1. The intersection of two vector spaces

The key idea goes back to the definition of a vector space and a subspace. New questions arise from considering not just a single subspace or a single matrix \( A \), but the interconnections between two subspaces or two matrices. The first point is the most important:

**Example 1** The intersection of two orthogonal subspaces \( V \) and \( W \) is the one-point subspace \( \{0\} \). Only the zero vector is orthogonal to itself.

**Example 2** If the sets of \( n \times n \) upper and lower triangular matrices are the subspaces \( V \) and \( W \), their intersection is the set of diagonal matrices. This is certainly a subspace. Adding two diagonal matrices, or multiplying by a scalar, leaves us with a diagonal matrix.

**Example 3** Suppose \( V \) is the nullspace of \( A \) and \( W \) is the nullspace of \( B \). Then \( V \cap W \) is the smaller nullspace of the larger matrix

\[
C = \begin{bmatrix} A \\ B \end{bmatrix}
\]

\( Cx = 0 \) requires both \( Ax = 0 \) and \( Bx = 0 \), so \( x \) has to be in both nullspaces.

2. The sum of two vector spaces

Usually, after discussing and illustrating the intersection of two sets, it is natural to look at their union. With vector spaces, however, it is not natural. The union \( V \cup W \) of two subspaces will not in general be a subspace. Consider the \( x \) axis and the \( y \) axis in the plane. Each axis by itself is a subspace, but taken together they are not. The sum of \( (1, 0) \) and \( (0, 1) \) is not on either axis. This will always happen unless one of the subspaces is contained
in the other, only then is their union (which coincides with the larger one) again a subspace.

Nevertheless, we do want to combine two subspaces, and therefore in place of their union we turn to their sum.

**Definition.** If $V$ and $W$ are both subspaces of a given space, then so is their sum $V + W$. It is made up of all possible combinations $x = v + w$, where $v$ is an arbitrary vector in $V$ and $w$ is an arbitrary vector in $W$.

This is nothing but the space spanned by $V \cup W$. It is the smallest vector space that contains both $V$ and $W$. The sum of the $x$ axis and the $y$ axis is the whole $x$-$y$ plane; so is the sum of any two different lines, *perpendicular or not*. If $V$ is the $x$ axis and $W$ is the $45^\circ$ line $x = y$, then any vector like $(5, 3)$ can be split into $v + w = (2, 0) + (3, 3)$. Thus $V + W$ is all of $\mathbb{R}^2$.

**Example 4** Suppose $V$ and $W$ are orthogonal complements of one another in $\mathbb{R}^n$. Then their sum is $V + W = \mathbb{R}^n$. Every $x$ is the sum of its projection $v$ in $V$ and its projection $w$ in $W$.

**Example 5** If $V$ is the space of upper triangular matrices, and $W$ is the space of lower triangular matrices, then $V + W$ is the space of all matrices. Every matrix can be written as the sum of an upper and a lower triangular matrix—in many ways, because the diagonals are not uniquely determined.

**Example 6** If $V$ is the column space of a matrix $A$, and $W$ is the column space of $B$, then $V + W$ is the column space of the larger matrix $D = [A \ B]$. The dimension of $V + W$ may be less than the combined dimensions of $V$ and $W$ (because the two spaces may overlap), but it is easy to find:

$$\dim(V + W) = \text{rank of } D. \quad (1)$$

Surprisingly, the computation of $V \cap W$ is much more subtle. Suppose we are given the two bases $v_1, \ldots, v_k$ and $w_1, \ldots, w_h$; this time we want a basis for the intersection of the two subspaces. Certainly it is not enough just to check whether any of the $v$'s is equal to any of the $w$'s. The two spaces could even be identical, $V = W$, and still the bases might be completely different.

The most efficient method is this. Form the same matrix $D$ whose columns are $v_1, \ldots, v_k, w_1, \ldots, w_h$ and compute its nullspace $\mathcal{N}(D)$. We shall show that a basis for this nullspace leads to a basis for $V \cap W$, and that the two spaces have the same dimension. The dimension of the nullspace is called the "nullity," so

$$\dim(V \cap W) = \text{nullity of } D. \quad (2)$$

This leads to a formula which is important in its own right. Adding (1) and (2),

$$\dim(V + W) + \dim(V \cap W) = \text{rank of } D + \text{nullity of } D.$$
From our computations with the four fundamental subspaces, we know that the rank plus the nullity equals the number of columns. In this case $D$ has $k + l$ columns, and since $k = \dim V$ and $l = \dim W$, we are led to the following conclusion:

Not a bad formula.

**Example 7** The spaces $V$ and $W$ of upper and lower triangular matrices both have dimension $n(n + 1)/2$. The space $V + W$ of all matrices has dimension $n^2$, and the space $V \cap W$ of diagonal matrices has dimension $n$. As predicted by (3), $n^2 - n = n(n + 1)/2 + n(n + 1)/2$.

We now look at the proof of (3). For once in this book, the interest is less in the actual computation than in the technique of proof. It is the only time we will use the trick of understanding one space by matching it with another. Note first that the nullspace of $D$ is a subspace of $\mathbb{R}^{k + l}$, whereas $V \cap W$ is a subspace of $\mathbb{R}^n$. We have to prove that these two spaces have the same dimension. The trick is to show that these two subspaces are perfectly matched by the following correspondence.

Given any vector $x$ in the nullspace of $D$, write the equation $Dx = 0$ in terms of the columns as follows:

$$x_1v_1 + \cdots + x_kv_k + x_{k+1}w_1 + \cdots + x_{k+l}w_l = 0,$$

or

$$x_1v_1 + \cdots + x_kv_k = -x_{k+1}w_1 - \cdots - x_{k+l}w_l.$$  

The left side of this last equation is in $V$, being a combination of the $v_k$, and the right side is in $W$. Since the two are equal, they represent a vector $y$ in $V \cap W$. This provides the correspondence between the vector $x$ in $\mathcal{N}(D)$ and the vector $y$ in $V \cap W$. It is easy to check that the correspondence preserves addition and scalar multiplication: If $x$ corresponds to $y$ and $x'$ to $y'$, then $x + x'$ corresponds to $y + y'$ and $cx$ corresponds to $cy$. Furthermore, every $y$ in $V \cap W$ comes from one and only one $x$ in $\mathcal{N}(D)$ (Exercise 3.6.18).

This is a perfect illustration of an *isomorphism* between two vector spaces. The spaces are different, but for all algebraic purposes they are exactly the same. They match completely: Linearly independent sets correspond to linearly independent sets, and a basis in one corresponds to a basis in the other. So their dimensions are equal, which completes the proof of (2) and (3). This is the kind of result an algebraist is after, to identify two different mathematical objects as being fundamentally the same.† It is a fact that any two spaces with the same scalars and the same (finite) dimension are always isomorphic, but this is too general to be very

† Another isomorphism is between the row space and column space, both of dimension $r$. 
exciting. The interest comes in matching two superficially dissimilar spaces, like \( \mathcal{N}(D) \) and \( V \cap W \).

**EXAMPLE 8** \( V \) is the \( x\text{-}y \) plane and \( W \) is the \( x\text{-}z \) plane:

\[
D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

first 2 columns: basis for \( V \)

last 2 columns: basis for \( W \)

The rank of \( D \) is 3, and \( V + W \) is all of \( \mathbb{R}^3 \). The nullspace contains \( x = (1, 0, -1, 0) \), and has dimension 1. The corresponding vector \( y \) is \( 1 \text{(column 1)} + 0 \text{(column 2)} \), pointing along the \( x \)-axis—which is the intersection \( V \cap W \). Formula (3) for the dimensions of \( V + W \) and \( V \cap W \) becomes \( 3 + 1 = 2 + 2 \).
EXERCISES

3.6.1 Suppose $S$ and $T$ are subspaces of $\mathbb{R}^3$, with $\dim S = 7$ and $\dim T = 8$.
(a) What is the largest possible dimension of $S \cap T$?
(b) What is the smallest possible dimension of $S \cap T$?
(c) What is the smallest possible dimension of $S + T$?
(d) What is the largest possible dimension of $S + T$?

3.6.2 What are the intersections of the following pairs of subspaces?
(a) The $x$-$y$ plane and the $y$-$z$ plane in $\mathbb{R}^3$.
(b) The line through $(1, 1, 1)$ and the plane through $(1, 0, 0)$ and $(0, 1, 1)$.
(c) The zero vector and the whole space $\mathbb{R}^3$.
(d) The plane perpendicular to $(1, 1, 0)$ and the plane perpendicular to $(0, 1, 1)$ in $\mathbb{R}^3$.
What are the sums of those pairs of subspaces?

3.6.3 Within the space of all $4 \times 4$ matrices, let $V$ be the subspace of tridiagonal matrices and $W$ the subspace of upper triangular matrices. Describe the subspace $V + W$, whose members are the upper Hessenberg matrices, and the subspace $V \cap W$. Verify formula (3).

3.6.4 If $V \cap W$ contains only the zero vector then (3) becomes $\dim(V + W) = \dim V + \dim W$. Check this when $V$ is the row space of $A$, $W$ is the nullspace, and $A$ is $m \times n$ of rank $r$. What are the dimensions?

3.6.5 Give an example in $\mathbb{R}^3$ for which $V \cap W$ contains only the zero vector but $V$ is not orthogonal to $W$. 
3.6.6 If \( V \cap W = \{0\} \) then \( V + W \) is called the direct sum of \( V \) and \( W \), with the special notation \( V \oplus W \). If \( V \) is spanned by \((1, 1, 1)\) and \((1, 0, 1)\), choose a subspace \( W \) so that \( V \oplus W = \mathbb{R}^3 \).

3.6.7 Explain why any vector \( x \) in the direct sum \( V \oplus W \) can be written in one and only one way as \( x = v + w \) (with \( v \) in \( V \) and \( w \) in \( W \)).

3.6.8 Find a basis for the sum \( V + W \) of the space \( V \) spanned by \( v_1 = (1, 1, 0, 0) \), \( v_2 = (1, 0, 1, 0) \) and the space \( W \) spanned by \( w_1 = (0, 1, 0, 1) \), \( w_2 = (0, 0, 1, 1) \). Find also the dimension of \( V \cap W \) and a basis for it.

3.6.9 Show by example that the nullspace of \( AB \) need not contain the nullspace of \( A \), and the column space of \( AB \) is not necessarily contained in the column space of \( B \).

3.6.10 Find the largest invertible submatrix and the rank of

\[
A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 0 & 4 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

3.6.11 Suppose \( A \) is \( m \) by \( n \) and \( B \) is \( n \) by \( m \), with \( n < m \). Prove that their product \( AB \) is singular.

3.6.12 Prove from (3) that \( \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \).

3.6.13 If \( A \) is square and invertible prove that \( AB \) has the same nullspace (and the same row space and the same rank) as \( B \) itself. \textit{Hint:} Apply relationship (i) also to the product of \( A^{-1} \) and \( AB \).

3.6.14 Factor \( A \) into an \( m \) by \( r \) matrix \( L \) times an \( r \) by \( n \) matrix \( U \):

\[
A = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

3.6.15 Multiplying each column of \( L \) by the corresponding row of \( U \), and adding, gives the product \( A = LU \) as the sum of \( r \) matrices of rank one. Construct \( L \) and \( U \) and the two matrices of rank one that add to

\[
A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}
\]

3.6.16 Prove that the intersection of three 6-dimensional subspaces of \( \mathbb{R}^8 \) is not the single point \( \{0\} \). \textit{Hint:} How small can the intersection of the first two subspaces be?

3.6.17 Find the factorization \( A = LDL^T \), and then the two Cholesky factors in \((LD^{1/2})(LD^{1/2})^T\), for

\[
A = \begin{bmatrix} 4 & 12 \\ 12 & 45 \end{bmatrix}
\]
3.6.18 Verify the statement that "every \( y \) in \( V \cap W \) comes from one and only one \( x \) in \( \mathcal{N}(D) \)—by describing, for a given \( y \), how to go back to equation (5) and find \( x \).

3.6.19 What happens to the weighted average \( \bar{x}_w = (w_1^2b_1 + w_2^2b_2)/(w_1^2 + w_2^2) \) if the first weight \( w_1 \) approaches zero? The measurement \( b_1 \) is totally unreliable.

3.6.20 From \( m \) independent measurements \( b_1, \ldots, b_m \) of your pulse rate, weighted by \( w_1, \ldots, w_m \), what is the weighted average that replaces (6)? It is the best estimate when the statistical variances are \( \sigma_i^2 = 1/w_i^2 \).

3.6.21 If \( W = \begin{bmatrix} a & b \end{bmatrix} \), find the \( W \)-inner product of \( x = (2, 3) \) and \( y = (1, 1) \) and the \( W \)-length of \( x \). What line of vectors is \( W \)-perpendicular to \( y \)?

3.6.22 Find the weighted least squares solution \( \bar{x}_w \) to \( Ax = b \):

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Check that the projection \( A\bar{x}_w \) is still perpendicular (in the \( W \)-inner product!) to the error \( b - A\bar{x}_w \).

3.6.23 (a) Suppose you guess your professor's age, making errors \( e = -2, -1, 5 \) with probabilities \( \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \). Check that the expected error \( E(e) \) is zero and find the variance \( E(e^2) \).

(b) If the professor guesses too (or tries to remember), making errors \( -1, 0, 1 \) with probabilities \( \frac{1}{4}, \frac{2}{4}, \frac{1}{4} \), what weights \( w_1 \) and \( w_2 \) give the reliability of your guess and the professor's guess?

3.6.24 Suppose \( p \) rows and \( q \) columns, taken together, contain all the nonzero entries of \( A \). Show that the rank is not greater than \( p + q \). How large does a square block of zeros have to be, in a corner of a 9 by 9 matrix, to guarantee that the matrix is singular?