tag{10.2}$$

Consequently, the function φ in (1.9), given, for $d = 2$, by

$$\begin{aligned} &\varphi(x, \xi) \;=\; e^{-\xi}[(1-\xi)^2 - x^2]^{-1/2}, \text{ equals the Laplace transform of} \\ &\mathbb{R}^2 \ni (y, \eta) \mapsto (\pi e)^{-1} e^{-\eta}[(1+\eta)^2 - y^2]^{-1/2} u(1+\eta+y) u(1+\eta-y). \end{aligned} \tag{10.3}$$

In fact, in terms of the new variables $\lambda = (\eta + y)/2$ and $\tau = (\eta - y)/2$, with the Jacobian determinant $\partial(y, \eta)/\partial(\lambda, \tau) = 2$, and $xy + \xi\eta = (\xi + x)\lambda + (\xi - x)\tau$, the integral of the second line of (10.3) multiplied by $e^{xy+\xi\eta} dy\, d\eta$ has separated variables, and hence equals $(\pi e)^{-1}$ times the product of two integrals, namely, of $e^{(\xi-1+x)\lambda}(\lambda+1/2)^{-1/2} u(\lambda+1/2)\, d\lambda$ and $e^{(\xi-1-x)\tau}(\tau+1/2)^{-1/2} u(\tau+1/2)\, d\tau$ which, by (10.2) with (x, y) replaced by $(\xi + x, \lambda)$ or $(\xi - x, \tau)$, are equal, respectively, to $\sqrt{\pi}\, e^{(1-\xi\mp x)/2}(1 - \xi \mp x)^{-1/2}$. Thus, their product multiplied by $(\pi e)^{-1}$ coincides with the right-hand side in the first line of (10.3).

For $d = 3$ the inverse Laplace transform of the function φ in (1.9) is *not* a function, but rather a measure supported in a half of a Lorentzian light cone [**12**, Proposition 7], as described below.

Generally, given a d-dimensional oriented, time-oriented Lorentzian vector space V with the $-+\ldots+$ inner product $(\,,)$ and the volume form Θ, the future light cone $\partial C_+ \subseteq V$ is well known [**5**, p. 62] to carry a canonical smooth measure Λ, invariant under the isochronous Lorentz group. Explicitly: Λ_y, as an exterior $(d-1)$-form, at any $y \in \partial C_+$, equals the restriction to $y^\perp$ of $\Theta(y', \cdot, \ldots, \cdot)$, where the Lorentzian orthogonal complement $y^\perp$ obviously coincides with the tangent space of ∂C_+ at y, and $y' \in V$ is any vector such that $(y, y') = -1$. For obvious dimensional reasons, due to uniqueness of y' up to adding vectors from $y^\perp$, this definition is correct in the sense that Λ_y does not depend on the choice of y'.

The push-forward of Λ under the orthogonal projection $\partial C_+ \to W \smallsetminus \{0\}$ onto any spacelike hyperplane $W \subseteq V$ equals $dw/|w|$, in terms of the variable $w \in W$, which in turn becomes

$$\Lambda \,=\, r^{d-3}\,dr\,d\zeta \tag{10.4}$$

if one further identifies W with $\mathbb{R}^{d-1}$, letting r and ζ range, respectively, over $[0,\infty)$ and the unit sphere S^{d-2} centered at 0 in $\mathbb{R}^{d-1}$.

For our purposes $V = \mathbb{R}^d$ and the future light cone ∂C_+ is given by $1+\eta > r$ with $\eta = y_d$ and $r \geq 0$ such that $r^2 = \sum_{i<d} y_i^2$, where $y = (y_1,\ldots,y_d)$. Thus, $(0,\ldots,0,-1)$ serves as the origin (vertex of the cone). The measure Λ on ∂C_+ is, by (10.4), the push-forward, under the mapping $S^{d-2}\times(0,\infty) \to \partial C_+ \subseteq \mathbb{R}^d$ which sends (ζ, r) to $(r\zeta, r-1)$, of the standard product measure on $S^{d-2}\times(0,\infty)$ multiplied by r^{d-3}. When $d=3$, in the notation of (1.11),

$$\begin{aligned}&\text{our } \ \varphi \,=\, e^{-\xi}[(1-\xi)^2 - s^2]^{-1/2} \ \text{ is the Laplace trans-}\\ &\text{form of the product of } \Lambda \text{ and the function } (2\pi e)^{-1}e^{-\eta}.\end{aligned} \tag{10.5}$$

We will derive (10.5) from the following fact, verified in the next paragraph: whenever $\xi, s \in \mathbb{R}$ and $1-\xi > s \geq 0$, so that $1-\xi-s\cos\theta > 0$,

$$\int_0^\pi \frac{d\theta}{1-\xi-s\cos\theta} \,=\, \pi\,[(1-\xi)^2-s^2]^{-1/2}. \tag{10.6}$$

Namely, by continuity, we are free to assume that $s>0$. The left-hand side of (10.6) equals one-half of the analogous integral over $[0,2\pi]$, and so it coincides with $1/2$ times the contour integral over the circle $|z|=1$ in $\mathbb{C}$ of the holomorphic 1-form $f(z)\,dz$, where $f(z) = (iz)^{-1}/[1-\xi-s(z+z^{-1})/2] = 2i/[sz^2+2(\xi-1)z+s]$. The polynomial $z \mapsto sz^2+2(\xi-1)z+s$ has the roots $z_\pm = (1-\xi\pm[(1-\xi)^2-s^2]^{1/2})/s$ and $0<z_-<1<z_+$. In fact, $z_+>1$ as $1-\xi>s$, while $0<z_-<1$ since $z_+z_- = s/s = 1$. Consequently, the integral in (10.6) equals πi times the residue of $f(z)$ at $z=z_-$. The residue, being clearly the limit of $2i(z-z_-)/[sz^2+2(\xi-1)z+s]$ as $z\to z_-$, is thus $2i/[s(z_- - z_+)] = -i[(1-\xi)^2-s^2]^{-1/2}$, and (10.6) follows.

Denoting by V_m the m-dimensional volume of the unit m-sphere S^m centered at 0 in $\mathbb{R}^{m+1}$, we see that, for any fixed vector $\chi \in \mathbb{R}^{m+1}$ with $|\chi| = t$, the integral over S^m of the function $\zeta \mapsto e^{\langle\chi,\zeta\rangle}$ equals $\mathrm{V}_{m-1}Q_m(t)$, where

$$Q_m(t) \,=\, \int_0^\pi e^{t\cos\theta}\sin^{m-1}\theta\,d\theta, \quad \text{and so} \quad \mathrm{V}_{m-1}Q_m(0) = \mathrm{V}_m, \tag{10.7}$$

while $\mathrm{V}_{m-1} = 2\pi^{m/2}/\Gamma(m/2)$. Given $x,y \in \mathbb{R}^d$, let us now write $x=(v,\xi)$ and $y=(w,\eta)$, where $v,w\in\mathbb{R}^{d-1}$ and $\xi,\eta\in\mathbb{R}$, as well as $s=|v|$ and $r=|w|$.

In any dimension $d\geq 3$, the Laplace transform of the measure $(2\pi e)^{-1}e^{-\eta}\Lambda$, mentioned in (10.5) for $d=3$, depends on $x\in\mathbb{R}^d$ only through ξ and s: due to (10.4), with $v=r\zeta$, where $\zeta\in S^{d-2}$, as $\eta = r-1$ on ∂C_+, the transform

$$\text{sends } x=(v,\xi) \text{ to } (2\pi e)^{-1}\int_0^\infty\int_{S^{d-2}} e^{r\langle v,\zeta\rangle}\,e^{(\xi-1)(r-1)}r^{d-3}\,d\zeta\,dr,$$

which amounts to $(s,\xi) \mapsto (2\pi e)^{-1}\mathrm{V}_{d-3}\int_0^\infty Q_{d-2}(rs)\,e^{(\xi-1)(r-1)}r^{d-3}dr$ for Q_{d-2} and V_{d-3} as in (10.7) and, using (10.7), we rewrite it as

$$(s,\xi) \,\mapsto\, (2\pi)^{-1}e^{-\xi}\mathrm{V}_{d-3}\int_0^\pi\int_0^\infty e^{r(\xi-1+s\cos\theta)}\,r^{d-3}\sin^{d-3}\theta\,dr\,d\theta.$$

For $d = 3$, with $\mathrm{V}_0 = 2$, this becomes

$$(s, \xi) \mapsto \pi^{-1} e^{-\xi} \int_0^\pi \int_0^\infty e^{-r(1-\xi-s\cos\theta)} dr\, d\theta.$$

Integrating over r first, with $1 - \xi - s\cos\theta > 0$, we obtain

$$(s, \xi) \mapsto \pi^{-1} e^{-\xi} \int_0^\pi \frac{d\theta}{1 - \xi - s\cos\theta}.$$

By (10.6), the right-hand side equals $e^{-\xi}[(1-\xi)^2 - s^2]^{-1/2}$, proving (10.5).

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