MATH 4552

The Riemann sphere

We treat the complex plane $\mathbb{C}$ as the xy-plane in a Cartesian 3-space with the coordinates x, y, ζ. Rather than representing space points as triples (x, y, ζ) of real numbers, we write them as pairs (z, ζ), where $z = x + iy$ is complex, and ζ real.

The *Riemann sphere* is the sphere Σ in the 3-space with the radius $1/2$, centered at $(0, 1/2)$. The equation $|w|^2 + (\zeta - 1/2)^2 = 1/4$, characterizing $w \in \mathbb{C}$ and $\zeta \in \mathbb{R}$ such that the point (w, ζ) lies on Σ, obviously amounts to

$$|w|^2 = (1 - \zeta)\zeta. \tag{1}$$

Complex numbers z are identified with points $(z, 0)$. On Σ there are two distinguished points: $(0, 0)$, that is, the complex number 0, which we also call the *south pole* and denote by S, and $\infty = (0, 1)$, referred to as the point *at infinity*, or the *north pole* N.

Since the right-hand side of (1) is negative when $\zeta < 0$ or $\zeta > 1$ (being the product of one negative and one positive factor), all points $(w, \zeta) \in \Sigma$ have $0 \le \zeta \le 1$. The extremum values $\zeta = 0$ and $\zeta = 1$ are attained only at S and, respectively, at N, since either of those values in (1) gives $w = 0$.

The *stereographic projection* is the mapping

$$\Sigma \setminus \{N\} \ni (w, \zeta) = P \mapsto z = \frac{w}{1 - \zeta} \in \mathbb{C}, \tag{2}$$

assigning to every point $P \in \Sigma$ other than the north pole N the unique point z (that is, $(z, 0)$) at which the half-line emanating from N and passing through P intersects the xy-plane $\mathbb{C}$. It is explained below that a unique such z does in fact exist, and it equals $w/(1 - \zeta)$ (while, as we just saw, $1 - \zeta > 0$).

Specifically, points of the line passing through $N = (0, 1)$ and $P = (w, \zeta)$ have the form $N + t(P - N) = (tw, 1 + t(\zeta - 1))$, the last component of which is 0 for one and only one real t, namely, $t = 1/(1 - \zeta)$ (which is also positive, so that $(tw, 0) = (w/(1 - \zeta), 0)$ lies on the half-line mentioned above).

The stereographic projection (2) is a one-to-one correspondence between $\Sigma \setminus \{N\}$ and $\mathbb{C}$. In other words, any $z \in \mathbb{C}$ equals $w/(1 - \zeta)$ for a unique $(w, \zeta) \in \Sigma \setminus \{N\}$. This unique (w, ζ) appears in the following description of the inverse stereographic projection:

$$\mathbb{C} \ni z \mapsto P = (w, \zeta) = \left(\frac{z}{|z|^2 + 1}, \frac{|z|^2}{|z|^2 + 1} \right) \in \Sigma \setminus \{N\}. \tag{3}$$

Theorem. *The images of circles contained in Σ under the stereographic projection* (2) *are lines or circles in $\mathbb{C}$, and every line or circle in $\mathbb{C}$ arises in this way. More precisely, lines in $\mathbb{C}$ are the stereographic-projection images of circles contained in Σ and passing through N, from which N has been removed.*

Proof. If (p,c) is a (nonzero) vector normal to a fixed plane in 3-space, then, for a suitable constant d, the equation of the plane reads

$$\mathrm{Re}\, p\overline{w} + c\zeta = d \tag{4}$$

that is, (w,ζ) lies in the plane if and only if it satisfies (4). (This becomes obvious when you recall that, with the traditional notation x,y,z for Cartesian coordinates, a plane with a normal vector $(a,b,c) \neq (0,0,0)$ has the equation $ax+by+cz=d$.) Note that

$$\text{the plane (4) passes through } N \text{ precisely when } c = d, \tag{5}$$

as one sees setting $(w,\zeta) = N = (0,1)$ in (4). A plane (4) may intersect Σ along a circle, or at a single point, or not intersect it at all, depending on whether the distance δ between the plane and $(0,1/2)$ (the center of Σ) is less, equal, or greater than $1/2$, the radius of Σ. The first of these three cases occurs if and only if

$$(2d - c)^2 < |p|^2 + c^2. \tag{6}$$

In fact, δ is the length of a vector parallel to the normal vector (p,c), which added to $(0,1/2)$ produces a point lying in the plane (4). In other words, $\delta^2 = |t(p,c)|^2$ for real t chosen so that $(w,\zeta) = (0,1/2) + t(p,c) = (tp, tc+1/2)$ satisfies (4). Thus, $t = (d - c/2)/(|p|^2+c^2)$ and, as $\delta^2 = t^2(|p|^2+c^2)$, (6) means precisely that $\delta < 1/2$.

Given $p \in \mathbb{C}$ and $c,d \in \mathbb{R}$ with $(p,c) \neq (0,0)$, satisfying (6), the image under the stereographic projection (2) of the circle contained in Σ, which is the intersection of Σ with the plane (4), consist precisely of those $z \in \mathbb{C}$ satisfying the equation

$$(c-d)|z|^2 + \mathrm{Re}\, p\overline{z} - d = 0. \tag{7}$$

To see this, impose condition (4) on (w,ζ) given by (3) (that is, on $P = (w,\zeta)$ to which $z \in \mathbb{C}$ corresponds under the stereographic projection): multiplying both sides by $|z|^2+1$, you obtain $(|z|^2+1)d = \mathrm{Re}\, p\overline{z} + c|z|^2$ or, equivalently, (7).

We will use the fact that, for any $z,q \in \mathbb{C}$, one has

$$|z|^2 + 2\mathrm{Re}\, q\overline{z} = |z+q|^2 - |q|^2, \tag{8}$$

which is clear since the right-hand side equals $(z+q)(\overline{z}+\overline{q}) - q\overline{q} = z\overline{z} + q\overline{z} + \overline{q\overline{z}}$.

First, suppose that the plane (4) does not pass through N, so that $p \in \mathbb{C}$, $c,d \in \mathbb{R}$, $(p,c) \neq (0,0)$ and, by (5), $c \neq d$. Now (7) divided by $c - d \neq 0$ reads

$$|z|^2 + \mathrm{Re}\,\frac{p\overline{z}}{c-d} - \frac{d}{c-d} = 0,$$

which, in view of (8) for $q = p/[2(c-d)]$, amounts to $|z+q|^2 = a^2$, that is,

$$|z+q| = a, \quad \text{where } q = p/[2(c-d)] \text{ and } a = \sqrt{|p|^2 + c^2 - (2d-c)^2}\,. \tag{9}$$

Here $a > 0$ by (6), so that the image is the circle of radius a centered at q.

Finally, let the plane (4) pass through N. Thus, $p \in \mathbb{C}$, $c \in \mathbb{R}$, $(p,c) \neq (0,0)$ and, by (5), $d = c$. Hence (7) becomes

$$\mathrm{Re}\, p\overline{z} = c, \tag{10}$$

and so the image is a line: $p \neq 0$, or else (6) with $d = c$ would give $c^2 < c^2$. **Q.E.D.**