Left-invariant Einstein metrics on Lie groups

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LEFT-INVARIANT CONNECTIONS

The *Lie algebra* $\mathfrak{g}$ of a given Lie group G is, by definition, the space of left-invariant vector fields on G.

Left-invariant connections ∇ on G are the same as bilinear mappings $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, that is, elements of $[\mathfrak{g}^*]^{\otimes 2} \otimes \mathfrak{g}$.

Torsion-free left-invariant connections ∇ have the form

$$\nabla = \mathrm{D} + \mathrm{S},$$

where $\mathrm{S} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is symmetric, and D is the *standard* (bi-invariant) torsion-free connection, with

$$\mathrm{D}_v w = \frac{1}{2}[v, w].$$

THE KILLING FORM β

Let $\mathcal{T} = [\mathfrak{g}^*]^{\odot 2}$ be the space of real-valued symmetric bilinear forms on $\mathfrak{g}$, that is, of left-invariant symmetric 2-tensors on G.

A distinguished element of $\mathcal{T}$ is the (bi-invariant) *Killing form* β, defined by

$$\beta(v, w) \,=\, \mathrm{tr}\,(\mathrm{Ad}\, v)(\mathrm{Ad}\, w),$$

where $\mathrm{Ad}\, v = [v, \,\cdot\,]$.

Positive-definite elements of $\mathcal{T}$ coincide with left-invariant Riemannian metrics on G.

More generally, nondegenerate elements of $\mathcal{T}$ are the same as *left-invariant pseudo-Riemannian metrics* on G.

THE CURVATURE AND RICCI TENSORS

$R = R^\nabla$ is the *curvature tensor* of a connection ∇ on the given manifold, with the sign convention

$$R^\nabla(v,w)u \,=\, \nabla_w\nabla_v u \,-\, \nabla_v\nabla_w u \,+\, \nabla_{[v,w]}u$$

for vector fields u, v, w.

For the *Ricci tensor* ρ^∇ and vector u, v, w tangent at any point,

$$\rho^\nabla(v,w) \,=\, \mathrm{tr}\,\{u \mapsto R^\nabla(v,u)w\}.$$

The Jacobi identity gives

$$R^{\mathrm{D}}(v,w)u \,=\, \frac{1}{4}\,[[v,w],u], \qquad \rho^{\mathrm{D}} \,=\, -\,\frac{1}{4}\,\beta.$$

LOCAL CARTAN-SYMMETRY OF D

For any Lie algebra $\mathfrak{g}$, bi-invariance of β implies that β is D-parallel ($\mathrm{D}\beta = 0$):

$$\beta([u, v], w) + \beta(v, [u, w]) = 0.$$

More generally, R^{D} is always D-parallel, since

$$[\mathrm{Ad}\, u, \mathrm{Ad}\, [v, w]] = \mathrm{Ad}\, [u, v], w] + \mathrm{Ad}\, [v, [u, w]].$$

(The outer bracket on the left-hand side is the commutator; the equality is a consequence of the Jacobi identity and the equivalent fact that Ad is a Lie-algebra homomorphism.)

β AS A (LOCALLY SYMMETRIC) EINSTEIN METRIC

The *Levi-Civita connection* of a pseudo-Riemannian metric γ on a manifold is the unique torsion-free connection ∇ making the metric parallel ($\nabla\gamma = 0$).

An *Einstein metric* on a manifold: any pseudo-Riemannian metric γ with $\rho = \kappa\gamma$ for the Ricci tensor ρ of the Levi-Civita connection of γ and some $\kappa \in \mathbb{R}$ (the *Einstein constant* of γ).

If our Lie algebra $\mathfrak{g}$ is *semisimple* (that is, β is nondegenerate), then, consequently, D is the Levi-Civita connection of the pseudo-Riemannian metric β.

In other words, β *then is a locally symmetric Einstein metric*.

THE CONNECTION-TO-EINSTEIN-METRIC APPROACH

On any given manifold, there is a bijective correspondence between

- non-Ricci-flat pseudo-Riemannian Einstein metrics, modulo multiplication by nonzero constants,

and

- torsion-free connections ∇ having symmetric, nondegenerate, ∇-parallel Ricci tensors.

(In the case of Lie groups, one may add 'left-invariant' to both.)

THE RICCI-TENSOR FORMULA

Let $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a left-invariant torsion-free connection on G.

We denote by ∇w the linear operator $\mathfrak{g} \to \mathfrak{g}$ sending v to $\nabla_v w$.

Then

$$\rho^\nabla(v, w) = \delta^\nabla(\nabla_v w) - (\nabla v, \nabla w).$$

where $(a, b) = \mathrm{tr}\, ab$ and $\delta^\nabla \in \mathfrak{g}^*$ is given by

$$\delta^\nabla(w) = \mathrm{tr}\, \nabla w\,.$$

THE EINSTEIN CONDITION IN TERMS OF ∇ ALONE

Let $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a bilinear operation in a finite-dimensional real vector space $\mathfrak{g}$. As before, ∇w is the operator $v \mapsto \nabla_v w$.

We set $\delta(w) = \mathrm{tr}\,\nabla w$ and require that $\rho : \mathfrak{g} \times \mathfrak{g} \to \mathrm{I\!R}$ given by $\rho(v, w) = \delta(\nabla_v w) - (\nabla v, \nabla w)$ be

(i) symmetric and nondegenerate,
(ii) ∇-parallel (meaning: $\rho(\nabla_v w, u)$ is skew-symmetric in w, u),

and, at the same time, also require

(iii) the Jacobi identity for the bracket $[v, w] = \nabla_v w - \nabla_w v$.

EXAMPLE: HYPERBOLIC SPACES OVER $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$

Let $(\mathfrak{g}, \langle\,,\rangle)$ be an inner product space with $\dim_{\mathbb{F}} \mathfrak{g} = n < \infty$ over the scalar field $\mathbb{F}$ (one of $\mathbb{R}, \mathbb{C}, \mathbb{H}$). Sesquilinearity of $\langle\,,\rangle$ means that $\langle xu, yv\rangle = x\langle u, v\rangle \bar{y}$ for $u, v \in \mathfrak{g}$ and $x, y \in \mathbb{F}$.

We fix a unit vector $u \in \mathfrak{g}$ and define $\nabla : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\begin{array}{l} \nabla_u u = \nabla_u v = \nabla_u w = 0, \quad \nabla_v u = 2v, \quad \nabla_v v' = -2\,\mathrm{Re}\langle v, v'\rangle\, u, \\ \nabla_v w = \nabla_w v = \langle v, u\rangle\, w, \quad \nabla_w u = w, \quad \nabla_w w' = \langle w', w\rangle\, u, \end{array}$$

for any $\mathbb{F}$-imaginary multiples v, v' of u and any w, w' that are $\langle\,,\rangle$-orthogonal to u. Conditions (i) – (iii) now hold, with $\delta = [(n+1)d - 2]\,\mathrm{Re}\langle u, \cdot\,\rangle$ and $\rho^{\nabla} = [4 - (n+3)d]\,\mathrm{Re}\langle\,,\rangle$, where $d = \dim_{\mathbb{R}} \mathbb{F}$, while the bracket in (iii) turns $\mathfrak{g}$ into a (nonunimodular) solvable Lie algebra of step 3 with a step 2 nilpotent commutant ideal.

SOLVABILITY AND NILPOTENCY

The *commutant ideal* $[\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ is the vector sub spanned by all brackets.

The *upper* (or, *lower*) *central series* $\mathfrak{g}$ is the sequence $\mathfrak{g}_j$, $j \geq 0$, with $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_j = [\mathfrak{g}, \mathfrak{g}_{j-1}]$ (or, respectively, $\mathfrak{g}_j = [\mathfrak{g}_{j-1}, \mathfrak{g}_{j-1}]$) for $j \geq 1$.

One calls $\mathfrak{g}$ *nilpotent* (or, *solvable*) if in its upper (or, lower) central series one has $\mathfrak{g}_j = \{0\}$ for some j. The smallest such j is referred to as the *step* of nilpotency or solvability.

IS SOLVABILITY NECESSARY?

All known examples of Lie groups admitting left-invariant Riemannian Einstein metrics with negative Einstein constants ($\kappa < 0$) are solvable.

It is an open question whether a metric as above exists on any nonsolvable Lie group.

However, groups with such metrics are always noncompact as a consequence of Bochner's vanishing theorem.

METRICS ON SOLVABLE LIE GROUPS

Much is understood about left-invariant Riemannian Einstein metrics with $\kappa < 0$ on solvable Lie groups G.

Such a metric γ is called *standard* if the γ-orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ is an Abelian Lie subalgebra.

Heber (*Inventiones Math.*, 1998) proved that if a standard metric exists on G, then it is unique up to isometries and scaling.

Lauret (*Annals of Math.*, 2010) later showed that all left-invariant Riemannian Einstein metrics with $\kappa < 0$ on solvable Lie groups are standard.

SPECIAL LINEAR GROUPS

Nobody knows if the (nonsolvable) group $\mathrm{SL}(n, \mathbb{R})$, for any $n \geq 3$, admits a left-invariant Riemannian Einstein metric with $\kappa < 0$.

On the other hand, Leite and Dotti de Miatello constructed, in 1982, left-invariant Riemannian metrics of negative Ricci curvature on $\mathrm{SL}(n, \mathbb{R})$, for all $n \geq 3$.

Alekseevsky's still-open 1975 conjecture states that a homogeneous space G/K may carry a G-invariant Riemannian Einstein metric with $\kappa < 0$ only if K is a maximal compact subgroup of G.

If true, Alekseevsky's conjecture would imply a negative answer to the above question about $\mathrm{SL}(n, \mathbb{R})$.

THE D'ATRI-ZILLER THEOREM

On any compact simple Lie group, $-\beta$ is a bi-invariant Riemannian Einstein metric with $\kappa > 0$.

D'Atri and Ziller proved in 1979 that every compact simple Lie group other than $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ admits a left-invariant Riemannian Einstein metric (necessarily, with $\kappa > 0$) which is not a multiple of β.
Their argument realizes such a metric via its eigenspace decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_r,$$

relative to the Killing form β, where $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_r$ is chosen to be the Lie algebra of a suitable compact Lie subgroup K, and $\mathfrak{k}_0$ is the center of $\mathfrak{k}$, while $\mathfrak{k}_j$, $j \geq 1$, are simple ideals.