Harmonic-curvature warped products over surfaces

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Abstract. For warped products with harmonic curvature, nonconstant warping functions ϕ, and compact two-dimensional bases (M, h), we establish a dichotomy: either the Gaussian curvature K of the metric $g = \phi^{-2}h$ is constant and negative, or ϕ equals a specific elementary function of K, also depending on the dimension p and Einstein constant ε of the fibre. In both cases the fibre must be an Einstein manifold with $p > 1$ and $\varepsilon > 0$, while the function $f = \phi^{p/2}$ satisfies a Yamabe-type second-order differential equation on (M, g). We prove that both possibilities are realized on every closed orientable surface of genus greater than 1, and in the latter case – which also occurs on the 2-sphere and real projective plane – the metrics in question constitute uncountably many distinct homothety types.

Introduction

One says that a Riemannian manifold with the curvature tensor R has *harmonic curvature* [**4**, Sect. 16.33] if $\operatorname{div} R = 0$ or, in local coordinates, $R_{ijl}{}^{k}{}_{,k} = 0$. This condition amounts to the Codazzi equation (1.3) imposed on the Ricci tensor, and it implies constancy of the scalar curvature [**4**, Sect. 16.4(ii)]. A compact Riemannian manifold has harmonic curvature if and only if its Levi-Civita connection is a critical point of its Yang-Mills functional [**4**, Sect. 16.34].

The known examples of Riemannian manifolds with harmonic curvature include five non-disjoint classes, consisting of: *Einstein manifolds, conformally flat manifolds of constant scalar curvature, locally reducible manifolds having* $\operatorname{div} R = 0$, *certain nontrivial warped products of dimensions* $n > 4$ *with Einstein fibres and one-dimensional or hyperbolic bases, and some four-manifolds that are, locally or globally, nontrivial warped products of surfaces.* See [**4**, Sect. 16.34, 16.40], [**9**, Sect. 4]. Every known *compact* example belongs to one of these five classes.

In the construction of the last two classes of the preceding paragraph, except for the case of hyperbolic bases, the (local) warped-product structure, rather than being an Ansatz, follows from a purely geometric assumption, namely, that a certain tensor field B should have no more than two distinct eigenvalues at each point. Specifically, in the fourth (or, fifth) class, B is the Ricci tensor [**4**, Sect. 16.38] or, respectively, the self-dual Weyl tensor acting on self-dual bivectors [**9**, p. 145].

2010 *Mathematics Subject Classification.* Primary 53C25; Secondary 53B20.

Both authors' research was supported in part by a FAPESP-OSU 2015 Regular Research Award (FAPESP grant: 2015/50265-6).

Warped products with harmonic curvature were studied by Kim, Cho and Hwang [**15**], and there are interesting results on Einstein warped products [**14, 16**].

It is therefore natural to consider the following problem.

QUESTION 0.1. *Which compact warped-product Riemannian manifolds of dimensions greater than* 3 *have harmonic curvature, without belonging to the first three classes italicized above?*

Question 0.1 remains open in general, and its complexity clearly increases with the dimension m of the base. For $m = 1$ the answer is well known [**7**, Lemma 1(ii) and Theorem 1]. The present paper deals with the case $m = 2$.

We begin by proving a dichotomy result (Theorem 3.1): if a warped product has harmonic curvature, a compact-surface base (M, h), and a nonconstant warping function ϕ, while K denotes the Gaussian curvature of the conformally-related metric $g = \phi^{-2}h$ on M, then

$$\text{the fibre must be an Einstein manifold of some dimension } p > 1 \text{ with an Einstein constant } \varepsilon > 0 \tag{0.1}$$

(for compact bases of all dimensions; see Remark 2.2), the function $f = \phi^{p/2}$ satisfies a Yamabe-type second-order differential equation (6.2.iii) on (M, g), and

$$\text{either } K \text{ is the negative constant } -\varepsilon/(p-1), \text{ or } \phi \text{ equals a positive constant times } |(p-1)K+\varepsilon|^{1/(1-p)}, \text{ with } K \text{ (necessarily) nonconstant.} \tag{0.2}$$

Conversely, these conditions imply harmonic curvature for the warped product.

Theorem 3.1 is a the first step toward answering Question 0.1 for $m = 2$, and the two cases of (0.2) amount to two very different problems.

The first one concerns finding, for p, ε fixed as in (0.1), *nonconstant positive solutions* f *to the quasilinear elliptic equation*

$$\Delta f - af = -cf^{1+4/p} \text{ with constants } a = p(p-2)\varepsilon/[4(p-1)] \text{ and } c > 0 \tag{0.3}$$

on a given closed surface of negative constant Gaussian curvature $K = -\varepsilon/(p-1)$. (This is equation (6.2.iii), with $r = 0 < c$ due to (6.3) – (6.4.i).) Yamabe [**22**] has shown – cf. Lemma 13.1 below – that such f exist, on any compact Riemannian surface, if the parameters a, p satisfy a specific inequality, which here reads

$$p > 2 - \lambda_1/K. \tag{0.4}$$

As we point out in Remark 1.10, a result of Schoen, Wolpert and Yau [**19**] yields

$$\lambda_1 < 2|K| \tag{0.5}$$

whenever the metric of constant curvature $K < 0$, on any closed orientable surface of genus $\mathbf{g} > 1$, is confined to a suitable nonempty open subset of the Teichmüller space; (0.5) gives (0.4) for all $p \geq 4$, and the fibre dimensions $p \geq 4$ are the only ones of interest for the "constant K case" of Question 0.1 (see Remark 3.5).

Consequently, *the first possibility in* (0.2) *is realized, with warped products of all relevant fibre dimensions, by a Teichmüller-open nonempty set of metrics of constant curvatures* $K < 0$, *on closed orientable surfaces of all genera* $\mathbf{g} > 1$.

In the remaining, second case of (0.2) we look for *metrics* g *on compact surfaces* M *having nonconstant Gaussian curvatures* K *such that there exist positive constants* $\varepsilon, \mu \in \mathrm{I\!R}$ *with* $(p-1)K + \varepsilon \neq 0$ *everywhere in* M *and*

$$\begin{aligned} &2(p+1)[(p-1)K+\varepsilon]\Delta K - (3p-2)(p+1)g(\nabla K, \nabla K)] \\ &\quad = \mu|(p-1)K+\varepsilon|^{2(p-2)/(p-1)} - (2K+p\varepsilon)[(p-1)K+\varepsilon]^2. \end{aligned} \tag{0.6}$$

(Equation (0.6), that is, (3.3), requires a normalization of the warping function, described in Section 3.) Let us emphasize that the existence of $\varepsilon, \mu \in (0,\infty)$ for which (0.6) holds and $|(p-1)K+\varepsilon| > 0$ on M is *a property of the metric g alone*. Using a bifurcation argument, we prove, in Section 12, that metrics g with this property exist for M diffeomorphic to S^2, $\mathrm{I\!RP}^2$ or any closed orientable surface of genus greater than 1. More precisely, curves of such metrics, emanating from a given metric $\hat{g}$ of (nonzero) constant Gaussian curvature $\hat{K}$ on M, are naturally associated with certain positive eigenvalues λ of $-\hat{\Delta}$, for the Laplacian $\hat{\Delta}$ of $\hat{g}$. Each of the curves in question, which we call *λ-branches*, consists of metrics representing uncountably many distinct homothety types and, if $\lambda' \neq \lambda$, a metric from the λ-branch, close to $\hat{g}$, cannot be homothetic to any metric near $\hat{g}$ belonging to the λ'-branch. Here are some further details.

If $\hat{K} > 0$, the eigenvalues $\lambda > 0$ that give rise to λ-branches may be completely arbitrary (on $\mathrm{I\!RP}^2$), or even-numbered and otherwise arbitrary (on S^2). For $\hat{K} < 0$ (that is, on any closed orientable surface of genus greater than 1) these λ have to be simple and different from $(p-2)|\hat{K}|$, and so, according to the theorem of Schoen, Wolpert and Yau [**19**] mentioned in Remark 1.10, constant-curvature metrics $\hat{g}$ admitting such eigenvalues λ fill a nonempty open subset of the Teichmüller space.

As a result, *warped products of all fibre dimensions $p > 1$ realize the second case of* (0.2) *with $M = S^2$, or $M = \mathrm{I\!RP}^2$, or M closed, orientable and of any genus greater than* 1, *while – in the last case – the conformal types of the metrics g form a nonempty Teichmüller-open set.*

Two subcases of the second case of (0.2) need commenting on. One, characterized by $p = 2$, has already been settled in [**9**]. The other, in which $M = T^2$, is still an open problem, even though one can easily provide examples of nontrivial compact warped products with harmonic curvature and bases diffeomorphic to T^2 that are neither Einstein nor conformally flat: namely, Riemannian products of S^1 and suitably chosen harmonic-curvature warped-product manifolds having the base S^1, classified in [**7**, Lemma 1(ii) and Theorem 1]. However, being reducible, such examples do not lie within the scope of Question 0.1.

1. Notations and preliminaries

Manifolds (always assumed connected), mappings and tensor fields, including Riemannian metrics and functions, are by definition of class C^∞, except in Sections 10 – 11 where, for technical reasons, we require that, rather than being smooth, they should belong to suitable L^2 Sobolev spaces. Given a Riemannian metric g, we let ∇ stand for the Levi-Civita connection of g as well as the g-gradient, and $\Delta, \mathrm{Ric}, \mathrm{div}, K$ for the g-Laplacian, the Ricci tensor of g, the g-divergence and, in the case of a surface metric g, its Gaussian curvature. When a metric is denoted by h, the analogous symbols will be $D, \Delta^h, \mathrm{Ric}^h, \mathrm{div}^h$ and K^h.

One calls a function β on Riemannian manifold (M,g) *isoparametric* [**21**] if $\Delta\beta$ and $g(\nabla\beta,\nabla\beta)$ are functions of β. It is well-known that, when $\dim M = 2$, the existence of nonconstant isoparametric functions amounts (locally, at generic points) to "rotational symmetry." More precisely, for $\beta : M \to \mathrm{I\!R}$ and a Killing field v with $d_v\beta = 0$ and $\nabla\beta \neq 0 \neq v$ everywhere, β must be isoparametric since the flow of v leaves $\Delta\beta$ and $g(\nabla\beta,\nabla\beta)$ invariant as well. Conversely, on an oriented Riemannian surface (M,g), isoparametricity of a function β without critical points leads to an explicit construction of a Killing field v without zeros, orthogonal to $\nabla\beta$. See, e.g., [**9**, Lemma 7], or formula (1.6.ii) below.

Here is another well-known fact, cf. [**11**, Remark 2.5] or the end of this section:

(1.1) any Killing vector field defined on a nonempty connected open subset of a simply connected, real-analytic, Riemannian manifold (M,h), can be uniquely extended to a Killing field on M.

LEMMA 1.1. *Let a compact real-analytic Riemannian surface (M,h) have nonconstant Gaussian curvature K^h and nonzero Euler characteristic $\chi(M)$. Any h-Killing vector field v defined on a nonempty connected open set $U \subseteq M$ has a unique extension to an h-Killing field on M, provided that, if necessary, one replaces (M,h) by a two-fold isometric covering thereof, and U by a connected component of the pre-image of U under the covering projection.*

PROOF. Assuming v to be nontrivial and denoting by (M',h') the Riemannian universal covering of (M,h), we see that v gives rise to a h'-Killing field v' on a suitable (connected) open submanifold U' of M, and (1.1) allows us to treat v' as defined on all of M'. Then, push-forwards of v' under deck transformations are constant multiples of v' (or else K^h would be constant), and v' has zeros (or else it would span a tangent-direction field on M', descending to M, even though $\chi(M) \neq 0$). As the flow of v' is periodic due to its obvious periodicity on a neighborhood of a zero of v', the push-forwards of v' under deck transformations, having the same flow period as v' itself, must all equal $\pm v'$. $\square$

For any function β on a Riemannian manifold (M,g) one clearly has

$$2[\nabla d\beta](v,\,\cdot\,) = dQ, \quad \text{where } v = \nabla\beta \text{ and } Q = g(\nabla\beta,\nabla\beta). \tag{1.2}$$

Suppose now that we are given a 1-form ξ and a twice-covariant symmetric tensor field b on a Riemannian manifold (M,h). Treating b as a 1-form valued in 1-forms, we define the exterior product $\xi\wedge b$ and the exterior derivative db to be the 2-forms valued in 1-forms with the local-coordinate expressions $[\xi\wedge b]_{qrs} = \xi_q b_{rs} - \xi_r b_{qs}$ and $[db]_{qrs} = b_{rs,q} - b_{qs,r}$ or, in coordinate-free notation, $[\xi\wedge b](u,v) = \xi(u)b(v,\,\cdot\,) - \xi(v)b(u,\,\cdot\,)$ and $[db](u,v) = (D_u b)(v,\,\cdot\,) - (D_v b)(u,\,\cdot\,)$ for tangent vector fields u,v and the Levi-Civita connection D of h. Then, cf. [**4**, Sect. 16.3],

$$db = 0 \text{ if and only if } b \text{ is a Codazzi tensor on } (M,h) \tag{1.3}$$

while, for any functions $f,\phi : M \to \mathrm{I\!R}$, with $\dim M = 2$ in (1.4.b) – (1.4.c),

$$\begin{aligned} &\text{a)}\quad d[fb] \,=\, f db \,+\, df\wedge b, \qquad \text{b)}\quad d[Dd\phi] \,=\, -K^h d\phi\wedge h,\\ &\text{c)}\quad K^h d\phi \,=\, \mathrm{div}^h[Dd\phi] \,-\, d\Delta^h\phi. \end{aligned} \tag{1.4}$$

Namely, (1.4.b) amounts to the Ricci identity for $d\phi$ expressed in terms of the Gaussian curvature K^h, that is, the two-dimensional case of the general formula $d\nabla\xi = \xi R$ (in coordinates: $\xi_{s,jq} - \xi_{s,qj} = R_{qjs}{}^p\xi_p$), applied here to $\xi = d\phi$, and valid for any 1-form ξ on a manifold with a torsion-free connection ∇ having the curvature tensor R. (The exterior derivative db of $b = \nabla\xi$ is defined, as above, by $[db]_{qrs} = b_{rs,q} - b_{qs,r}$, but this time the twice-covariant tensor field b need not be symmetric.) Contracting (1.4.b), one gets the Bochner identity (1.4.c).

LEMMA 1.2. *Let J and α be the complex-structure tensor and the area 2-form of an oriented two-dimensional Riemannian manifold (M,h), with the convention* *that* $\alpha_{qr} = J_q^s h_{sr}$ *or, equivalently,* $\alpha = h(J\,\cdot\,,\,\cdot\,)$. *Any* 1*-form* ξ *and twice-covariant symmetric tensor field* b *on* M *then satisfy the relation*

$$J^*(\xi \wedge b) \,=\, \alpha \otimes [(\mathrm{tr}_h b)\xi - b(v,\,\cdot\,)], \quad \text{for } v \text{ characterized by } \xi = h(v,\,\cdot\,),$$

the local-coordinate version of which reads $(\xi_q b_{rs} - \xi_r b_{qs})J_i^s = (b_s^s \xi_i - b_i^s \xi_s)\alpha_{qr}$. *Two special cases arise when* $b = h$ *or, respectively,* $\xi = d\phi$ *and* $b = Dd\phi$ *with a function* $\phi : M \to \mathrm{I\!R}$. *Namely, if* D *denotes both the Levi-Civita connection of* (M, h) *and the* h*-gradient,* Δ^h *is the* h*-Laplacian, and* $H = h(D\phi, D\phi) : M \to \mathrm{I\!R}$,

$$J^*(\xi \wedge h) \,=\, \alpha \otimes \xi, \qquad J^*(d\phi \wedge Dd\phi) \,=\, \alpha \otimes [(\Delta^h\phi)\,d\phi - dH/2]. \tag{1.5}$$

Proof. Being skew-symmetric in q, r, the expression $\xi_q b_{rs} - \xi_r b_{qs}$ must equal $\rho_s \alpha_{qr}$, for some 1-form ρ. Contracting this equality against h^{rs} we obtain $b_s^s \xi_i - b_i^s \xi_s = \alpha_{ir} w^r = J_i^s h_{sr} w^r = J_i^s \rho_s$, with the vector field w given by $\rho = h(w,\,\cdot\,)$, which proves our formula for $J^*(\xi \wedge b)$, and (1.5) now follows from (1.2). $\square$

Given a Riemannian surface (M, g) and $\beta, \sigma, \zeta : M \to \mathrm{I\!R}$ such that σ and ζ are functions of β, while $\Delta\beta = \sigma$ and $g(\nabla\beta, \nabla\beta) = 2\zeta$, one has

$$\begin{array}{rl} \text{i)} & 2\zeta\,\nabla d\beta \,=\, 2(\sigma - \zeta')\zeta g \,+\, (2\zeta' - \sigma)\,d\beta \otimes d\beta, \quad \text{with } \zeta' = d\zeta/d\beta, \\ \text{ii)} & \text{if the surface } (M, g) \text{ is oriented, then } e^\kappa Jv \text{ is a } g\text{-Killing field,} \end{array} \tag{1.6}$$

for $v = \nabla\beta$ and J as in Lemma 1.2. Here (1.6.ii) holds on the open set U on which $d\beta \neq 0$, the function κ of β being any antiderivative of $(\sigma - 2\zeta')/(2\zeta)$, defined away from zeros of ζ. Namely, both sides of (1.6.i) are symmetric, have the same g-trace, and agree, when evaluated on $\nabla\beta$, as a consequence of (1.2), which yields (1.6.i) both on our open set U, and in the interior of the zero set of $d\beta$, while the union of the two sets is dense. To obtain (1.6.ii), note that $2\zeta e^{-\kappa}\nabla(e^\kappa Jv) = 2(\sigma - \zeta')\zeta J$ in view of the relations $2\zeta\,d\kappa \otimes Jv = (\sigma - 2\zeta')\,d\beta \otimes Jv$ and $2\zeta\,J\nabla v = 2(\sigma - \zeta')\zeta J + (2\zeta' - \sigma)\,d\beta \otimes Jv$ due, respectively, to our choice of κ, and to (1.6.i) rewritten as $2\zeta\,\nabla v = 2(\sigma - \zeta')\zeta + (2\zeta' - \sigma)\,d\beta \otimes v$, where $2(\sigma - \zeta')\zeta$ stands for $2(\sigma - \zeta')\zeta$ times the identity.

Lemma 1.3. *For a Riemannian surface* (M, g) *with the Gaussian curvature* K,

(i) *whenever* $\psi, \nu : M \to \mathrm{I\!R}$ *are functions with* $\nabla dK = \psi g + \nu\,dK \otimes dK$, *one necessarily has* $(K - \psi\nu)\,dK + d\psi = g(\nabla K, \nabla\nu)\,dK - g(\nabla K, \nabla K)\,d\nu$,

(ii) *if functions* $\varSigma, Z$ *defined on an interval containing the range of* K *satisfy the relations* $\Delta K = \varSigma(K)$ *and* $g(\nabla K, \nabla K) = 2Z(K)$, *then*

$$(2Z' - \varSigma)(Z' - \varSigma) = 2(Z'' - \varSigma' - K)Z, \quad \text{where } (\,)' = d/dK. \tag{1.7}$$

Proof. In (i), $\Delta K = 2\psi + \nu g(\nabla K, \nabla K)$, and our claim immediately follows from (1.4.c) for $h = g$ and $\phi = K$, combined with (1.2). Under the hypotheses of (ii), we may apply (1.6.i), at points with $d\beta \neq 0$, to $(\beta, \sigma, \zeta) = (K, \varSigma, Z)$, obtaining the assumption of (i) for $\psi = \varSigma - Z'$ and $\nu = (Z' - \varSigma/2)/Z$. The conclusion of (i) now reads $(K - \psi\nu)\,dK + d\psi = 0$, since ν is a function of K, and it easily gives (1.7) wherever $Z \neq 0$. Thus, (1.7) holds on the whole interval in question, due to a dense-union argument analogous to the one following (1.6); note that $\varSigma = 0$ on every open interval on which $Z = 0$. $\square$

Remark 1.4. The scalar curvatures s, s^h and Laplacians Δ, Δ^h of conformally related Riemannian metrics g and $h = g/\tau^2$ in dimension m are given by

$$\mathrm{s}^h = \tau^2\mathrm{s} + 2(m-1)\tau\Delta\tau - m(m-1)g(v,v), \qquad \Delta^h = \tau^2\Delta - (m-2)\tau d_v, \tag{1.8}$$

where $v = \nabla\tau$ is the g-gradient of τ. Cf. [**4**, Theorem 1.159]. For $m = 2$, this becomes $K^h = \tau^2 K + \tau\Delta\tau - g(v,v)$ and $\Delta^h = \tau^2\Delta$, with $\mathrm{s} = 2K$ and $\mathrm{s}^h = 2K^h$ expressed in terms of the Gaussian curvatures K, K^h of g and h.

Remark 1.5. Under the assumptions of Remark 1.4, if τ assumes its extremum values $\tau_{\max}$, $\tau_{\min}$, while s, s^h are both constant and $\mathrm{s} < 0 < \tau$, then τ is constant and $\mathrm{s}^h < 0$. This well-known conclusion follows since, by (1.8), $\tau_{\max}^2 \le \mathrm{s}^h/\mathrm{s} \le \tau_{\min}^2$.

Remark 1.6. The existence of a nontrivial h-Killing vector field v, for a Riemannian metric h on a compact surface M, precludes negativity of the Gaussian curvature K of *any* metric g on M. In fact, passing to a two-fold covering, if necessary, we may assume M oriented, which turns h into a Kähler metric on a compact complex curve of some genus $\mathbf{g}$, admitting a nontrivial real-holomorphic vector field v (so that $\mathbf{g} \le 1$), while the condition $K < 0$ would give $\mathbf{g} > 1$.

Remark 1.7. A Riemannian product with factors of dimensions n and n' is conformally flat if and only if both factors have constant sectional curvatures K, K' and $(n-1)(n'-1)(K+K') = 0$. See [**23**, Section 5], as well as [**4**, subsection 1.167].

Remark 1.8. Let $\Delta f = \varOmega(f)$ for a function f on a compact Riemannian manifold and a function $\varOmega$ on an interval containing the range of f. If $\varOmega$ is strictly increasing or constant, then f must be constant, since $\varOmega(f_{\max}) \le 0 \le \varOmega(f_{\min})$.

Remark 1.9. Let λ_j be the jth eigenvalue of $-\hat\Delta$, for the Laplacian $\hat\Delta$ of the 2-sphere (or, projective plane) of constant Gaussian curvature $\hat K$, with

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots . \tag{1.9}$$

Then $\lambda_j = j(j+1)\hat K$ (or, $\lambda_j = 2j(2j+1)\hat K$). The spectrum of $-\hat\Delta$ acting on rotationally invariant functions is the same, but with one-dimensional eigenspaces, spanned by the *zonal spherical harmonics* [**24**, Sect. 2.3].

Remark 1.10. On every closed orientable surface M of genus greater than 1 there exist metrics g having constant Gaussian curvature $K < 0$ and an arbitrarily large number of eigenvalues of $-\Delta$ in $(|K|/4, t + |K|/4]$, for any $t \in (0,\infty)$, where Δ is the Laplacian; all such metrics obviously satisfy (0.5). See [**5**, p. 211, Theorem 8.12], [**6**, p. 251, Theorem 2]. Also, M then admits a sequence of metrics g with $K = -1$ for which the lowest positive eigenvalue λ_1 of $-\Delta$ tends to 0 and has multiplicity 1. This follows from a result of Schoen, Wolpert and Yau [**19**, three final lines of the first paragraph on p. 279], applied to $n = 1$.

Finally, (1.1) follows since one can treat Killing fields as sections of a certain vector bundle, parallel relative to a natural connection. This is why the same conclusion holds, more generally, both for conformal vector fields in the pseudo-Riemannian case, and for infinitesimal affine transformations on a manifold with a connection, cf. [**10**, lines following Lemma 9.1], [**12**, text surrounding formula (1.5)].

2. Warped products and harmonic curvature

Given Riemannian manifolds (M,h), $(\varPi,\eta)$ of positive dimensions m, p and a nonconstant function $\phi : M \to (0,\infty)$, consider the nontrivial warped product

$$(M \times \varPi,\ h + \phi^2\eta) \tag{2.1}$$

with the base (M,h), fibre $(\varPi,\eta)$ and warping function ϕ. (The word 'nontrivial' refers to nonconstancy of ϕ, and the same symbols h,η,ϕ represent the pullbacks of h,η,ϕ to the product $M\times\varPi$.) The warped-product metric of (2.1) is obviously conformal to a product metric: $h+\phi^2\eta=\phi^2[g+\eta]$, where $g=\phi^{-2}h$. As one easily verifies [**11**, Lemma 1.2], cf. also [**15**], the (nontrivial) warped product (2.1) has harmonic curvature if and only if there exists a constant $\varepsilon\in\mathrm{I\!R}$ such that

(i) $\mathrm{Ric}^h-p\phi^{-1}Dd\phi$ is a Codazzi tensor on (M,h),
(ii) $\mathrm{div}^h(\phi^{p-2}Dd\phi)=[(p-1)\varLambda-\varepsilon]\,\phi^{p-4}d\phi$, where $\varLambda=h(D\phi,D\phi)$,
(iii) $(\varPi,\eta)$ is an Einstein manifold with the Einstein constant ε,

Ric^h and div^h being the Ricci tensor of h and the h-divergence, and D denoting both the Levi-Civita connection of (M,h) and the h-gradient.

Let us point out that, except for notations, (iii) and (i) are precisely (a)–(b) in [**11**, Lemma 1.2], while (ii), with $\varLambda=h(D\phi,D\phi)$, is equivalent to the condition

$$\phi^3\mathrm{div}^h(\phi^{-1}Dd\phi)=[(p-1)\varLambda-\varepsilon]\,d\phi+(1-p)\phi\,d\varLambda/2 \tag{2.2}$$

of [**11**, Lemma 1.2(c)], as one sees differentiating by parts, and also to the equality

$$\phi^2[\mathrm{Ric}^h(D\phi,\,\cdot\,)+d\Delta^h\phi]=[(p-1)h(D\phi,D\phi)-\varepsilon]\,d\phi+(1-p/2)\,\phi\,d[h(D\phi,D\phi)], \tag{2.3}$$

where Δ^h denotes the h-Laplacian. See [**11**, Lemma 1.2(e)].

Using, for instance, the components of the Ricci tensor of $h+\phi^2\eta$ evaluated in [**11**, the Appendix], and noting that, if $\varLambda=h(D\phi,D\phi)$,

$$2\phi^{-1}\Delta^h\phi\,+\,(p-1)\phi^{-2}\varLambda\,=\,4(p+1)^{-1}\phi^{-(p+1)/2}\Delta^h\phi^{(p+1)/2}, \tag{2.4}$$

we express the (necessarily constant) scalar curvature μ of $h+\phi^2\eta$ as follows:

$$\mathrm{s}^h\,+\,p[\varepsilon\phi^{-2}-4(p+1)^{-1}\phi^{-(p+1)/2}\Delta^h\phi^{(p+1)/2}]\,=\,\mu\in\mathrm{I\!R}. \tag{2.5}$$

Constancy of the scalar curvature μ is a general property of every metric with harmonic curvature [**4**, Sect. 16.4(ii)]. Here we can also derive it from (i) and (2.3): any h-Codazzi tensor b obviously has $\mathrm{div}^h b=d(\mathrm{tr}_h b)$ which, in the case of $b=\mathrm{Ric}^h-p\phi^{-1}Dd\phi$ amounts to $-2p\,\mathrm{div}^h(\phi^{-1}Dd\phi)=d[\mathrm{s}^h-2p\phi^{-1}\Delta^h\phi]$, where we have used the Bianchi identity for the Ricci tensor [**4**, Proposition 1.94]. At the same time, (2.2) states that $-2p\,\mathrm{div}^h(\phi^{-1}Dd\phi)$ equals p times the differential of $[(p-1)h(D\phi,D\phi)-\varepsilon]\,\phi^{-2}$. Subtracting these two equalities one gets

$$\text{constancy of }\ \mathrm{s}^h-2p\phi^{-1}\Delta^h\phi-p[(p-1)h(D\phi,D\phi)-\varepsilon]\,\phi^{-2}, \tag{2.6}$$

that is, by (2.4) – (2.5), of μ. Next, for a warped product (2.1) with $\mathrm{div}\,R=0$,

(2.7) the base metric h is real-analytic in suitable local coordinates.

In other words, the C^∞-manifold structure of the base M contains a unique real-analytic structure making h analytic. Namely, as shown by DeTurck and Goldschmidt [**13**], the analog of (2.7) holds for harmonic-curvature metrics, while the base (M,h) is isometric to a totally geodesic submanifold of the warped product (2.1).

Remark 2.1. If $\dim M=2$, condition (i) amounts to $(dK^h+pK^h\phi^{-1}d\phi)\wedge h$ $+p\phi^{-2}d\phi\wedge Dd\phi=0$, as one sees using (1.3) and (1.4.a) for the pair (f,b) equal to (K^h,h) or $(\phi^{-1},Dd\phi)$, with $\mathrm{Ric}^h=K^hh$, followed by (1.4.b).

Remark 2.2. Any nontrivial warped product (2.1) with a compact base (M,h) and harmonic curvature has $p=\dim\varPi\ge 2$, as the Einstein constant ε of the fibre $(\varPi,\eta)$ must be positive [**11**, Theorem 1.4]: the h-inner product of the left-hand side of (ii) with the h-gradient $D\phi$ obviously differs by an h-divergence from $-\phi^{p-2}$ times the h-norm squared of $Dd\phi$ while, with $\Lambda = h(D\phi, D\phi)$, the analogous inner product for the right-hand side equals $[(p-1)\Lambda - \varepsilon]\Lambda\phi^{p-4}$.

Remark 2.3. The case of a one-dimensional fibre ($p = 1$), for nontrivial warped products with harmonic curvature, is of very limited interest: it precludes compactness of the base (Remark 2.2) and, for two-dimensional bases – the main focus of this paper – the resulting three-manifolds (2.1) are conformally flat [**4**, Sect. 16.4(e)].

3. Warped products with two-dimensional bases

Recall that the warped-product metric in (2.1) is conformal to a product metric:

$$h + \phi^2\eta \;=\; [g+\eta]/\tau^2, \quad \text{where } g = \phi^{-2}h \text{ and } \tau = 1/\phi. \tag{3.1}$$

The question of which nontrivial warped products (2.1) with two-dimensional bases have harmonic curvature may obviously be rephrased in terms of the surface metric $g = \phi^{-2}h$ and the function $\tau = 1/\phi$. Remark 2.3 and condition (iii) of Section 2 allow us to assume that the fibre (Π, η) is an Einstein manifold of dimension $p \geq 2$ with some Einstein constant ε. In Section 4 we prove the following result.

Theorem 3.1. *Given a Riemannian surface (M, g), a nonconstant function $\tau : M \to (0, \infty)$, and an Einstein manifold (Π, η) of dimension $p \geq 2$ with the Einstein constant ε, the warped-product metric $[g+\eta]/\tau^2$ on $M \times \Pi$ has harmonic curvature if and only if the Gaussian curvature K of g satisfies the equation*

$$(2K + p\varepsilon)\tau^2 + 2(p+1)\tau\Delta\tau - (p+1)(p+2)g(\nabla\tau, \nabla\tau) \;=\; \mu \tag{3.2}$$

for a constant $\mu \in \mathrm{I\!R}$, and one of the following two conditions occurs.

(a) *K is constant, and equal to $-\varepsilon/(p-1)$,*
(b) *K is nonconstant, $(p-1)K + \varepsilon \neq 0$ everywhere in M, and τ equals a positive constant times $|(p-1)K + \varepsilon|^{1/(p-1)}$.*

The constant μ in (3.2) then coincides with the scalar curvature of $[g+\eta]/\tau^2$.

The positive constant mentioned of Theorem 3.1(b) may always be assumed equal to 1 by simultaneously rescaling τ and μ, so that (3.2) still holds. The resulting *normalized version* of case (b) in Theorem 3.1 amounts to a condition imposed on K alone, with no reference to τ at all. Explicitly, it reads

$$(p+1)[2\omega\Delta K \;-\; (3p-2)g(\nabla K, \nabla K)] \;=\; \mu|\omega|^{2(p-2)/(p-1)} \;-\; (2K + p\varepsilon)\omega^2 \tag{3.3}$$

for $\omega = (p-1)K + \varepsilon$, with constants $\varepsilon, \mu \in \mathrm{I\!R}$, where K is the (nonconstant) Gaussian curvature of the Riemannian surface (M, g), and $\omega \neq 0$ everywhere.

Theorem 3.2. *Under the assumptions stated in the preceding three lines, if M is compact, ε and μ are uniquely determined by g and p.*

Theorem 3.2, which will be proved in Section 5, has an obvious consequence: the product $\varepsilon\mathrm{A}$, for $\mathrm{A} = \mathrm{area}(M, g)$, is a homothety invariant of g. Note that multiplying g by $z \in (0, \infty)$ causes the quintuple $(\mathrm{A}, K, \varepsilon, \omega, \mu)$ to be replaced with $(z\mathrm{A}, z^{-1}K, z^{-1}\varepsilon, z^{-1}\omega, z^{(1+p)/(1-p)}\mu)$.

Remark 3.3. Another homothety invariant, naturally associated with any nonflat compact Riemannian surface having the Gaussian curvature K, is the point $[K_{\max} : K_{\min}]$ of the real projective line $\mathrm{I\!RP}^1$, where $[\,:\,]$ are the homogeneous coordinates. Clearly, K is constant if and only if $[K_{\max} : K_{\min}] = [1:1]$.

REMARK 3.4. Whenever the hypotheses of Theorem 3.1 are satisfied, along with (3.2) for a constant μ, and one of conditions (a) – (b) holds, compactness of M implies positivity of both ε and μ. See Remarks 2.2 and 6.2.

REMARK 3.5. In the context of Question 0.1, case (a) of Theorem 3.1 is of interest only for $p \geq 4$, since an Einstein manifold (Π, η) of dimension $p \in \{2, 3\}$ with the Einstein constant ε has constant sectional curvature $\varepsilon/(p-1)$. According to Remark 1.7, this implies conformal flatness of the harmonic-curvature metric $[g+\eta]/\tau^2$ (while, in case (b), $[g+\eta]/\tau^2$ is never conformally flat).

4. Proof of Theorem 3.1

Due to (3.1), $[g+\eta]/\tau^2$ has harmonic curvature if and only if (M, h) satisfies (i) and (ii) in Section 2 or, equivalently, (i) and (2.3). This further amounts to

$$\text{(4.1)}\qquad \begin{array}{ll} \text{a)} & \phi^2 dK^h + p[K^h\phi\, d\phi + (\Delta^h\phi)\, d\phi - d\varLambda/2] = 0, \\ \text{b)} & \phi^2(K^h d\phi + d\Delta^h\phi) - [(p-1)\varLambda - \varepsilon]\, d\phi - (1-p/2)\phi\, d\varLambda = 0, \end{array}$$

with $\varLambda = h(D\phi, D\phi)$, and K^h denoting the Gaussian curvature of h. Namely, $\mathrm{Ric}^h = K^h h$, so that (2.3) becomes (4.1.b), while Remark 2.1 and (1.5) easily yield the equivalence between (i) and (4.1.a).

As a consequence of (4.1), we obtain the relation (2.6), which now reads

$$\text{(4.2)}\qquad 2K^h + p[\varepsilon\phi^{-2} - 2\phi^{-1}\Delta^h\phi - (p-1)\phi^{-2}\varLambda] = \mu \ \text{ for some } \mu \in \mathrm{I\!R}.$$

Explicitly, subtracting (4.1.b) multiplied by $2p\phi^{-3}$ from $2\phi^{-2}$ times (4.1.a), we see that $d\{\ldots\} = 0$, with $\{\ldots\}$ denoting the left-hand side in (4.2). The system (4.1) is thus equivalent to one consisting of (4.1.a) and (4.2), namely

$$\text{(4.3)}\qquad \begin{array}{ll} \text{i)} & \phi^2 dK^h + p[K^h\phi\, d\phi + (\Delta^h\phi)\, d\phi - d\varLambda/2] = 0, \\ \text{ii)} & 2K^h + p[\varepsilon\phi^{-2} - 2\phi^{-1}\Delta^h\phi - (p-1)\phi^{-2}\varLambda] \text{ is constant.} \end{array}$$

Let us now rewrite (4.3) in terms of the conformally related metric $g = \phi^{-2}h$ and the function $\tau = 1/\phi$ on M, using the symbols K, ∇, Δ for the Gaussian curvature of g, the g-gradient, and the g-Laplacian, as well as setting $Q = g(\nabla\tau, \nabla\tau)$ and $Y = \Delta\tau$. Since $\varLambda = Q/\tau^2$ and $\Delta(1/\tau) = (2Q - \tau Y)/\tau^3$, Remark 1.4 yields

$$\text{(4.4)}\qquad \begin{array}{ll} \text{a)} & \tau^3 dK + \tau^2 dY - (1+p/2)\tau\, dQ + [(2-p)K\tau + Y]\tau\, d\tau = 0, \\ \text{b)} & (2K + p\varepsilon)\tau^2 + 2(p+1)\tau Y - (p+1)(p+2)Q \text{ is constant.} \end{array}$$

Finally, we may replace (4.4) with the (obviously equivalent) system consisting of (4.4.b) and the equality $2p(p-1)^{-1}\tau^{p+2} d\{\tau^{1-p}[(p-1)K + \varepsilon]\} = 0$ obtained by applying d to (4.4.b), multiplying the result by τ, and then subtracting it from $2(p+1)$ times (4.4.a). This proves Theorem 3.1, with the cases (a), (b) depending on whether the constant $\tau^{1-p}[(p-1)K + \varepsilon]$ is or is not equal to 0.

5. Proof of Theorem 3.2

We assume that $p > 2$, as the case $p = 2$ is already settled in [**9**, Remark 4].

It suffices to establish uniqueness of ε, since (3.3) provides an expression for μ in terms of ε and geometric invariants of g. Suppose that, on the contrary, in addition to (3.3) with $\omega = (p-1)K + \varepsilon$ one also has

$$(p+1)[2\tilde{\omega}\Delta K - (3p-2)g(\nabla K, \nabla K)] = \tilde{\mu}|\tilde{\omega}|^{2(p-2)/(p-1)} - (2K + p\tilde{\varepsilon})\tilde{\omega}^2$$

for $\tilde{\omega} = (p-1)K + \tilde{\varepsilon}$ and constants $\tilde{\varepsilon}, \tilde{\mu}$, while $\tilde{\varepsilon} < \varepsilon$ and $\omega\tilde{\omega} \neq 0$ everywhere, cf. the lines following (3.3). As $\omega - \tilde{\omega} = \varepsilon - \tilde{\varepsilon}$, subtracting the last equality (or,

ω times it) from (3.3) or, respectively, from $\tilde\omega$ times (3.3), one gets $\Delta K=\varSigma(K)$ and $g(\nabla K,\nabla K)=2Z(K)$ with the functions $\varSigma$ and Z of the variable K given by

$$(5.1)\qquad \begin{array}{rl}\text{i)} & 2(p+1)(\varepsilon-\tilde\varepsilon)\varSigma(K)=\tilde\omega^2\tilde\varTheta-\omega^2\varTheta,\\ \text{ii)} & 2(p+1)(3p-2)(\varepsilon-\tilde\varepsilon)Z(K)=\omega\tilde\omega[\tilde\omega\tilde\varTheta-\omega\varTheta],\quad\text{for}\\ \text{iii)} & \omega=(p-1)K+\varepsilon,\qquad \tilde\omega=(p-1)K+\tilde\varepsilon,\quad\text{and}\\ \text{iv)} & \varTheta=2K+p\varepsilon-\mu|\omega|^{2/(1-p)},\qquad \tilde\varTheta=2K+p\tilde\varepsilon-\tilde\mu|\tilde\omega|^{2/(1-p)}.\end{array}$$

Thus, since $\omega\tilde\omega\neq 0$ everywhere,

(5.2) the values of K lie in I_*, which is one of the intervals $(-\infty,\varepsilon/(1-p))$, $(\varepsilon/(1-p),\tilde\varepsilon/(1-p))$, $(\tilde\varepsilon/(1-p),\infty)$, while $\omega,\tilde\omega,\varTheta,\tilde\varTheta,\varSigma$ and Z are real-analytic functions of the real variable K, defined on the whole interval I_*.

As $(p-1)K=\omega-\varepsilon=\tilde\omega-\tilde\varepsilon$, (5.1.iii) – (5.1.iv), with $(\ )'=d/dK$, give

$$(5.3)\qquad \begin{array}{l}(p-1)\varTheta=2\omega+(p+1)(p-2)\varepsilon-(p-1)\mu|\omega|^{2/(1-p)},\\ (p-1)\tilde\varTheta=2\tilde\omega+(p+1)(p-2)\tilde\varepsilon-(p-1)\tilde\mu|\tilde\omega|^{2/(1-p)},\\ \omega'=\tilde\omega'=p-1,\quad \varTheta'=2+2(\operatorname{sgn}\omega)\mu|\omega|^{(1+p)/(1-p)},\\ \tilde\varTheta'=2+2(\operatorname{sgn}\tilde\omega)\tilde\mu|\tilde\omega|^{(1+p)/(1-p)}.\end{array}$$

Multiplying (5.1.i) and (5.1.ii) by $p-1$, then using (5.3), we obtain

$$(5.4)\qquad \begin{array}{rl}\text{i)} & 2(p^2-1)(\varepsilon-\tilde\varepsilon)\varSigma=2(\tilde\omega^3-\omega^3)+(p+1)(p-2)(\tilde\varepsilon\tilde\omega^2-\varepsilon\omega^2)\\ & \qquad+(p-1)[\mu|\omega|^{2(p-2)/(p-1)}-\tilde\mu|\tilde\omega|^{2(p-2)/(p-1)}],\\ \text{ii)} & 2(p^2-1)(3p-2)(\varepsilon-\tilde\varepsilon)Z\\ & \qquad=[2\tilde\omega^3+(p+1)(p-2)\tilde\varepsilon\tilde\omega^2-(p-1)\tilde\mu|\tilde\omega|^{2(p-2)/(p-1)}]\omega\\ & \qquad-[2\omega^3+(p+1)(p-2)\varepsilon\omega^2-(p-1)\mu|\omega|^{2(p-2)/(p-1)}]\tilde\omega.\end{array}$$

Next, as a consequence of (5.3),

$$(5.5)\qquad \begin{array}{l}[\omega\varTheta]'=4\omega+(p+1)(p-2)\varepsilon-(p-3)\mu|\omega|^{2/(1-p)},\\ [\tilde\omega\tilde\varTheta]'=4\tilde\omega+(p+1)(p-2)\tilde\varepsilon-(p-3)\tilde\mu|\tilde\omega|^{2/(1-p)}.\end{array}$$

Thus, $[\omega^2\varTheta]'=\omega'\omega\varTheta+\omega[\omega\varTheta]'=\omega\{(p-1)\varTheta+[\omega\varTheta]'\}$. Now, by (5.3) and (5.5),

$$(5.6)\qquad \begin{array}{l}[\omega^2\varTheta]'=2[3\omega+(p+1)(p-2)\varepsilon-(p-2)\mu|\omega|^{2/(1-p)}]\omega,\\ [\tilde\omega^2\tilde\varTheta]'=2[3\tilde\omega+(p+1)(p-2)\tilde\varepsilon-(p-2)\tilde\mu|\tilde\omega|^{2/(1-p)}]\tilde\omega.\end{array}$$

From (5.1.ii), $2(p+1)(3p-2)(\varepsilon-\tilde\varepsilon)Z'=\{\omega[\tilde\omega^2\tilde\varTheta]-\tilde\omega[\omega^2\varTheta]\}'$. The Leibniz rule applied to both products $\omega[\tilde\omega^2\tilde\varTheta]$ and $\tilde\omega[\omega^2\varTheta]$ yields, via (5.3), (5.1.i) and (5.6),

$$\begin{array}{l}2(p+1)(3p-2)(\varepsilon-\tilde\varepsilon)Z'=2(p^2-1)(\varepsilon-\tilde\varepsilon)\varSigma\\ \qquad+2\omega\tilde\omega[(p^2-p+1)(\tilde\varepsilon-\varepsilon)+(p-2)\mu|\omega|^{2/(1-p)}-(p-2)\tilde\mu|\tilde\omega|^{2/(1-p)}],\end{array}$$

as $\omega-\tilde\omega=\varepsilon-\tilde\varepsilon$, which, setting $\gamma=\operatorname{sgn}\omega$ and $\tilde\gamma=\operatorname{sgn}\tilde\omega$, we can rewrite as

$$(5.7)\qquad \begin{array}{l}2(p+1)(\varepsilon-\tilde\varepsilon)[(3p-2)Z'-(p-1)\varSigma]=2(p^2-p+1)(\tilde\varepsilon-\varepsilon)\omega\tilde\omega\\ \qquad+2(p-2)[\gamma\mu\tilde\omega|\omega|^{(p-3)/(p-1)}-\tilde\gamma\tilde\mu\omega|\tilde\omega|^{(p-3)/(p-1)}].\end{array}$$

With the same meaning of γ and $\tilde\gamma$, using (5.1.i) and (5.6), or (5.3), we get

$$(5.8)\qquad \begin{array}{rl}\text{i)} & (p+1)(\varepsilon-\tilde\varepsilon)\varSigma'=3(\tilde\omega^2-\omega^2)+(p+1)(p-2)(\tilde\varepsilon\tilde\omega-\varepsilon\omega)\\ & \qquad+(p-2)[\gamma\mu|\omega|^{(p-3)/(p-1)}-\tilde\gamma\tilde\mu|\tilde\omega|^{(p-3)/(p-1)}],\\ \text{ii)} & \varTheta''=-2(p+1)\mu|\omega|^{2p/(1-p)},\quad \tilde\varTheta''=-2(p+1)\tilde\mu|\tilde\omega|^{2p/(1-p)}.\end{array}$$

Consequently, (5.1.ii) gives $2(p+1)(3p-2)(\varepsilon-\tilde\varepsilon)Z''=[(\omega\tilde\omega^2)\tilde\varTheta]''-[(\omega^2\tilde\omega)\varTheta]''=(\omega\tilde\omega^2)\tilde\varTheta''-(\omega^2\tilde\omega)\varTheta''+2(\omega\tilde\omega^2)'\tilde\varTheta'-2(\omega^2\tilde\omega)'\varTheta'+(\omega\tilde\omega^2)''\tilde\varTheta-(\omega^2\tilde\omega)''\varTheta$. As (5.3) easily implies that $(\omega\tilde\omega^2)', (\omega^2\tilde\omega)', (\omega\tilde\omega^2)''$ and $(\omega^2\tilde\omega)''$ are, respectively, equal to $(p-1)(\tilde\omega+2\omega)\tilde\omega$, $(p-1)(\omega+2\tilde\omega)\omega$, $2(p-1)^2(2\tilde\omega+\omega)$ and $2(p-1)^2(2\omega+\tilde\omega)$, we obtain $2(p+1)(3p-2)(\varepsilon-\tilde\varepsilon)Z'' = (\omega\tilde\omega^2)\tilde\Theta'' - (\omega^2\tilde\omega)\Theta'' + 2(p-1)(\tilde\omega+2\omega)\tilde\omega\tilde\Theta' - 2(p-1)(\omega+2\tilde\omega)\omega\Theta' + 2(p-1)(2\tilde\omega+\omega)[(p-1)\tilde\Theta] - 2(p-1)(2\omega+\tilde\omega)[(p-1)\Theta]$. Therefore, replacing $\tilde\Theta'', \Theta'', \tilde\Theta', \Theta', (p-1)\tilde\Theta$ and $(p-1)\Theta$ with the expressions provided by (5.8.ii) and (5.3), we see that, since $\omega-\tilde\omega=\varepsilon-\tilde\varepsilon$,

(5.9) $(p+1)(3p-2)Z'' + 3(p-1)(p^2-p+2)\omega$ equals the sum of a constant and a constant-coefficient combination of the functions $\mu\tilde\omega|\omega|^{2/(1-p)} - \tilde\mu\omega|\tilde\omega|^{2/(1-p)}$ and $\gamma\mu|\omega|^{(p-3)/(p-1)} - \tilde\gamma\tilde\mu|\tilde\omega|^{(p-3)/(p-1)}$.

LEMMA 5.1. *As $K\to\pm\infty$, one has the following limit relations.*

(a) $2(p+1)\Sigma/K^2$ *and* $(p+1)\Sigma'/K$ *both tend to* $-(p-1)(p^2-p+4)$,
(b) $2(p+1)(3p-2)Z/K^3 \to -(p-1)^2(p^2-p+2)$,
(c) $2(p+1)(3p-2)Z'/K^2 \to -3(p-1)^2(p^2-p+2)$,
(d) $(p+1)(3p-2)Z''/K \to -3(p-1)(p^2-p+2)$.

The limits, as $K\to\pm\infty$, of $4[(p+1)(3p-2)]^2(2Z'-\Sigma)(Z'-\Sigma)/K^4$ and of $8[(p+1)(3p-2)]^2(Z''-\Sigma'-K)Z/K^4$ are $-(p-1)^2(p-2)(p^2+5p-2)(3p^2-p+2)$ and, respectively, $-4(p-1)^2(p-2)(p^2-p+2)(3p^3-5p^2+12p-8)$. The difference between the former and the latter limits equals $(p-1)^3(p-2)$ times the positive function $12p^4-23p^3+55p^2-56p+60$ of the real variable $p\ge 1$.

PROOF. By (5.1.iii), ω/K and $\tilde\omega/K$ tend to $p-1$ as $K\to\pm\infty$. Since $\tilde\omega^3-\omega^3 = (\tilde\omega-\omega)(\tilde\omega^2+\tilde\omega\omega+\omega^2)$ and $\omega-\tilde\omega=\varepsilon-\tilde\varepsilon$, (5.4.i) and (5.8.i), with $2(p-2)/(p-1)<2$ and $(p-3)/(p-1)<1$, yield (a). Similarly, (5.4.ii), (5.7) combined with (a), and (5.9) give (b), (c) and, respectively, (d). Now (a) – (d) imply positivity in the final clause as $12p^4-23p^3+55p^2-56p+60 = p^2(p-1)(12p-11)+4(11p^2-14p+15)$. $\square$

We now derive the contradiction that proves Theorem 3.2. By Lemma 1.3(ii), our Z and Σ satisfy the differential equation (1.7), and so the interval I_* in (5.2) must be bounded since, due to the positivity claim at the very end of Lemma 5.1, the two sides of (1.7), divided by K^4, have different limits as $|K|\to\infty$. (At the beginning of this section we assumed that $p>2$.) Thus, $I_*=(\varepsilon/(1-p),\tilde\varepsilon/(1-p))$ and $K<\tilde\varepsilon/(1-p)$, so that $K<0$, as Remark 2.2 yields $\tilde\varepsilon>0$. According to the lines preceding (1.1), our equalities $\Delta K=\Sigma(K)$ and $g(\nabla K,\nabla K)=2Z(K)$ imply the existence of a g-Killing field v without zeros, defined on a nonempty connected open set $U\subseteq M$, which is also an h-Killing field, for the metric $h=g/\tau^2$ in (3.1), since the normalization of (3.3) gives $\tau=|(p-1)K+\varepsilon|^{1/(p-1)}$, that is, $\tau=|\omega|^{1/(p-1)}$, and the local flow of v preserves the Gaussian curvature K. The same obviously applies to the metric $\tilde h=g/\tilde\tau^2$, where $\tilde\tau=|\tilde\omega|^{1/(p-1)}$. By (2.7) and Theorem 3.1, h and $\tilde h$ are real-analytic. At least one of them has nonconstant Gaussian curvature. Otherwise, their constant Gaussian curvatures would be negative (from the Gauss-Bonnet theorem – note that $K<0$) and, as h and $\tilde h$ are conformally related, Remark 1.5 would imply constancy of their conformal factor $\tilde\tau/\tau$, thus making K constant. Finally, $\chi(M)<0$ since $K<0$, so that by Lemma 1.1 a nontrivial Killing field exists on (M,h), or $(M,\tilde h)$, or on a two-fold isometric covering. This in turn contradicts Remark 1.6.

6. Theorem 3.1, rephrased

Let us rewrite Theorem 3.1 in terms of the positive function $f = \tau^{-p/2}$, the parameter $\gamma \in \mathrm{I\!R}$ characterized by $(p-1)K + \varepsilon = \gamma\tau^{p-1}$, and the following triple of real constants:

$$(a, c, r) = (p(p-2)\varepsilon/[4(p-1)],\, p\mu/[4(p+1)],\, p\gamma/[2(p^2-1)]). \tag{6.1}$$

THEOREM 6.1. *Given a nonconstant function $f : M \to (0, \infty)$ on a Riemannian surface (M, g), and an Einstein manifold (Π, η) of dimension $p \geq 2$, the metric $f^{4/p}[g + \eta]$ on $M \times \Pi$ has harmonic curvature if and only if, for the Gaussian curvature K of g, some $a, c, r \in \mathrm{I\!R}$, and the Einstein constant ε of η,*

$$\begin{array}{ll} \text{i)} & K = 2r(1 + 1/p)f^{-2(1-1/p)} - \varepsilon/(p-1), \quad \text{ii)} \;\; p(p-2)\varepsilon = 4(p-1)a, \\ \text{iii)} & \Delta f - af = -cf^{1+4/p} + rf^{-1+2/p}. \end{array} \tag{6.2}$$

The constant scalar curvature of $f^{4/p}[g + \eta]$ then equals $4(1 + 1/p)c$. Also,

$$\text{cases (a) and (b) in Theorem 3.1 correspond to } r = 0 \text{ and } r \neq 0. \tag{6.3}$$

Here (6.3) is obvious since $2(p^2 - 1)r = p\gamma$, cf. (6.1), and $p \geq 2$.

When M is compact, and $f : M \to (0, \infty)$ nonconstant, (6.2) implies that

$$\begin{array}{rl} \text{i)} & c > 0, \quad \text{ii)} \;\; p-2 \text{ and } a \text{ are both zero, or both positive,} \\ \text{iii)} & \text{if } a = 0, \text{ then } K \text{ is nonconstant, } r > 0, \text{ and } p = 2, \\ \text{iv)} & \text{whenever } r < 0, \text{ one has } p > 2 \text{ and } K < 0 \text{ everywhere.} \end{array} \tag{6.4}$$

In fact, (6.4.ii) follows from (6.2.ii), as $\varepsilon > 0$ (see Remark 2.2) and $p \geq 2$. Next, one of c and r is positive: by (6.4.ii), $a \geq 0$, so that if we had $r \leq 0$ and $c \leq 0$, (6.2.iii) would make Δf the sum of three constant or increasing functions of the variable $f > 0$ resulting, via Remark 1.8, in constancy of f. Nonpositivity of c would thus lead to positivity of r, with (6.2.iii) expressing Δf as the sum of three nonnegative terms and, again, contradicting nonconstancy of f. This yields (6.4.i). If $a = 0$, we get $r > 0$ (or else Δf would, by (6.2.iii) and (6.4.i), be negative), so that (6.2.i) and (6.4.ii) yield (6.4.iii). To prove (6.4.iv), let $r < 0$. Hence, by (6.2.i), $K < 0$, as $\varepsilon > 0$ (Remark 2.2), while $p > 2$, or else (6.4.ii) with $p = 2$ and (6.4.iii) would give $r > 0$.

REMARK 6.2. Positivity of ε (or, μ) in the compact case follows from Remark 2.2 or, respectively, (6.4.i) and (6.1).

7. Vanishing differentials and Hessians

For a manifold $\mathcal{W}$, an interval $I \subseteq \mathrm{I\!R}$, a C^2 curve $I \ni t \mapsto y(t) \in \mathcal{W}$, and a parameter $c \in I$ such that $\dot{y}(c) = 0$, the *acceleration* vector $w = \ddot{y}(c) \in T_{y(c)}\mathcal{W}$ with the components $w^a = \ddot{y}^a(c)$ in any local coordinates at $y(c)$ is clearly well defined, a coordinate-free description being: $d_w\phi = d^2[\phi(y(t))]/dt^2$, evaluated at $t = c$, whenever ϕ is a C^2 function on a neighborhood of $y(c)$ in $\mathcal{W}$. Thus,

$$\ddot{y}(c) \text{ equals the ordinary second derivative of } y(t) \text{ at } t = c \tag{7.1}$$

if $\mathcal{W}$ happens to be a C^2 submanifold of a Banach space $\hat{\mathcal{V}}$, making $I \ni t \mapsto y(t)$ a curve in $\hat{\mathcal{V}}$. This is immediate if one diffeomorphically identifies a neighborhood $\hat{\mathcal{U}}$ of $y(c)$ in $\hat{\mathcal{V}}$ with $U \times \hat{\mathcal{U}}'$, for open subsets $\hat{\mathcal{U}}'$ of some Banach space and U of $\mathrm{I\!R}^n$, where $n = \dim \mathcal{W}$ and $0 \in \hat{\mathcal{U}}'$, so as to make $\hat{\mathcal{W}} \cap \hat{\mathcal{U}}$ correspond to $U \times \{0\}$, and then treats the projection $U\times\hat{\mathcal{U}}'\to U$, restricted to $\hat{\mathcal{W}}\cap\hat{\mathcal{U}}=U\times\{0\}$, as a local coordinate system for $\hat{\mathcal{W}}$.

Given a C^2 mapping $F:\mathcal{N}\to\mathcal{W}$ between manifolds and a point $z\in\mathcal{N}$ such that $dF_z=0$, one defines the *Hessian* of F at z to be the symmetric bilinear mapping $H:T_z\mathcal{N}\times T_z\mathcal{N}\to T_{F(z)}\mathcal{W}$ characterized by the component formula $[H(u,v)]^a=H^a_{jk}u^jv^k$ with $H^a_{jk}=(\partial_j\partial_kF^a)(z)$, whenever $u,v\in T_z\mathcal{N}$ and x^j (or, y^a) are local coordinates in $\mathcal{N}$ at z, or in $\mathcal{W}$ at $F(z)$. An obviously equivalent definition of $H(u,v)$, where symmetry allows us to set $u=v$, reads

$$\begin{aligned}&H(v,v)=\ddot{y}(c)\ \text{ for the curve }\ y(t)=F(x(t))\ \text{ if }\ t\mapsto x(t)\\&\text{is a } C^2 \text{ curve in } \mathcal{N} \text{ such that }\ x(c)=z\ \text{ and }\ \dot{x}(c)=v.\end{aligned}\tag{7.2}$$

The acceleration $w=\ddot{y}(c)$ in the lines preceding (7.1) involves a special case of the Hessian; specifically, $w=H(u,u)$ at $z=c$ in $\mathcal{N}=I$, the mapping F and u being the curve and, respectively, 1 treated as a vector tangent to I.

8. Fredholm differentials and bifurcations

Suppose that we are given real Banach spaces $\mathcal{V},\hat{\mathcal{V}}$ and a C^k mapping L, $1\le k\le\infty$, from a neighborhood of 0 in $\mathcal{V}$ into $\hat{\mathcal{V}}$, such that $L(0)=0$ and the differential of L at 0 is a Fredholm operator $\varPhi=dL_0:\mathcal{V}\to\hat{\mathcal{V}}$. Thus, $\operatorname{Ker}\varPhi$ and $\hat{\mathcal{V}}/\varPhi(\mathcal{V})$ are finite-dimensional, from which closedness of the image $\varPhi(\mathcal{V})$ in $\mathcal{V}$ follows [**1**, p. 156]. We fix closed subspaces $\mathcal{Y}\subseteq\mathcal{V}$ and $\mathcal{W}\subseteq\hat{\mathcal{V}}$ with $\mathcal{V}=\mathcal{Y}\oplus\operatorname{Ker}\varPhi$ and $\hat{\mathcal{V}}=\mathcal{W}\oplus\varPhi(\mathcal{V})$, so that $\dim\mathcal{W}<\infty$. Due to Banach's open mapping theorem,

$$\varPhi=dL_0 \text{ restricted to } \mathcal{Y} \text{ is a linear homeomorphism } \mathcal{Y}\to\varPhi(\mathcal{V}).\tag{8.1}$$

The problem of understanding the preimage $L^{-1}(0)$, clearly contained in $L^{-1}(\mathcal{W})$, has a *local finite-dimensional reduction.*

Lemma 8.1. *Under the above assumptions, the intersection $\mathcal{N}$ of $L^{-1}(\mathcal{W})$ and a suitable neighborhood of 0 in $\mathcal{V}$ forms a C^k manifold of the finite dimension* $\dim\operatorname{Ker}\varPhi$*, having at 0 the tangent space $T_0\mathcal{N}=\operatorname{Ker}\varPhi$, while L restricted to $\mathcal{N}$ constitutes a C^k mapping $F:\mathcal{N}\to\mathcal{W}$ with $F(0)=0$ such that $dF_0=0$ and, if $k\ge2$, the Hessian H of F at 0 is given by $H(v,v')=\pi(d[dL]_0v)v'$ for any $v,v'\in T_0\mathcal{N}=\operatorname{Ker}\varPhi$ and the projection $\pi:\hat{\mathcal{V}}\to\mathcal{W}$ having the kernel $\varPhi(\mathcal{V})$.*

Proof. Let $\mathrm{pr}=\mathrm{Id}-\pi$ be the projection $\hat{\mathcal{V}}\to\varPhi(\mathcal{V})$ with the kernel $\mathcal{W}$. Setting $S(y,z)=(\mathrm{pr}\,L(y+z),z)$, we obtain a C^k mapping S from a neighborhood of 0 in $\mathcal{Y}\times\operatorname{Ker}\varPhi$ into $\varPhi(\mathcal{V})\times\operatorname{Ker}\varPhi$. The assignment $(\dot{y},\dot{z})\mapsto(\varPhi\dot{y},\dot{z})$ represents the differential of S at $(0,0)$ which – due to (8.1) – is a linear homeomorphism. Our claim about $F:\mathcal{N}\to\mathcal{W}$ now follows from the inverse mapping theorem: $\mathcal{N}$ corresponds via S to a neighborhood of $(0,0)$ in $\{0\}\times\operatorname{Ker}\varPhi$, while $dS^{-1}_{(0,0)}(0,\dot{z})=(0,\dot{z})$, and $dF_0=0$ since $T_0\mathcal{N}=\operatorname{Ker}\varPhi=\operatorname{Ker}dL_0$. To evaluate H, we choose a curve $t\mapsto y(t)+z(t)\in\mathcal{N}\subseteq L^{-1}(\mathcal{W})$ with $y(t)\in\mathcal{Y}$ and $z(t)\in\operatorname{Ker}\varPhi$, having at $t=0$ the value 0 and velocity $v\in T_0\mathcal{N}=\operatorname{Ker}\varPhi$. Thus, $y(0)=z(0)=\dot{y}(0)=0$ and $\dot{z}(0)=v$, as well as $L(y+z)=\pi L(y+z)$ for all t, due to $\mathcal{W}$-valuedness of L, where – from now on – we write $y,z,\dot{y},\dot{z}$ rather than $y(t)$, etc. Applying d/dt twice to the last equality, one gets $dL_{y+z}(\dot{y}+\dot{z})=\pi\,dL_{y+z}(\dot{y}+\dot{z})$ at any t, and $d[dL_{y+z}(\dot{y}+\dot{z})]/dt=\pi(d[dL]_0v)v$ at $t=0$, since $d[dL_{y+z}]/dt=d[dL]_{y+z}(\dot{y}+\dot{z})$ (with $y=z=\dot{y}=0$ and $\dot{z}=v$ when $t=0$), while $\pi\,dL_0(\ddot{y}+\ddot{z})=\pi\varPhi(\ddot{y}+\ddot{z})=$

0 (note that $\Phi(\mathcal{V}) = \operatorname{Ker}\pi$). By (7.2) and (7.1), $H(v,v) = \pi(d[dL]_0 v)v$, and symmetry of H implies the required formula for $H(v,v')$. $\square$

9. The saddle-point case

A simple special case of Lemma 8.1 arises when

(i) $\Phi(\mathcal{V})$ has the codimension 1 in $\hat{\mathcal{V}}$ and, consequently, $\dim\mathcal{W} = 1$,
(ii) $k \geq 2$ and $\dim\operatorname{Ker}\Phi = 2$, so that $\mathcal{N}$ is a surface,
(iii) there is an embedded C^1 curve $\mathcal{C} \subseteq \mathcal{N}$ with $0 \in \mathcal{C} \subseteq L^{-1}(0)$,
(iv) we identify $\mathcal{W}$ with $\mathrm{I\!R}$, which turns $0 \in \mathcal{N}$ into a critical point of the C^2 function $F : \mathcal{N} \to \mathrm{I\!R}$ on the surface $\mathcal{N}$, having $F(0) = 0$,
(v) $H(v,v') \neq 0$ for the Hessian H of F at 0, some vector v tangent to the curve $\mathcal{C}$ at 0, and some $v' \in T_0\mathcal{N} = \operatorname{Ker}\Phi$.

Then H is indefinite. Namely, $H \neq 0$, while $H(v,v) = 0$ since $L = 0$ along $\mathcal{C}$, and so H cannot be definite (or semidefinite), or else we would have $v = 0$ (or $H(v,v') = 0$). As a result, F has a saddle point at 0, and

(9.1) the intersection of $L^{-1}(0)$ with a neighborhood of 0 in $\mathcal{N}$ is the union of two embedded curves intersecting transversally at $0 \in \mathcal{N}$ and having no other points in common; one of these curves is contained in our $\mathcal{C}$, the other has the tangent line $\mathrm{I\!R}w$ at 0, for some $w \in \operatorname{Ker}\Phi \smallsetminus \mathrm{I\!R}v$ with $H(w,w) = 0$.

We will refer to these two curves, respectively, as

(9.2) the curve of trivial solutions (contained in $\mathcal{C}$), and the bifurcating branch.

Next, given real Banach spaces $\mathcal{V}, \hat{\mathcal{V}}$ and a mapping L of class C^k, $2 \leq k \leq \infty$, from a neighborhood of 0 in $\mathcal{V}$ into $\hat{\mathcal{V}}$ with $L(0) = 0$, suppose that

(a) $\mathcal{V} = \mathcal{V}' \times \mathrm{I\!R}$, for a Banach space $\mathcal{V}'$,
(b) $L^t(0) = 0$ for all t near 0 in $\mathrm{I\!R}$, where we set $L^t(x) = L(x,t)$,
(c) dL^0_0 (the differential of L^0 at $0 \in \mathcal{V}'$) is a Fredholm operator,
(d) $\dim\operatorname{Ker} dL^0_0 = \dim[\hat{\mathcal{V}}/dL^0_0(\mathcal{V}')] = 1$,
(e) $dL^0_0(\mathcal{V}') \cap d\dot{L}^0_0(\operatorname{Ker} dL^0_0) = \{0\} \neq d\dot{L}^0_0(\operatorname{Ker} dL^0_0)$, with $\dot{L}^t = dL^t/dt$.

Lemma 9.1. *Under the assumptions* (a) – (e), *the hypotheses of Lemma* 8.1 *along with conditions* (i) – (v) *above are all satisfied, and hence so are their conclusions, including* (9.1), *while* $T_{(0,0)}\mathcal{N} = \operatorname{Ker} dL^0_0 \times \mathrm{I\!R}$. *For the Hessian H of F at* $(0,0)$ *and any vectors* $v, v' \in \operatorname{Ker} dL^0_0 \times \mathrm{I\!R}$ *of the form* $v = (0,1)$ *and* $v' = (u,0)$, *where* $u \in \operatorname{Ker} dL^0_0$, *one has* $H(v,v') = d\dot{L}^0_0 u$. *The curve $\mathcal{C}$ of condition* (iii) *is a neighborhood of* $(0,0)$ *in* $\{0\} \times \mathrm{I\!R}$.

Proof. The hypotheses of Lemma 8.1 easily follow from (a) – (e), and so do (i) – (iv): the Fredholm property of $\Phi = dL_{(0,0)}$, with the dimensions required in (i) – (ii), is obvious since Φ has the kernel $\operatorname{Ker} dL^0_0 \times \mathrm{I\!R}$ and the image $dL^0_0(\mathcal{V}')$. Finally, for $v, v' \in T_{(0,0)}\mathcal{N} = \operatorname{Ker}\Phi = \operatorname{Ker} dL^0_0 \times \mathrm{I\!R}$ as in the statement of the lemma, with a nonzero vector $u \in \operatorname{Ker} dL^0_0$, the formula $H(v,v') = \pi(d[dL]_0 v)v'$ of Lemma 8.1 reads $H(v,v') = \pi d\dot{L}^0_0 u$ while, by (e), $d\dot{L}^0_0 u \notin dL^0_0(\mathcal{V}')$. The relation $dL^0_0(\mathcal{V}') = \Phi(\mathcal{V}) = \operatorname{Ker}\pi$ now yields $H(v,v') \neq 0$, proving (v). $\square$

10. Nonconstant Gaussian curvature: part one

Compact warped products $(M\times\varPi,\bar g)$ with harmonic curvature, nonconstant warping functions, and two-dimensional bases (M,g) represent two separate cases, (a) and (b) in Theorem 3.1. Case (b), discussed here, amounts, by Theorem 6.1, to having $\bar g = f^{4/p}[g+\eta]$ for an Einstein metric η with some Einstein constant ε and a nonconstant function $f : M \to (0,\infty)$ satisfying (6.2), that is,

$$\text{(10.1)}\qquad \text{a)}\;\; \Delta f = \varOmega(f), \qquad \text{b)}\;\; K = 2r(1+1/p)f^{-2(1-1/p)} - \varepsilon/(p-1),$$

K and $\varOmega$ being the Gaussian curvature of g and the function on $(0,\infty)$ given by

$$\text{(10.2)}\qquad \varOmega(f) = af - cf^{1+4/p} + rf^{-1+2/p} \quad \text{with} \quad a = p(p-2)\varepsilon/[4(p-1)].$$

Here $p\ge 2$ is the dimension of the fibre, $c,r\in\mathrm{I\!R}$, and $r\ne 0$, cf. (6.3), while K must be nonconstant due to nonconstancy of f and (10.1.b).

In the next section we will use the bifurcation method of Lemma 9.1 to prove the existence of Riemannian metrics g on compact surfaces M admitting nonconstant functions $f : M\to(0,\infty)$ with (10.1) – (10.2). Such g will arise from conformal changes of the form $g = e^{2x}\hat g$, where the metric $\hat g$ on M has constant Gaussian curvature $\hat K$, and $x : M\to\mathrm{I\!R}$. However, rather than being smooth, x is only required to lie in a suitable L^2 Sobolev space, chosen so as to ensure C^4-differentiability of x.

Our approach uses a fixed choice of the data $M,\hat g,\hat K,p,i,r,\lambda$ consisting of a compact Riemannian surface $(M,\hat g)$ of constant Gaussian curvature $\hat K\ne 0$, integers $p\ge 2$ and $i\ge 6$, a real parameter $r\ne 0$, and a suitable eigenvalue λ of $-\hat\Delta$, for the $\hat g$-Laplacian $\hat\Delta$. The Gauss-Bonnet theorem and (6.4.iv) make it necessary to assume that

$$\text{(10.3)}\qquad \text{if}\;\; r<0, \quad \text{then}\;\; p>2 \;\;\text{and}\;\; \hat K<0.$$

By a *solution of* (10.1) we then mean a quadruple (x,f,ε,c) formed by a C^4 function $x : M\to\mathrm{I\!R}$, a C^2 function $f : M\to(0,\infty)$, and constants $\varepsilon,c\in\mathrm{I\!R}$ such that (10.1), with (10.2), holds for the Gaussian curvature K of the C^4 metric $g = e^{2x}\hat g$ on M and the g-Laplacian Δ (the objects $M,\hat g,\hat K,p,r$ still being fixed).

In contrast with the lines surrounding (10.1) – (10.2), f and K are this time allowed to be constant: in fact, there exist *trivial solutions* of (10.1), namely, (x,f,ε,c) having $x=0$, a constant $f>0$, and $\varepsilon,c\in\mathrm{I\!R}$ chosen so as to yield (10.1) – (10.2) with $K=\hat K$ and $\varOmega(f)=0$, that is,

$$\text{(10.4)}\qquad \begin{array}{l} \varepsilon = (p-1)[2r(1+1/p)f^{-2(1-1/p)} - \hat K], \quad \text{and} \\ c = af^{-4/p} + rf^{-2-2/p} \;\; \text{for} \;\; a = p(p-2)\varepsilon/[4(p-1)]. \end{array}$$

This curve of trivial solutions is parametrized by $f\in(0,\infty)$, and some of them can be deformed to bifurcating branches of solutions with nonconstant f and K. There are obstructions to such a deformation, in the form of *three positivity conditions* imposed on the constant $f>0$. The first two reflect the fact that nonconstancy of f gives $\varepsilon,c\in(0,\infty)$, cf. Remark 2.2 and (6.4.i), while – in trivial solutions – ε,c depend on f via (10.4). The third condition arises since a bifurcation can only occur at f if the value of f is quite specifically related to a nonzero (and hence *positive*) eigenvalue λ of $-\hat\Delta$, for the $\hat g$-Laplacian $\hat\Delta$. See formula (10.6.i) below.

It is convenient to replace the parameter $f\in(0,\infty)$ mentioned above with the positive real variable $\theta = f^{-2(1-1/p)}$. For the trivial solution (x,f,ε,c) of (10.1) corresponding to θ one then has $x = 0$ and $f = \theta^{p/(2-2p)}$, whereas (10.4) reads

$$\varepsilon = 2(p-1/p)r\theta - (p-1)\hat{K}, \qquad 4c/p = [2(p-1)r\theta - (p-2)\hat{K}]\theta^{2/(p-1)}. \tag{10.5}$$

In terms of θ, the relation between f and the eigenvalue λ of $-\hat{\Delta}$ takes the form

$$\text{i)}\ \ \lambda = 2(p-1/p)r\theta - (p-2)\hat{K}, \quad \text{that is,} \quad \text{ii)}\ \ \lambda = \varepsilon + \hat{K}, \tag{10.6}$$

justified later by (11.6) and Lemma 11.2. If $\theta \in (0,\infty)$, simultaneous positivity of the three constants ε, c, λ in (10.5) – (10.6.i) clearly amounts to

$$2(p^2-1)r\theta > \max\{p(p-1)\hat{K}, (p+1)(p-2)\hat{K}, p(p-2)\hat{K}\}. \tag{10.7}$$

With $M, \hat{g}, \hat{K}, p, r$ still fixed, let $I_r \subseteq (0,\infty)$ be the open interval defined by

$$I_r = (\theta_+, \infty), \ \text{when} \ r > 0, \ \text{or} \ I_r = (0, \theta_-), \ \text{for} \ r < 0, \tag{10.8}$$

where $\theta_+ = \max\{p\hat{K}/[2(p+1)r], 0\}$ and $\theta_- = p(p-2)\hat{K}/[2(p^2-1)r]$. Note that $\theta_- > 0$ if $r < 0$, due to (10.3), while

$$p(p-2) \le (p+1)(p-2) < p(p-1) \quad \text{whenever} \ p \ge 2. \tag{10.9}$$

Our three positivity conditions mean precisely that $\theta \in I_r$. Namely, we have

Lemma 10.1. *The interval I_r is the set of all $\theta \in (0,\infty)$ for which the three expressions ε, c and λ, given by* (10.5) – (10.6.i)*, are simultaneously positive.*

Proof. Depending on whether $r > 0$ and $\hat{K} > 0$ (or, $r > 0$ and $\hat{K} < 0$ or, respectively, $r < 0$, so that (10.3) gives $\hat{K} < 0$), condition (10.7) imposed on $\theta \in (0,\infty)$ reads, by (10.9), $\theta > p\hat{K}/[2(p+1)r]$, or $\theta > 0$ or, respectively, $\theta < p(p-2)\hat{K}/[2(p^2-1)r]$, as required. □

Remark 10.2. Given a compact Riemannian manifold (M, g) of any dimension m and an open interval $I \subseteq \mathrm{I\!R}$, the Sobolev embedding theorem implies that, if $i > m$, the Sobolev space $L^2_i(M, \mathrm{I\!R})$ of functions with i derivatives in L^2 can be turned into a Banach algebra, while the I-valued functions in $L^2_i(M, \mathrm{I\!R})$ form an open subset $L^2_i(M, I)$ of $L^2_i(M, \mathrm{I\!R})$. On the other hand, for any Banach algebra $\mathcal{A}$, convergent power series define $\mathcal{A}$-valued C^∞ functions on open subsets of $\mathcal{A}$. Applied to $\mathcal{A} = L^2_i(M, \mathrm{I\!R})$, this yields $\mathcal{A}$-valuedness and C^∞-differentiability of the mapping $L^2_i(M, I) \ni x \mapsto \varphi \circ x$, whenever the function $\varphi : I \to \mathrm{I\!R}$ is real-analytic.

11. Nonconstant Gaussian curvature: part two

We now proceed to construct metrics on closed surfaces realizing case (b) in Theorem 3.1. Curves of such metrics g, emanating from a constant-curvature metric $\hat{g}$, will arise via the bifurcation argument of Lemma 9.1. As outlined in Section 10, the construction uses a fixed septuple $M, \hat{g}, \hat{K}, p, i, r, \lambda$ formed by

(i) a closed Riemannian surface $(M, \hat{g})$ of nonzero constant Gaussian curvature $\hat{K}$, along with integers $p \ge 2$ and $i \ge 6$,

(ii) a real parameter $r \ne 0$, satisfying (10.3): $r > 0$ unless $\hat{K} < 0$ and $p > 2$,

(iii) a constant $\lambda \in (0,\infty)$ such that, for the $\hat{g}$-Laplacian $\hat{\Delta}$,
  - (a) $\lambda = 2l(2l+1)\hat{K}$, where l is a positive integer, if $\hat{K} > 0$,
  - (b) $[\lambda + (p-2)\hat{K}]r > 0$ and $\dim \mathrm{Ker}\,(\hat{\Delta} + \lambda) = 1$, when $\hat{K} < 0$.

In both cases (iii.a) – (iii.b), λ is a positive eigenvalue of $-\hat{\Delta}$ (see Section 12).

For $I_r \subseteq (0,\infty)$ as in (10.8), let $\delta \in \mathrm{I\!R}$ and $I \subseteq \mathrm{I\!R}$ be given by

$$\delta \,=\, p[\lambda + (p-2)\hat{K}]/[2(p^2-1)r], \qquad I \,=\, \{t \in \mathrm{I\!R} : \delta + t \in I_r\}. \tag{11.1}$$

Lemma 11.1. *Under the assumptions* (i) – (iii), $\delta \in I_r$ *and* I *is an open interval containing* 0.

Proof. The condition $\delta \in I_r$ reads $r\delta > r\theta_+ = \max\{p\hat{K}/[2(p+1)], 0\}$ when $r > 0$, and $0 > r\delta > r\theta_- = p(p-2)\hat{K}/[2(p^2-1)]$ if $r < 0$. Thus, $\delta \in I_r$ by (iii), since $p \geq 2$, and (11.1) yields $2(p^2-1)r\delta/p = \lambda + (p-2)\hat{K}$. □

For our fixed septuple $M, \hat{g}, \hat{K}, p, i, r, \lambda$, any given $t \in I$, a function $x : M \to \mathrm{I\!R}$ having some further properties, named in the paragraph following (11.4), and δ as in (11.1), we let $\varepsilon, c, K, \Delta, \varOmega$ and f denote the constants in (10.5) with $\theta = \delta + t$, the Gaussian curvature of the metric $g = e^{2x}\hat{g}$, the g-Laplacian, the function (10.2), and f characterized by (10.1.b), that is, by $K = 2r(1+1/p)f^{-2(1-1/p)} - \varepsilon/(p-1)$. Using $K, \Delta, \varOmega$ and f depending on t, x as described here, we define $L^t(x) = L(x,t)$ to be $\Delta f - \varOmega(f)$. Explicitly,

$$\begin{aligned}
&L^t(x) \,=\, \Delta f \,-\, af \,+\, cf^{1+4/p} \,-\, rf^{-1+2/p}, \quad \text{where}\\
&a = p(p-2)\varepsilon/[4(p-1)] \ \text{ for } \ \varepsilon = 2(p-1/p)r(\delta+t) - (p-1)\hat{K},\\
&c \,=\, p[2(p-1)r(\delta+t) - (p-2)\hat{K}](\delta+t)^{2/(p-1)}/4,\\
&f \,=\, [2r(1+1/p)]^{-p/(2-2p)}[K \,+\, \varepsilon/(p-1)]^{p/(2-2p)}, \quad \text{and}\\
&g \,=\, e^{2x}\hat{g}\,, \quad \Delta \,=\, e^{-2x}\hat{\Delta}, \quad K \,=\, e^{-2x}(\hat{K} - \hat{\Delta}x), \quad \text{cf. Remark 1.4.}
\end{aligned} \tag{11.2}$$

Since (11.2) easily shows that, whenever $t \in I$,

$$K \text{ and } f \text{ have, at } x = 0 \text{ and } t, \text{ the values } \hat{K} \text{ and } (\delta+t)^{p/(2-2p)}, \tag{11.3}$$

relations (11.2) easily yield

$$L^t(0) \,=\, 0 \quad \text{for all} \quad t \in I. \tag{11.4}$$

As for x, we require that it be close to 0 in a subspace $\mathcal{V}'$ – described in the lines preceding (11.7) – of the Sobolev space $L^2_i(M, \mathrm{I\!R})$, with $i \geq 6$ derivatives in L^2. The Sobolev embedding theorem then guarantees C^{i-2} differentiability of x, while its closeness to 0 is meant to ensure positivity of f via that of $\delta + t \in I_r$ in (11.3), the latter due to the definition of I, cf. (11.1), and the inclusion $I_r \subseteq (0,\infty)$.

Our data $\hat{K}, p, r, \delta$ are constants, while ε and c depend only on t (not on x), K only on x, and f on both x, t. Therefore, by (11.2) and (11.3), for the differentials of $\hat{K}, p, r, \delta, \varepsilon, c, K$ and f with respect to the variable $x \in \mathcal{V}'$, at $x = 0$ and any $t \in I$, one has

$$\begin{aligned}
&dp_0 \,=\, dr_0 \,=\, d\hat{K}_0 \,=\, d\delta_0 \,=\, d\varepsilon_0 \,=\, dc_0 \,=\, 0,\\
&dK_0 = -4r(1-1/p^2)(\delta+t)^{(2-3p)/(2-2p)}df_0 = -(\hat{\Delta} + 2\hat{K}),
\end{aligned} \tag{11.5}$$

$2\hat{K}$ denoting here $2\hat{K}$ times the identity. From (11.1),

$$\lambda \,=\, 2(p-1/p)r\delta \,-\, (p-2)\hat{K} \,\in\, (0,\infty), \tag{11.6}$$

which is also the value of λ in Lemma 10.1 for $\theta = \delta$.

Lemma 11.2. *With notations of Section 9,* $4r(1-1/p^2)(\delta+t)^{1-p/(2-2p)}dL^t_0 = [\hat{\Delta} + \lambda + 2(p-1/p)rt](\hat{\Delta} + 2\hat{K})$, *as well as* $8r(p-1)(1-1/p^2)\delta^{2-p/(2-2p)}d\dot{L}^0_0 = [(2-3p)\hat{\Delta} - p\lambda + 2(p-1)(p-2)\hat{K}](\hat{\Delta} + 2\hat{K})$, *at any* $t \in I$, *or* $t = 0$, *and* $x = 0$.

Proof. By (11.2), $d[\Delta f]_0 = \hat{\Delta} df_0$ and $d[\Omega(f)]_0 = \Omega'(f)\, df_0$, with $\Omega'(f)$ denoting the derivative $d\Omega/df$ at $f = (\delta + t)^{p/(2-2p)}$ (the value of f for $x = 0$, which is a constant function on M, depending on t). From (10.2) and (11.2), $\Omega'(f) = a - (1+4/p)cf^{4/p} - (1-2/p)rf^{-2+2/p} = -[2(p-1/p)r(\delta+t) - (p-2)\hat{K}]$ which, by (11.6), equals $-2(p-1/p)rt - \lambda$. This yields $dL_0^t = d[\Delta f]_0 - d[\Omega(f)]_0 = [\hat{\Delta} + \lambda + 2(p-1/p)rt]\, df_0$, cf. (11.2), and the last line of (11.5) implies the first equality; applying d/dt to it and using (11.6), we obtain the second one. □

To use Lemma 9.1, we fix $M, \hat{g}, \hat{K}, p, i, r, \lambda, \delta$ as in (i) – (iii) and (11.1), along with specific vector subspaces $\mathcal{V}'$ of $L_i^2(M, \mathrm{I\!R})$ and $\hat{\mathcal{V}}$ of $L_{i-4}^2(M, \mathrm{I\!R})$ such that

(iv) $\hat{\mathcal{V}}$ contains $\hat{\mathcal{V}}' = (\hat{\Delta} + \lambda)(\mathcal{V}')$ and all $L^t(\mathcal{V}')$, $t \in I$, while $\dim[\hat{\mathcal{V}}/\hat{\mathcal{V}}'] = 1$.

Here is how we select $\mathcal{V}'$ and $\hat{\mathcal{V}}$. For $\hat{K} < 0$, we set $\mathcal{V}' = L_i^2(M, \mathrm{I\!R})$ and $\hat{\mathcal{V}} = L_{i-4}^2(M, \mathrm{I\!R})$. If $\hat{K} > 0$, we fix a nontrivial $\hat{g}$-isometric action of the circle group S^1 on $M = \mathrm{I\!RP}^2$ (or, $M = S^2$) and let the subspaces $\mathcal{V}', \hat{\mathcal{V}}$ of $L_i^2(M, \mathrm{I\!R})$ and $L_{i-4}^2(M, \mathrm{I\!R})$ consist of all S^1-invariant functions required, in the case of $M = S^2$, to be also invariant under the antipodal isometry. In both cases one has (iv), since

$$\dim[\mathcal{V}' \cap \mathrm{Ker}\,(\hat{\Delta} + \lambda)] = 1, \tag{11.7}$$

due to (iii.b) or, respectively, (iii.a) combined with Remark 1.9.

Lemma 11.3. *Conditions* (a) – (e) *of Section* 9 *are all satisfied by* $\mathcal{V}', \hat{\mathcal{V}}$ *and* L *chosen as above, with* $\mathcal{V} = \mathcal{V}' \times \mathrm{I\!R}$ *and* $k = \infty$*, while* $\mathrm{Ker}\, dL_0^0 \subseteq \mathrm{Ker}\,(\hat{\Delta} + \lambda)$.

Proof. First, C^∞-differentiability of L and (a) – (b) are obvious from Remark 10.2 and, respectively, (11.4). (The former also holds for a more general reason: one can derive it from Nemytsky's theorem [**18**, Section 10.3.4], without invoking real-analyticity.) Next, we have (c) – (d). Namely, due to (iii-a) and Remark 1.9, $\hat{\Delta} + 2\hat{K}$ is injective on $\mathcal{V}'$. Thus, Lemma 11.2 for $t = 0$ gives $\mathrm{Ker}\, dL_0^0 = \mathcal{V}' \cap \mathrm{Ker}\,(\hat{\Delta} + \lambda)$ and $dL_0^0(\mathcal{V}') = (\hat{\Delta} + \lambda)(\mathcal{V}')$, while (11.7) and (iv) imply one-dimensionality of both spaces in (d). Finally, (e) follows since the restriction of the factor $(2-3p)\hat{\Delta} - p\lambda + 2(p-1)(p-2)\hat{K}$ in the last equality of Lemma 11.2 to $\mathrm{Ker}\,(\hat{\Delta} + \lambda)$ equals the identity times $2(p-1)[\lambda + (p-2)\hat{K}]$, which is nonzero as a consequence of (iii). □

Lemma 11.3 allows us to apply Lemma 9.1 to our $\mathcal{V}', \hat{\mathcal{V}}, L$ and $\mathcal{V} = \mathcal{V}' \times \mathrm{I\!R}$, arising from a fixed septuple $M, \hat{g}, \hat{K}, p, i, r, \lambda$, which yields (i) – (v) of Section 9, along with (9.1). We define the *λ-branch* corresponding to these data to be the set of all $g = e^{2x}\hat{g}$, where $(x, t) \in \mathcal{V}' \times \mathrm{I\!R}$ ranges over the bifurcating branch of solutions introduced in (9.2). The λ-branches, associated with all positive eigenvalues λ of $-\hat{\Delta}$ satisfying condition (iii), are curves of metrics on our closed surface M, emanating from the fixed metric $\hat{g}$ of nonzero constant Gaussian curvature $\hat{K}$.

Lemma 11.4. *Every metric* $g \neq \hat{g}$ *in any λ-branch, close to* $\hat{g}$*, realizes case* (b) *of Theorem* 3.1 *and, in particular, has nonconstant Gaussian curvature* K.

Proof. By (9.1) – (9.2) the bifurcating branch is a subset of $L^{-1}(0)$, so that (11.2) gives (6.2) whenever $g = e^{2x}\hat{g}$ for any (x, t) from the bifurcating branch, with a, ε, c, f as in (11.2). Theorem 6.1, the lines preceding it, and (6.3) will now yield case (b) of Theorem 3.1, once K (or, equivalently, f) is shown to be nonconstant, which we do in the next paragraph; we cannot simply invoke (6.3), with our fixed $r \neq 0$, since Theorem 6.1 *assumes* nonconstancy of f.

The curve of trivial solutions – see (9.2) — is contained in $\{0\} \times \mathrm{I\!R}$, due to the final clause of Lemma 9.1. It intersects the bifurcating branch transversally, at $(0,0) \in \mathcal{V} = \mathcal{V}' \times \mathrm{I\!R}$, while both curves lie on the surface $\mathcal{N} \subseteq \mathcal{V}$. (Cf. (ii), (a) in Section 9 and (9.1).) A nonzero vector (x,t) tangent to the bifurcating branch at $(0,0)$ thus has $x \neq 0$ (or else it would be also tangent to the trivial-solutions curve), and the image of (x,t) under the differential $d\varPsi_{(0,0)}$, at $(0,0)$, of the mapping $\varPsi$ sending $(x,t) \in \mathcal{N}$ to the Gaussian curvature K of the metric $g = e^{2x}\hat{g}$ is, from the last line of (11.5), equal to $-(\hat{\Delta}+2\hat{K})x$. in view of Lemmas 9.1 and 11.3, $(x,t) \in T_{(0,0)}\mathcal{N} = \operatorname{Ker} dL_0^0 \times \mathrm{I\!R}$ and $\operatorname{Ker} dL_0^0 \subseteq \operatorname{Ker}(\hat{\Delta} + \lambda)$, so that $d\varPsi_{(0,0)}(x,t) = -(\hat{\Delta} + 2\hat{K})x = (\lambda - 2\hat{K})x$ is nonzero as $\lambda \neq 2\hat{K}$ by (iii), and hence also nonconstant, being an eigenfunction of $-\hat{\Delta}$ for the positive eigenvalue λ. This, combined with constancy of $\hat{K} = \varPsi(0,0)$, implies nonconstancy of $\varPsi(x,t)$ for all $(x,t) \neq (0,0)$ in the bifurcating branch, sufficiently close to $(0,0)$. □

The harmonic-curvature property of the metric $f^{4/p}[g+\eta]$ in Theorem 6.1 is obviously unaffected when one multiplies g and η by the same positive constant, or separately rescales f. Our approach removes this freedom, by insisting that ε and c be defined as in (11.2): the metric $g = e^{2x}\hat{g}$, for any $(x,t) \in L^{-1}(0)$ near $(0,0)$, either equals $\hat{g}$, or has nonconstant Gaussian curvature, depending on whether (x,t) lies in the trivial-solutions curve, or in the bifurcating branch with $(0,0)$ removed. Thus, such metrics include no nontrivial constant multiples of $\hat{g}$.

12. Nonconstant Gaussian curvature: conclusion

Lemma 11.4 implies the second case of (0.2), that is, (b) in Theorem 3.1, for M diffeomorphic to S^2, $\mathrm{I\!RP}^2$ or a closed orientable surface of any genus $\mathbf{g} > 1$, and metrics on M forming nontrivial curves of homothety types which, in the case $\mathbf{g} > 1$, also represent a Teichmüller-open nonempty set of conformal structures.

These metrics give rise to nontrivial compact warped products with harmonic curvature, having fibres of all dimensions $p \geq 2$, and any such M as the base.

Recall that the λ-branches appearing in Lemma 11.4, for eigenvalues $\lambda > 0$ of $-\hat{\Delta}$ satisfying (iii) in Section 11, constitute curves of metrics on the closed surface M, emanating from the metric $\hat{g}$ of constant Gaussian curvature $\hat{K} \neq 0$, and every metric g near $\hat{g}$ in the λ-branch, except $\hat{g}$, realizes case (b) of Theorem 3.1. The metrics in any given λ-branch

(a) represent uncountably many distinct homothety types, and
(b) when sufficiently close to $\hat{g}$, they cannot be homothetic to any metric from a λ'-branch, close to $\hat{g}$, provided that $\lambda' \neq \lambda$.

First, (a) follows since the homothety invariant $[K_{\max} : K_{\min}]$ of Remark 3.3, restricted to any neighborhood of $\hat{g}$ in the λ-branch, is nonconstant (and, obviously, continuous): its constancy would make it equal to $[1\!:\!1]$ (the value of the invariant for $\hat{g}$), and the Gaussian curvatures of all the metrics near $\hat{g}$ in the λ-branch would thus be constant, contrary to the final clause of Lemma 11.4.

On the other hand, when a metric $g \neq \hat{g}$ in the λ-branch approaches $\hat{g}$, the area A of (M,g) tends to the area $\hat{\mathrm{A}}$ of $(M,\hat{g})$ (clearly equal to $2\pi/\hat{K}$ times the Euler characteristic $\chi(M)$) and, consequently, for the homothety invariant $\varepsilon\mathrm{A}$

mentioned in the lines following Theorem 3.2, (10.6.ii) implies that

$$\varepsilon\mathrm{A} \to 2\pi(-1+\lambda/\hat{K})\chi(M) \quad \text{as } g \to \hat{g} \text{ in the } \lambda\text{-branch.} \tag{12.1}$$

The limits (12.1) are obviously different for different λ, which proves (b).

According to Theorem 3.1, for every positive eigenvalue λ of $-\hat{\Delta}$ having the property (iii-a) or (iii-b) of Section 11, the metrics $g \neq \hat{g}$ in the λ-branch give rise to nontrivial compact warped products with harmonic curvature.

In case (iii.a), λ is a positive eigenvalue of $-\hat{\Delta}$, which may be completely arbitrary (if $M = \mathrm{I\!RP}^2$), or of the form λ_j with j even and positive (if $M = S^2$ and (1.9) represents the spectrum of $-\hat{\Delta}$). See Remark 1.9.

Condition (iii.b) amounts to requiring that λ be simple and either greater than $(p-2)|\hat{K}|$ (when $r > 0$) or less than $(p-2)|\hat{K}|$ (for $r < 0$), while $\hat{K} < 0$.

If $r < 0$ (so that (ii) in Section 11 gives $p > 2$), or $r > 0$ and $p = 2$, the existence of such eigenvalues λ is immediate from the result of Schoen, Wolpert and Yau [**19**] mentioned in Remark 1.10.

Finally, when $r > 0$ and $p > 2$, we can only provide some anecdotal evidence for an analogous existence assertion: on the Bolza surface, with the convention (1.9), λ_{24} is greater than $23|\hat{K}|$ and simple [**20**]; therefore, (iii.b) holds in this case for all $p \in \{3, 4, \ldots, 21\}$.

To simplify the phrasing of the last two paragraphs, let us unify the two cases of condition (iii.b) by ignoring the sign of r. Then, (iii.b) states that, on a closed orientable surface of genus $\mathbf{g} > 1$, with a metric of constant Gaussian curvature $\hat{K} < 0$, the eigenvalue $\lambda > 0$ of $-\hat{\Delta}$ is simple and different from $(p-2)|\hat{K}|$. The result of [**19**] guarantees that, for every genus $\mathbf{g} > 1$, metrics admitting such eigenvalues λ realize a nonempty open subset of the Teichmüller space.

13. Constant Gaussian curvature: existence

We now proceed to verify that the first case of (0.2) – or, equivalently, (a) in Theorem 3.1 – holds for a Teichmüller-open, nonempty set of metrics of constant negative curvatures, on closed orientable surfaces M of all genera $\mathbf{g} > 1$.

This results in nontrivial compact warped products with harmonic curvature, having fibres of all relevant dimensions $p \geq 4$, and all such M as the bases.

The existence assertion needed here is provided by the following result of Yamabe [**22**], cf. also [**2**, pp. 115–119], [**4**, Lemma 16.37], which remains valid even if $\dim M = m > 2$, as long as $(m-2)q < 2m$. The sign of the g-Laplacian Δ in [**4**] is the opposite of ours.

Lemma 13.1. *Given a compact Riemannian surface (M, g), real numbers $q > 2$ and $c > 0$, and $a \in \mathrm{I\!R}$ such that $(q-2)a > \lambda_1$ for the lowest positive eigenvalue λ_1 of $-\Delta$, the equation*

$$\Delta f - af = -cf^{q-1} \tag{13.1}$$

admits a nonconstant positive C^∞ solution $f : M \to \mathrm{I\!R}$.

By (6.3), Lemma 13.1 can be applied to case (a) in Theorem 3.1 for compact bases M. The resulting construction of compact warped products with harmonic curvature is a special case of one in [**8**] and [**4**, Example 16.35(v)].

Due to (6.3), equation (6.2.iii) then becomes (13.1) for $q = 2 + 4/p$ (so that $q > 2$), while (6.2.i) with $r = 0$ reads $\varepsilon = (1-p)K$. Since the Einstein constant ε

of the fibre is positive (Remark 2.2), (M, g) has in this case the negative constant Gaussian curvature K and, as $a = p(p-2)\varepsilon/[4(p-1)]$, the condition $(q-2)a > \lambda_1$ needed in Lemma 13.1 is equivalent to $p > 2-\lambda_1/K$, cf. (0.4). However, we are free to assume that $p \geq 4$. (If $p \in \{2, 3\}$, the warped-product metric is conformally flat according to Remark 3.5.) Obviously, having $p > 2 - \lambda_1/K$ for *all* $p \geq 4$ amounts to the inequality $2 > -\lambda_1/K$.

According to the second part of Remark 1.10, every closed orientable surface of genus greater than 1 admits metrics with negative constant Gaussian curvature K satisfying this last inequality, which implies the existence of examples mentioned in the italicized statement at the beginning of this section.

14. Constant Gaussian curvature: multiplicity

In equation (13.1) we can always assume that $c = a$, rewriting it as

$$\Delta f - af = -af^{q-1}, \tag{14.1}$$

since f may be replaced with $(a/c)^{q-2} f$. This normalization removes the freedom of simultaneously rescaling f and c, which is of no geometric interest.

There are various known multiplicity results for positive solutions of (14.1) on compact Riemannian manifolds (M, g) of all dimensions $m \geq 2$. Consider

(14.2) the number $\#(M, g, a, q)$ of distinct nonconstant positive smooth solutions f to (14.1) on (M, g),

so that $0 \leq \#(M, g, a, q) \leq \infty$. Typically, a lower bound on $\#(M, g, a, q)$ is given in terms of the topology of M.

One defines the *Lusternik–Schnirelmann category* $\mathrm{cat}(X)$ of a topological space X to be the least integer $k \geq 1$ such that X is the union of k open contractible subsets. If no such k exists, one sets $\mathrm{cat}(X) = \infty$. Let us also denote by $\mathrm{b}_i(X, \mathbb{K})$ the ith Betti number of X with coefficients in any given field $\mathbb{K}$, and by $\mathrm{b}(X, \mathbb{K})$ the sum $\sum_i \mathrm{b}_i(X, \mathbb{K}) \leq \infty$. Thus, $\mathrm{cat}(S^m) = 2$, while any closed surface M of genus $\mathbf{g} \geq 1$ has $\mathrm{cat}(M) = 3$ and $b(M, \mathbb{Z}_2) = 2(1 + \mathbf{g})$.

Finally, one calls a solution f of (14.1) *nondegenerate* if it is nondegenerate as a critical point of the associated energy functional or, equivalently, if the linearized equation $\Delta\psi - a\psi = a(1 - q)f^{q-2}\psi$ holds only for the trivial solution $\psi = 0$.

Theorem 14.1. *Given a compact Riemannian manifold (M, g) of dimension $m \geq 2$, any sufficiently large $a \in (0, \infty)$, and any $q \in (2, 2m/(m-2))$, with $2m/(m-2) = \infty$ if $m = 2$, one has $\#(M, g, a, q) > \mathrm{cat}(M)$, in the notation of* (14.2). *For any field $\mathbb{K}$ and any sufficiently large a such that all nonconstant smooth solutions of* (14.1) *are nondegenerate, $\#(M, g, a, q) > 2\mathrm{b}(M, \mathbb{K}) - 2$.*

Proof. This is the central result of [**3**], where it is stated (for reasons not clear to us) only for $m \geq 3$. However, the proof remains completely valid also in the case $m = 2$, due to the Sobolev embedding theorem. The same result, with exactly the same proof, also appears in [**17**, Theorem 1.2], with no restriction on $m \geq 2$. $\square$

In the case of hyperbolic surfaces M, Theorem 14.1 with $\mathrm{cat}(M) = 3$ yields

Corollary 14.2. *On any closed orientable surface of genus greater than* 1, *endowed with a metric of negative constant Gaussian curvature, equation* (14.1) *has at least four distinct nonconstant positive smooth solutions f.*

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