ECHELON FORM AND GAUSS-JORDAN ELIMINATION

As we noted in the previous section, our method for solving a system of linear equations will be to pass to the augmented matrix, use elementary row operations to reduce the augmented matrix, and then solve the simpler but equivalent system represented by the reduced matrix. This procedure is illustrated in Fig. 1.3.

The objective of the Gauss-Jordan reduction process (represented by the middle block in Fig. 1.3) is to obtain a system of equations simplified to the point where we
1.2 Echelon Form and Gauss-Jordan Elimination

\[
\text{Given system of equations} \rightarrow \text{Augmented matrix} \rightarrow \text{Reduced matrix} \rightarrow \text{Reduced system of equations} \rightarrow \text{Solution}
\]

*Figure 1.3*  Procedure for solving a system of linear equations

can immediately describe the solution. See, for example, Examples 6 and 7 in Section 1.1. We turn now to the question of how to describe this objective in mathematical terms—that is, how do we know when the system has been simplified as much as it can be? The answer is: The system has been simplified as much as possible when it is in reduced echelon form.

**Echelon Form**

When an augmented matrix is reduced to the form known as *echelon form*, it is easy to solve the linear system represented by the reduced matrix. The formal description of echelon form is given in Definition 3. Then, in Definition 4, we describe an even simpler form known as *reduced echelon form*.

**Definition 3**

An \((m \times n)\) matrix \(B\) is in *echelon form* if:

1. All rows that consist entirely of zeros are grouped together at the bottom of the matrix.
2. In every nonzero row, the first nonzero entry (counting from left to right) is a 1.
3. If the \((i+1)\)-st row contains nonzero entries, then the first nonzero entry is in a column to the right of the first nonzero entry in the \(i\)th row.

Put informally, a matrix \(A\) is in echelon form if the nonzero entries in \(A\) form a staircase-like pattern, such as the four examples shown in Fig. 1.4. (Note: Exercise 46 shows that there are exactly seven different types of echelon form for a \((3 \times 3)\) matrix. Figure 1.4 illustrates four of the possible patterns. In Fig. 1.4, the entries marked \(*\) can be zero or nonzero.)

\[
A = \begin{bmatrix}
  1 & * & * \\
  0 & 1 & * \\
  0 & 0 & 1
\end{bmatrix} \quad A = \begin{bmatrix}
  1 & * & * \\
  0 & 1 & * \\
  0 & 0 & 0
\end{bmatrix} \quad A = \begin{bmatrix}
  1 & * & * \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{bmatrix} \quad A = \begin{bmatrix}
  0 & 1 & * \\
  0 & 0 & 1 \\
  0 & 0 & 0
\end{bmatrix}
\]

*Figure 1.4*  Patterns for four of the seven possible types of \((3 \times 3)\) matrices in echelon form. Entries marked \(*\) can be either zero or nonzero.
Two examples of matrices in echelon form are

\[
A = \begin{bmatrix}
1 & -1 & 4 & 3 & 0 & 2 & 0 \\
0 & 0 & 1 & 8 & -4 & 3 & 2 \\
0 & 0 & 0 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 & 1 & -1 & 4 & 3 \\
0 & 0 & 1 & 6 & -5 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

We show later that every matrix can be transformed to echelon form with elementary row operations. It turns out, however, that echelon form is not unique. In order to guarantee uniqueness, we therefore add one more constraint and define a form known as reduced echelon form. As noted in Theorem 2, reduced echelon form is unique.

**Definition 4**

A matrix that is in echelon form is in reduced echelon form provided that the first nonzero entry in any row is the only nonzero entry in its column.

Figure 1.5 gives four examples (corresponding to the examples in Fig. 1.4) of matrices in reduced echelon form.

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad
A = \begin{bmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & 0
\end{bmatrix}, \quad
A = \begin{bmatrix}
1 & * & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad
A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]

*Figure 1.5* Patterns for four of the seven possible types of \((3 \times 3)\) matrices in reduced echelon form. Entries marked * can be either zero or nonzero.

Two examples of matrices in reduced echelon form are

\[
A = \begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 3
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 2 & 0 & 1 & -1 \\
0 & 0 & 1 & 3 & 4 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

As can be seen from these examples and from Figs. 1.4 and 1.5, the feature that distinguishes reduced echelon form from echelon form is that the leading 1 in each nonzero row has only 0's above and below it.

**Example 1**

For each matrix shown, choose one of the following phrases to describe the matrix.

(a) The matrix is not in echelon form.

(b) The matrix is in echelon form, but not in reduced echelon form.

(c) The matrix is in reduced echelon form.
1.2 Echelon Form and Gauss-Jordan Elimination

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
3 & -4 & 1
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 3 & 2 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
D = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}, \quad
E = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\]

\[
F = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}, \quad
G = \{1 \ 0 \ 0\}, \quad
H = \{0 \ 0 \ 1\}.
\]

**Solution**

A, B, and F are not in echelon form; D is in echelon form but not in reduced echelon form; C, E, G, and H are in reduced echelon form.

**Solving a Linear System Whose Augmented Matrix Is in Reduced Echelon Form**

Software packages that can solve systems of equations typically include a command that produces the reduced echelon form of a matrix. Thus, to solve a linear system on a machine, we first enter the augmented matrix for the system and then apply the machine's reduce command. Once we get the machine output (that is, the reduced echelon form for the original augmented matrix), we have to interpret the output in order to find the solution. The next example illustrates this interpretation process.

**Example 2**

Each of the following matrices is in reduced echelon form and is the augmented matrix for a system of linear equations. In each case, give the system of equations and describe the solution.

\[
B = \begin{bmatrix}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & -2 \\
0 & 0 & 1 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
1 & -3 & 0 & 4 & 2 \\
0 & 0 & 1 & -5 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad
E = \begin{bmatrix}
1 & 2 & 0 & 5 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

**Solution**

**Matrix B:** Matrix B is the augmented matrix for the following system:

\[
x_1 = 3 \\
x_2 = -2 \\
x_3 = 7.
\]

Therefore, the system has the unique solution \(x_1 = 3, x_2 = -2,\) and \(x_3 = 7.\)
Matrix C: Matrix C is the augmented matrix for the following system

\[
\begin{align*}
    x_1 - x_3 &= 0 \\
    x_2 + 3x_3 &= 0 \\
    0x_1 + 0x_2 + 0x_3 &= 1.
\end{align*}
\]

Because no values for \(x_1, x_2,\) or \(x_3\) can satisfy the third equation, the system is inconsistent.

Matrix D: Matrix D is the augmented matrix for the following system

\[
\begin{align*}
    x_1 - 3x_2 + 4x_4 &= 2 \\
    x_3 - 5x_4 &= 1.
\end{align*}
\]

We solve each equation for the leading variable in its row, finding

\[
\begin{align*}
    x_1 &= 2 + 3x_2 - 4x_4 \\
    x_3 &= 1 + 5x_4.
\end{align*}
\]

In this case, \(x_1\) and \(x_3\) are the dependent (or constrained) variables whereas \(x_2\) and \(x_4\) are the independent (or unconstrained) variables. The system has infinitely many solutions, and particular solutions can be obtained by assigning values to \(x_2\) and \(x_4\). For example, setting \(x_2 = 1\) and \(x_4 = 2\) yields the solution \(x_1 = -3, x_2 = 1, x_3 = 11,\) and \(x_4 = 2\).

Matrix E: The second row of matrix E sometimes leads students to conclude erroneously that the system of equations is inconsistent. Note the critical difference between the third row of matrix C (which did represent an inconsistent system) and the second row of matrix E. In particular, if we write the system corresponding to E, we find

\[
\begin{align*}
    x_1 + 2x_2 &= 5 \\
    x_3 &= 0.
\end{align*}
\]

Thus, the system has infinitely many solutions described by

\[
\begin{align*}
    x_1 &= 5 - 2x_2 \\
    x_3 &= 0
\end{align*}
\]

where \(x_2\) is an independent variable.

As we noted in Example 2, if an augmented matrix has a row of zeros, we sometimes jump to the conclusion (an erroneous conclusion) that the corresponding system of equations is inconsistent (see the discussion of matrix E in Example 2). Similar confusion can arise when the augmented matrix has a column of zeros. For example, consider the matrix

\[
\begin{bmatrix}
    1 & 0 & 0 & -2 & 0 & 3 \\
    0 & 0 & 1 & -4 & 0 & 1 \\
    0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix},
\]

where \(F\) is the augmented matrix for a system of 3 equations in 5 unknowns. Thus, \(F\) represents the system
1.2 Echelon Form and Gauss-Jordan Elimination

\[
\begin{align*}
   x_1 - 2x_4 &= 3 \\
   x_3 - 4x_4 &= 2 \\
   x_5 &= 2.
\end{align*}
\]

The solution of this system is \( x_1 = 3 + 2x_4, \ x_3 = 1 + 4x_4, \ x_5 = 2, \) and \( x_4 \) is arbitrary. Note that the equations place no constraint whatsoever on the variable \( x_2 \). That does not mean that \( x_2 \) must be zero; instead, it means that \( x_2 \) is also arbitrary.

**Recognizing an Inconsistent System**

Suppose \([A \mid b]\) is the augmented matrix for an \((m \times n)\) linear system of equations. If \([A \mid b]\) is in reduced echelon form, you should be able to tell at a glance whether the linear system has any solutions. The idea was illustrated by matrix \(C\) in Example 2.

In particular, we can show that if the last nonzero row of \([A \mid b]\) has its leading 1 in the last column, then the linear system has no solution. To see why this is true, suppose the last nonzero row of \([A \mid b]\) has the form

\[
[0, 0, 0, \ldots, 0, 1].
\]

This row, then, represents the equation

\[
0x_1 + 0x_2 + 0x_3 + \cdots + 0x_n = 1.
\]

Because this equation cannot be satisfied, it follows that the linear system represented by \([A \mid b]\) is inconsistent. We list this observation formally in the following remark.

**Remark** Let \([A \mid b]\) be the augmented matrix for an \((m \times n)\) linear system of equations, and let \([A \mid b]\) be in reduced echelon form. If the last nonzero row of \([A \mid b]\) has its leading 1 in the last column, then the system of equations has no solution.

When you are carrying out the reduction of \([A \mid b]\) to echelon form by hand, you might encounter a row that consists entirely of zeros except for a nonzero entry in the last column. In such a case, there is no reason to continue the reduction process since you have found an equation in an equivalent system that has no solution; that is, the system represented by \([A \mid b]\) is inconsistent.

**Reduction to Echelon Form**

The following theorem guarantees that every matrix can be transformed to one and only one matrix that is in reduced echelon form.

---

**Theorem 2**

Let \(B\) be an \((m \times n)\) matrix. There is a unique \((m \times n)\) matrix \(C\) such that:

(a) \(C\) is in reduced echelon form

and

(b) \(C\) is row equivalent to \(B\).

Suppose \(B\) is the augmented matrix for an \((m \times n)\) system of linear equations. One important consequence of this theorem is that it shows we can always transform \(B\) by a
series of elementary row operations into a matrix $C$ which is in reduced echelon form. Then, because $C$ is in reduced echelon form, it is easy to solve the equivalent linear system represented by $C$ (recall Example 2).

The following steps show how to transform a given matrix $B$ to reduced echelon form. As such, this list of steps constitutes an informal proof of the existence portion of Theorem 2. We do not prove the uniqueness portion of Theorem 2. The steps listed assume that $B$ has at least one nonzero entry (because if $B$ has only zero entries, then $B$ is already in reduced row echelon form).

**Reduction to Reduced Echelon Form for an $(m \times n)$ Matrix**

**Step 1.** Locate the first (left-most) column that contains a nonzero entry.

**Step 2.** If necessary, interchange the first row with another row so that the first nonzero column has a nonzero entry in the first row.

**Step 3.** If $a$ denotes the leading nonzero entry in row one, multiply each entry in row one by $1/a$. (Thus, the leading nonzero entry in row one is a 1.)

**Step 4.** Add appropriate multiples of row one to each of the remaining rows so that every entry below the leading 1 in row one is a 0.

**Step 5.** Temporarily ignore the first row of this matrix and repeat Steps 1–4 on the submatrix that remains. Stop the process when the resulting matrix is in echelon form.

**Step 6.** Having reached echelon form in Step 5, continue on to reduced echelon form as follows: Proceeding upward, add multiples of each nonzero row to the rows above in order to zero all entries above the leading 1.

The next example illustrates an application of the six-step process just described. When doing a small problem by hand, however, it is customary to alter the steps slightly—instead of going all the way to echelon form (sweeping from left to right) and then going from echelon to reduced echelon form (sweeping from bottom to top), it is customary to make a single pass (moving from left to right) introducing 0's above and below the leading 1. Example 3 demonstrates this single-pass variation.

---

**Example 3** Use elementary row operations to transform the following matrix to reduced echelon form

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 2 & 8 & 4 \\
0 & 0 & 0 & 1 & 3 & 11 & 9 \\
0 & 3 & -12 & -3 & -9 & -24 & -33 \\
0 & -2 & 8 & 1 & 6 & 17 & 21
\end{bmatrix}
$$

**Solution** The following row operations will transform $A$ to reduced echelon form.
1.2 Echelon Form and Gauss-Jordan Elimination

\[ R_1 \leftrightarrow R_3, (1/3)R_1: \] Introduce a leading 1 into the first row of the first nonzero column.

\[
\begin{bmatrix}
0 & 1 & -4 & -1 & -3 & -8 & -11 \\
0 & 0 & 0 & 1 & 3 & 11 & 9 \\
0 & 0 & 0 & 0 & 2 & 8 & 4 \\
0 & 0 & 0 & -1 & 0 & 1 & -1
\end{bmatrix}
\]

\[ R_4 + 2R_1: \] Introduce 0's below the leading 1 in row 1.

\[
\begin{bmatrix}
0 & 1 & -4 & -1 & -3 & -8 & -11 \\
0 & 0 & 0 & 1 & 3 & 11 & 9 \\
0 & 0 & 0 & 0 & 2 & 8 & 4 \\
0 & 0 & 0 & -1 & 0 & 1 & -1
\end{bmatrix}
\]

\[ R_1 + R_2, R_4 + R_2: \] Introduce 0's above and below the leading 1 in row 2.

\[
\begin{bmatrix}
0 & 1 & -4 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1 & 3 & 11 & 9 \\
0 & 0 & 0 & 0 & 2 & 8 & 4 \\
0 & 0 & 0 & 0 & 3 & 12 & 8
\end{bmatrix}
\]

\[ (1/2)R_3: \] Introduce a leading 1 into row 3.

\[
\begin{bmatrix}
0 & 1 & -4 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1 & 3 & 11 & 9 \\
0 & 0 & 0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 3 & 12 & 8
\end{bmatrix}
\]

\[ R_2 - 3R_3, R_4 - 3R_3: \] Introduce 0's above and below the leading 1 in row 3.

\[
\begin{bmatrix}
0 & 1 & -4 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}
\]

\[ (1/2)R_4: \] Introduce a leading 1 into row 4.

\[
\begin{bmatrix}
0 & 1 & -4 & 0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[ R_1 + 2R_4, R_2 - 3R_4, R_3 - 2R_4: \] Introduce 0's above the leading 1 in row 4.

\[
\begin{bmatrix}
0 & 1 & -4 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
Having provided this example of how to transform a matrix to reduced echelon form, we can be more specific about the procedure for solving a system of equations that is diagrammed in Fig. 1.3.

### Solving a System of Equations

Given a system of equations:

**Step 1.** Create the augmented matrix for the system.

**Step 2.** Transform the matrix in Step 1 to reduced echelon form.

**Step 3.** Decode the reduced matrix found in Step 2 to obtain its associated system of equations. (This system is equivalent to the original system.)

**Step 4.** By examining the reduced system in Step 3, describe the solution set for the original system.

The next example illustrates the complete process.

---

**Example 4**  
Solve the following system of equations:

\[
\begin{align*}
2x_1 - 4x_2 + 3x_3 - 4x_4 - 11x_5 &= 28 \\
-x_1 + 2x_2 - x_3 + 2x_4 + 5x_5 &= -13 \\
-3x_3 + x_4 + 6x_5 &= -10 \\
3x_1 - 6x_2 + 10x_3 - 8x_4 - 28x_5 &= 61.
\end{align*}
\]

**Solution**  
We first create the augmented matrix and then transform it to reduced echelon form. The augmented matrix is

\[
\begin{bmatrix}
2 & -4 & 3 & -4 & -11 & 28 \\
-1 & 2 & -1 & 2 & 5 & -13 \\
0 & 0 & -3 & 1 & 6 & -10 \\
3 & -6 & 10 & -8 & -28 & 61
\end{bmatrix}
\]

The first step is to introduce a leading 1 into row 1. We can introduce the leading 1 if we multiply row 1 by 1/2, but that would create fractions that are undesirable for hand work. As an alternative, we can add row 2 to row 1 and avoid fractions.

**\(R_1 + R_2: \)**

\[
\begin{bmatrix}
1 & -2 & 2 & -2 & -6 & 15 \\
-1 & 2 & -1 & 2 & 5 & -13 \\
0 & 0 & -3 & 1 & 6 & -10 \\
3 & -6 & 10 & -8 & -28 & 61
\end{bmatrix}
\]
\( \mathbf{R}_2 + \mathbf{R}_1, \mathbf{R}_4 - 3\mathbf{R}_1: \) Introduce 0's below the leading 1 in row 1.

\[
\begin{bmatrix}
1 & -2 & 2 & -2 & -6 & 15 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & -3 & 1 & 6 & -10 \\
0 & 0 & 4 & -2 & -10 & 16
\end{bmatrix}
\]

\( \mathbf{R}_1 - 2\mathbf{R}_2, \mathbf{R}_3 + 3\mathbf{R}_2, \mathbf{R}_4 - 4\mathbf{R}_2: \) Introduce 0's above and below the leading 1 in row 2.

\[
\begin{bmatrix}
1 & -2 & 0 & -2 & -4 & 11 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 1 & 3 & -4 \\
0 & 0 & 0 & -2 & -6 & 8
\end{bmatrix}
\]

\( \mathbf{R}_1 + 2\mathbf{R}_3, \mathbf{R}_4 + 2\mathbf{R}_3: \) Introduce 0's above and below the leading 1 in row 3.

\[
\begin{bmatrix}
1 & -2 & 0 & 0 & 2 & 3 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 1 & 3 & -4 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix above represents the system of equations

\[
\begin{align*}
x_1 - 2x_2 & + 2x_5 = 3 \\
x_3 & - x_5 = 2 \\
x_4 + 3x_5 & = -4.
\end{align*}
\]

Solving the preceding system, we find:

\[
\begin{align*}
x_1 & = 3 + 2x_2 - 2x_5 \\
x_3 & = 2 + x_5 \\
x_4 & = -4 - 3x_5
\end{align*}
\]

(1)

In Eq. (1) we have a nice description of all of the infinitely many solutions to the original system—it is called the *general solution* for the system. For this example, \(x_2\) and \(x_5\) are viewed as independent (or unconstrained) variables and can be assigned values arbitrarily. The variables \(x_1, x_3,\) and \(x_4\) are dependent (or constrained) variables, and their values are determined by the values assigned to \(x_2\) and \(x_5\). For example, in Eq. (1), setting \(x_2 = 1\) and \(x_5 = -1\) yields a *particular solution* given by \(x_1 = 7, x_2 = 1, x_3 = 1, x_4 = -1,\) and \(x_5 = -1.\)
system is inconsistent.

In Exercises 21–24, solve the system by transforming the augmented matrix into reduced echelon form.

In Exercises 25–29, solve the system by transforming the augmented matrix into reduced echelon form.

In Exercises 1–10, each of the given matrices represents a system of linear equations.

Consider the matrices in Exercises 1–10.

10. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

7. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

1. \[
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & -1 & 3 \\
0 & 2 & 1 & 4 \\
\end{bmatrix}
\]

Exercise 21:

To the teacher’s amazement, Gauss had the only correct answer in the class. Young Gauss had recognized that the number could be partitioned into 50 sets of pairs such that the sum of each pair was 10:

\[
\begin{array}{c}
(1 + 1) + (2 + 2) + (3 + 3) + \cdots + (50 + 50) = 2500 \\
50 \cdot 10 = 500 \\
\end{array}
\]

Chapter 1: Matrices and Systems of Linear Equations

Adding Integers
24. \( x_1 - x_2 + x_3 = 3 \)
25. \( x_1 + x_2 = 2 \)
26. \( x_1 - x_2 + x_3 = 4 \)
27. \( x_1 + x_2 - x_3 = 2 \)
28. \( x_1 + 3x_2 - 4x_3 = 3 \)
29. \( x_1 + x_2 - x_3 = 1 \)
30. \( x_1 + x_2 - x_5 = 1 \)
31. \( x_1 + x_3 + 4x_4 - 2x_5 = 1 \)
32. \( x_1 + x_2 = 1 \)
33. \( x_1 + x_2 = 1 \)
34. \( x_1 + 2x_2 = 1 \)
35. \( x_1 - x_2 - x_3 = 1 \)
36. \( x_1 + 2x_2 = -3 \)
37. \( x_1 + 3x_2 = 4 \)
38. \( 2x_1 + 4x_2 = a \)
39. \( 3x_1 + ax_2 = 3 \)
40. \( x_1 + ax_2 = 6 \)

In Exercises 41 and 42, find all values \( \alpha \) and \( \beta \) where \( 0 \leq \alpha \leq 2\pi \) and \( 0 \leq \beta \leq 2\pi \):

41. \( 2 \cos \alpha + 4 \sin \beta = 3 \)
42. \( 2 \cos^2 \alpha - \sin^2 \beta = 1 \)

43. Describe the solution set of the following system in terms of \( x_3 \):

\[ x_1 + x_2 + x_3 = 3 \]
\[ x_1 + 2x_2 = 5. \]

For \( x_1, x_2, x_3 \) in the solution set:

\[ a ) \text{ Find the maximum value of } x_3 \text{ such that } x_1 \geq 0 \text{ and } x_2 \geq 0. \]
\[ b ) \text{ Find the maximum value of } y = 2x_1 - 4x_2 + x_3 \text{ subject to } x_1 \geq 0 \text{ and } x_2 \geq 0. \]
\[ c ) \text{ Find the minimum value of } y = (x_1 - 1)^2 + (x_2 + 3)^2 + (x_3 + 1)^2 \text{ with no restriction on } x_1 \text{ or } x_2. \text{ [Hint: Regard } y \text{ as a function of } x_3 \text{ and set the derivative equal to 0; then apply the second-derivative test to verify that you have found a minimum.] } \]

44. Let \( A \) and \( I \) be as follows:

\[ A = \begin{bmatrix} 1 & d \\ c & b \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

Prove that if \( b - cd \neq 0 \), then \( A \) is row equivalent to \( I \).

45. As in Fig. 14, display all the possible configurations for a \( (2 \times 3) \) matrix that is in echelon form. [Hint: There are seven such configurations. Consider the various positions that can be occupied by one, two, or none of the symbols.]

46. Repeat Exercise 45 for a \( (3 \times 2) \) matrix, for a \( (3 \times 3) \) matrix, and for a \( (3 \times 4) \) matrix.

47. Consider the matrices \( B \) and \( C \):

\[ B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \]

By Exercise 44, \( B \) and \( C \) are both row equivalent to matrix \( I \) in Exercise 44. Determine elementary row operations that demonstrate that \( B \) is row equivalent to \( C \).

48. Repeat Exercise 47 for the matrices

\[ B = \begin{bmatrix} 1 & 4 \\ 3 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \]

49. A certain three-digit number \( N \) equals fifteen times the sum of its digits. If its digits are reversed, the resulting number exceeds \( N \) by 396. The one's digit is one larger than the sum of the other two. Give a linear system of three equations whose three unknowns are the digits of \( N \). Solve the system and find \( N \).

50. Find the equation of the parabola, \( y = ax^2 + bx + c \), that passes through the points \((-1, 6), (1, 4), \) and \((2, 9). \text{ [Hint: For each point, give a linear equation in } a, b, \text{ and } c.) \]

51. Three people play a game in which there are always two winners and one loser. They have the
understanding that the loser gives each winner an amount equal to what the winner already has. After three games, each has lost just once and each has $24. With how much money did each begin?

52. Find three numbers whose sum is 34 when the sum of the first and second is 7, and the sum of the second and third is 22.

53. A zoo charges $6 for adults, $3 for students, and $.50 for children. One morning 79 people enter and pay a total of $207. Determine the possible numbers of adults, students, and children.

54. Find a cubic polynomial, \( p(x) = a + bx + cx^2 + dx^3 \), such that \( p(1) = 5, p'(1) = 5, p(2) = 17 \), and \( p'(2) = 21 \).

In Exercises 55–58, use Eq. (2) to find the formula for the sum. If available, use linear algebra software for Exercises 57 and 58.

55. \( 1 + 2 + 3 + \cdots + n \)

56. \( 1^2 + 2^2 + 3^2 + \cdots + n^2 \)

57. \( 1^4 + 2^4 + 3^4 + \cdots + n^4 \)

58. \( 1^5 + 2^5 + 3^5 + \cdots + n^5 \)