MATH 4552

Cubic equations and Cardano's formulae

Consider a cubic equation with the unknown z and fixed complex coefficients a, b, c, d (where $a \neq 0$):

$$az^3 + bz^2 + cz + d = 0\,. \tag{1}$$

To solve (1), it is convenient to divide both sides by a and complete the first two terms to a full cube $(z + b/3a)^3$. In other words, setting

$$w \,=\, z \,+\, \frac{b}{3a} \tag{2}$$

we replace (1) by the simpler equation

$$w^3 \,+\, pw \,+\, q \,=\, 0 \tag{3}$$

with the unknown w (and some constant coefficients p, q). However, as any pair of numbers u, v satisfies the binomial formula $(u + v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$, i.e.,

$$(u + v)^3 \,-\, 3uv(u + v) \,-\, (u^3 + v^3) \,=\, 0\,, \tag{4}$$

we will find a solution w to (3) in the form

$$w \,=\, u \,+\, v\,, \tag{5}$$

provided that we have managed to choose (complex) numbers u, v in such a way that

$$p \,=\, -3uv \tag{6}$$

and

$$q \,=\, -(u^3 + v^3)\,. \tag{7}$$

Numbers u, v with (6) and (7) will also satisfy

$$-\,\frac{p^3}{27} \,=\, u^3v^3 \tag{8}$$

and so by (7) their cubes $u^3,\ v^3$ will be the two roots of the quadratic equation

$$t^2 \,+\, qt \,-\, \frac{p^3}{27} \,=\, 0 \tag{9}$$

with the (complex) unknown t; in fact, we have the identity

$$(t - u^3)(t - v^3) \;=\; t^2 - (u^3 + v^3)t + u^3v^3\,. \tag{10}$$

We now proceed as follows. First, we find the two complex solutions t to (9) and write them as u^3, v^3 (i.e., choose cubic roots u, v of these t). This will guarantee (8) and (7), but not necessarily (6). (The expressions in (6) then have equal cubes, so they need not be equal; what follows is that either both sides of (6) are zero, or their quotient is a cubic root of unity.) To obtain (6), change u by multiplying it by a suitable cubic root of unity; then, both (6) and (7) will be satisfied. Formula (5) now gives a solution $w = w_1$ to (3).

The other two solutions to (3) could be found via factoring out $w - w_1$ from (3) and solving the resulting quadratic equation, but we can proceed more directly. Let $\varepsilon = \omega_3$ be the primitive cubic root of unity, so that $1, \varepsilon, \overline{\varepsilon}$ are all cubic roots of unity. (We know that $\varepsilon = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\overline{\varepsilon} = \varepsilon^2 = e^{-2\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.) Our choice of u, v with (6) and (7) is not unique: given such u, v we can replace them with εu, $\overline{\varepsilon} v$ as well as $\overline{\varepsilon} u$, εv (and also switch the roles of u and v, which is not relevant here). Now we obtain the following expressions for all solutions to (3), known as *Cardano's formulae*:

$$w_1 \;=\; u \,+\, v\,, \qquad w_2 \;=\; \varepsilon u \,+\, \overline{\varepsilon} v\,, \qquad w_3 \;=\; \overline{\varepsilon} u \,+\, \varepsilon v\,. \tag{11}$$

Example. To solve

$$z^3 + 6z^2 + 9z + 3 = 0\,, \tag{12}$$

complete $z^3 + 6z^2$ to a full cube: $(z + 2)^3 = z^3 + 6z^2 + 12z + 8$, i.e., rewrite (12) as the simpler equation

$$w^3 \,-\, 3w \,+\, 1 \;=\; 0 \tag{13}$$

with the unknown $w = z + 2$. To cast (13) in the form (4) with $w = u + v$, we need to find u, v with

$$uv \;=\; 1\,, \qquad u^3 + v^3 \;=\; -1\,.$$

Hence

$$(t - u^3)(t - v^3) \;=\; t^2 \,+\, t \,+\, 1\,,$$

and u^3, v^3 are the roots of the equation

$$t^2 \,+\, t \,+\, 1 \;=\; 0\,. \tag{14}$$

Solving (14) we obtain $t = e^{\pm 2\pi i/3} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ (i.e., the solutions happen to be ε and $\overline{\varepsilon}$.) Choosing the cubic roots of these solutions to be $u = e^{2\pi i/9}$ and $v = e^{-2\pi i/9}$, we obtain

$$w_1 \;=\; u \,+\, v \;=\; 2\cos\frac{2\pi}{9}\,, \quad w_2 \;=\; \varepsilon u \,+\, \overline{\varepsilon} v \;=\; 2\cos\frac{8\pi}{9}\,, \quad w_3 \;=\; \overline{\varepsilon} u \,+\, \varepsilon v \;=\; 2\cos\frac{4\pi}{9}\,.$$

The solutions to (12) thus are $z_1 = 2\cos 40° - 2$, $z_2 = -2\cos\cos 20° - 2$, $z_3 = 2\sin 10° - 2$, i.e.,

$$z_1 \,=\, 2\cos\frac{2\pi}{9} \,-\, 2\,, \quad z_2 \,=\, 2\cos\frac{8\pi}{9} \,-\, 2 \,=\, -2\cos\frac{\pi}{9} \,-\, 2\,, \quad z_3 \,=\, 2\sin\frac{\pi}{18} \,-\, 2\,.$$

Quartic (fourth degree) equations and Ferrari's method

To solve a quartic equation

$$az^4 + bz^3 + cz^2 + kz + l = 0 \tag{15}$$

with the unknown z and fixed complex coefficients a, b, c, k, l (where $a \neq 0$), one proceeds as follows. First, we divide both sides by a and complete the highest two terms to a full fourth power $(z + b/4a)^4$. This means that by setting

$$w \,=\, z \,+\, \frac{b}{4a} \tag{16}$$

we replace (15) by the simpler equation

$$w^4 \,+\, pw^2 \,+\, qw \,+\, r \,=\, 0 \tag{17}$$

with the unknown w and some constant coefficients p, q, r. The next step is to find a factorization

$$w^4 \,+\, pw^2 \,+\, qw \,+\, r \,=\, (w^2 \,-\, \alpha w \,+\, \beta)(w^2 \,+\, \alpha w \,+\, \gamma) \tag{18}$$

of the polynomial $w^4 + pw^2 + qw + r$ into a product of two quadratic polynomials. Note that the coefficients of w^2 in both factors can be made equal to 1 by multiplying the factors by suitable constants, and the coefficients of w in the factors then must add up to zero (i.e., have the form $-\alpha$ and α) as their sum is the coefficient of w^3 in (17).

Equating the coefficients in (18), we see that our problem is to find, for the given p, q, r, some numbers α, β, γ with

$$\beta \,+\, \gamma \,=\, p \,+\, \alpha^2\,, \qquad \beta\gamma \,=\, r \tag{19}$$

and

$$\alpha(\beta \,-\, \gamma) \,=\, q\,. \tag{20}$$

For any fixed α, numbers β, γ with (19) will be the two roots of the quadratic equation

$$t^2 \,-\, (p + \alpha^2)t \,+\, r \tag{21}$$

with the complex unknown t, since we have the identity $(t-\beta)(t-\gamma) = t^2-(\beta+\gamma)t+\beta\gamma$ (see also (10)). Thus, solving (21), we see that β and γ are the numbers

$$\frac{p+\alpha^2 \pm \sqrt{(p+\alpha^2)^2-4r}}{2}, \tag{22}$$

where $\pm$ indicates that the square root of a complex number is unique only up to multiplication by -1. Thus,

$$\beta - \gamma = \pm\sqrt{(p+\alpha^2)^2-4r}\,, \tag{23}$$

so that

$$(\beta-\gamma)^2 = (p+\alpha^2)^2-4r \tag{24}$$

and so, by (20),

$$\alpha^2\left[(p+\alpha^2)^2-4r\right] = q^2\,, \tag{25}$$

i.e.,

$$\alpha^6 + 2p\alpha^4 + (p^2-4r)\alpha^2 - q^2 = 0\,. \tag{26}$$

This is a cubic equation with the unknown α^2. Denoting α a fixed square root of a fixed solution to (26) (which we may find using Cardano's formulae) and then defining β and γ to be the numbers (22), we obtain (19), while in (20) the expressions are either equal or differ only by sign (as their squares coincide in view of (25)). Replacing α with $-\alpha$ if necessary, we thus find complex solutions α, β, γ to the system (19), (20), which gives rise to the decomposition (18). Solving each of the equations $w^2 - \alpha w + \beta = 0$ and $w^2 + \alpha w + \gamma = 0$, we now find all solutions w to (17).

Example. To solve

$$48z^4 - 72z^2 + 16\sqrt{6}z - 1 = 0\,, \tag{27}$$

note that the coefficient of z^3 already is zero, so a shift of the unknown as in (16) is not needed. A factorization (18), i.e.,

$$48z^4 - 72z^2 + 16\sqrt{6}z - 1 = 48(z^2 - \alpha z + \beta)(z^2 + \alpha z + \gamma) \tag{28}$$

amounts to solving for α, β, γ the system (19), (20) with

$$p = -\frac{3}{2}, \qquad q = \sqrt{\frac{2}{3}}, \quad r = -\frac{1}{48}\,, \tag{29}$$

that is,

$$\beta + \gamma = \alpha^2 - \frac{3}{2}, \qquad \beta\gamma = -\frac{1}{48}, \qquad \alpha(\beta - \gamma) = \sqrt{\frac{2}{3}}. \tag{30}$$

Our β, γ thus coincide with the roots

$$\frac{\alpha^2 - \frac{3}{2} \pm \sqrt{(\frac{3}{2} - \alpha^2)^2 + \frac{1}{12}}}{2}. \tag{31}$$

of the quadratic equation

$$t^2 + (\frac{3}{2} - \alpha^2)t - \frac{1}{48}, \tag{32}$$

while α must satisfy (26) with (29), i.e.,

$$\alpha^6 - 3\alpha^4 + \frac{7}{3}\alpha^2 - \frac{2}{3} = 0. \tag{33}$$

We now solve this cubic equation for α^2, using Cardano's formulae. Specifically, setting

$$w = \alpha^2 - 1 \tag{34}$$

we replace (32) by the simpler equation

$$w^3 - \frac{2}{3}w - \frac{1}{3} = 0 \tag{35}$$

with the unknown w. We now can rewrite (35) in the form (4) with $w = u + v$, provided that we find u, v with (6) and (7) (where p, q now both stand for $-\frac{8}{3}$), that is,

$$uv = \frac{2}{9}, \qquad u^3 + v^3 = \frac{1}{3}.$$

As before (in (10)), $(t - u^3)(t - v^3) = t^2 - t/3 + 8/3^6$, so the cubes u^3, v^3 of u and v must be the roots of the quadratic equation

$$t^2 - \frac{1}{3}t + \frac{8}{3^6} = 0. \tag{36}$$

Solving (36) we obtain

$$t = \frac{9 \pm 7}{54},$$

and we may choose the cubic roots of these solutions to be

$$u = \frac{2}{3}, \qquad v = \frac{1}{3}.$$

The corresponding solution $w = u + v$ to (35) is $w = 1$ and, by (34), it yields $\alpha^2 = 2$. Selecting the suitable sign for α, we obtain from (31) the numbers

$$\alpha = \sqrt{2}\,, \qquad \beta = \frac{3 + 2\sqrt{3}}{12}\,, \qquad \gamma = \frac{3 - 2\sqrt{3}}{12}$$

satisfying (30) and hence leading to the decomposition (28) in the form

$$48z^4 - 72z^2 + 16\sqrt{6}z - 1 = 48\left[z^2 - \sqrt{2}z + \frac{3 + 2\sqrt{3}}{12}\right]\left[(z^2 + \sqrt{2}z + \frac{3 - 2\sqrt{3}}{12}\right].$$

The solutions z to (27) thus are

$$-\frac{\sqrt{2}}{2} \pm \frac{1}{2}\sqrt{\frac{2}{\sqrt{3}} + 1}\,, \qquad \frac{\sqrt{2}}{2} \pm \frac{i}{2}\sqrt{\frac{2}{\sqrt{3}} - 1}\,.$$