5.5 DIMENSION

We now use Theorem 5 to generalize the idea of dimension to the general vector-space setting. We begin with two theorems that will be needed to show that dimension is a well-defined concept. These theorems are direct applications of the corollary to Theorem 5, and the proofs are left to the exercises because they are essentially the same as the proofs of the analogous theorems from Section 3.5.

THEOREM 6 If V is a vector space and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is a basis of V, then any set of $p + 1$ vectors in V is linearly dependent.

THEOREM 7 Let V be a vector space, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a basis for V. If $Q = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ is also a basis for V, then $m = p$.

If V is a vector space that has a basis of p vectors, then no ambiguity can arise if we define the dimension of V to be p (since the number of vectors in a basis for V is an invariant property of V by Theorem 7). There is, however, one extreme case, which is also included in Definition 6. That is, there may not be a finite set of vectors that spans V; in this case we call V an infinite-dimensional vector space.

DEFINITION 6 Let V be a vector space.

1. If V has a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of n vectors, then V has ***dimension n***, and we write $\dim(V) = n$. [If $V = \{\boldsymbol{\theta}\}$, then $\dim(V) = 0$.]
2. If V is nontrivial and does not have a basis containing a finite number of vectors, then V is an ***infinite-dimensional*** vector space.

We already know from Chapter 3 that R^n has dimension n. In the preceding section it was shown that $\{1, x, x^2\}$ is a basis for $\mathcal{P}_2$, so $\dim(\mathcal{P}_2) = 3$. Similarly, the set $\{1, x, \dots, x^n\}$ is a basis for $\mathcal{P}_n$, so $\dim(\mathcal{P}_n) = n + 1$. The vector space V consisting of all (2×2) real matrices has a basis with four vectors, namely, $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. Therefore, $\dim(V) = 4$. More generally, the space of all $(m \times n)$ real matrices has dimension mn because the $(m \times n)$ matrices E_{ij}, $1 \le i \le m$, $1 \le j \le n$, constitute a basis for the space.

EXAMPLE 1 Let W be the subspace of the set of all (2×2) matrices defined by

$$W = \{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 2a - b + 3c + d = 0\}.$$

Determine the dimension of W.

Solution The algebraic specification for W can be rewritten as $d = -2a + b - 3c$. Thus an element of W is completely determined by the three independent variables a, b, and c.

In succession, let $a = 1, b = 0, c = 0$; $a = 0, b = 1, c = 0$; and $a = 0, b = 0, c = 1$. This yields three matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$$

in W. The matrix A is in W if and only if $A = aA_1 + bA_2 + cA_3$, so $\{A_1, A_2, A_3\}$ is a spanning set for W. It is easy to show that the set $\{A_1, A_2, A_3\}$ is linearly independent, so it is a basis for W. It follows that $\dim(W) = 3$. ■

An example of an infinite-dimensional vector space is given next, in Example 2. As Example 2 illustrates, we can show that a vector space V is infinite dimensional if we can show that V contains subspaces of dimension k for $k = 1, 2, 3, \ldots$.

If W is a subspace of a vector space V, and if $\dim(W) = k$, then it is almost obvious that $\dim(V) \geq \dim(W) = k$ (we leave the proof of this as an exercise). This observation can be used to show that $C[a, b]$ is an infinite-dimensional vector space.

Example 2 Show that $C[a, b]$ is an infinite-dimensional vector space.

Solution To show that $C[a, b]$ is not a finite-dimensional vector space, we merely note that $\mathcal{P}_n$ is a subspace of $C[a, b]$ for every n. But $\dim(\mathcal{P}_n) = n + 1$; and so $C[a, b]$ contains subspaces of arbitrarily large dimension. Thus $C[a, b]$ must be an infinite-dimensional vector space. ■

Properties of a p-Dimensional Vector Space

The next two theorems summarize some of the properties of a p-dimensional vector space V and show how properties of R^p carry over into V.

Theorem 8 Let V be a finite-dimensional vector space with $\dim(V) = p$.

1. Any set of $p + 1$ or more vectors in V is linearly dependent.
2. Any set of p linearly independent vectors in V is a basis for V. ■

This theorem is a direct generalization from R^p (Exercise 20). To complete our discussion of finite-dimensional vector spaces, we state the following lemma.

Lemma Let V be a vector space, and let $Q = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be a spanning set for V. Then there is a subset Q' of Q that is a basis for V.

Proof (We only sketch the proof of this lemma because the proof follows familiar lines.) If Q is linearly independent, then Q itself is a basis for V. If Q is linearly dependent, we can express some vector from Q in terms of the other $p - 1$ vectors in Q. Without loss of generality, let us suppose we can express $\mathbf{u}_1$ in terms of $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$. In that event we have

$$\mathrm{Sp}\{\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\} = \mathrm{Sp}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\} = V;$$

if $\{\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$ is linearly independent, it is a basis for V. If $\{\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p\}$ is linearly dependent, we continue discarding redundant vectors until we obtain a linearly independent spanning set, Q'. ■

The following theorem is a companion to Theorem 8.

THEOREM 9 Let V be a finite-dimensional vector space with $\dim(V) = p$.

1. Any spanning set for V must contain at least p vectors.
2. Any set of p vectors that spans V is a basis for V.

Proof Property 1 follows immediately from the preceding lemma, for if there were a spanning set Q for V that contained fewer than p vectors, then we could find a subset Q' of Q that is a basis for V containing fewer than p vectors. This finding would contradict Theorem 7, so property 1 must be valid.

Property 2 also follows from the lemma, because we know there is a subset Q' of Q such that Q' is a basis for V. Since $\dim(V) = p$, Q' must have p vectors, and since $Q' \subseteq Q$, where Q has p vectors, we must have $Q' = Q$.

EXAMPLE 3 Let V be the vector space of all (2×2) real matrices. In V, set

$$A_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad A_5 = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}.$$

For each of the sets $\{A_1, A_2, A_3\}$, $\{A_1, A_2, A_3, A_4\}$, and $\{A_1, A_2, A_3, A_4, A_5\}$, determine whether the set is a basis for V.

Solution We have already noted that $\dim(V) = 4$ and that $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for V. It follows from property 1 of Theorem 9 that the set $\{A_1, A_2, A_3\}$ does not span V. Likewise, property 1 of Theorem 8 implies that $\{A_1, A_2, A_3, A_4, A_5\}$ is a linearly dependent set. By property 2 of Theorem 8, the set $\{A_1, A_2, A_3, A_4\}$ is a basis for V if and only if it is a linearly independent set. It is straightforward to see that the set of coordinate vectors $\{[A_1]_B, [A_2]_B, [A_3]_B, [A_4]_B\}$ is a linearly independent set. By Theorem 5 of Section 5.4, the set $\{A_1, A_2, A_3, A_4\}$ is also linearly independent; thus the set is a basis for V.

5.5 EXERCISES

1. Let V be the set of all real (3×3) matrices, and let V_1 and V_2 be subsets of V, where V_1 consists of all the (3×3) lower-triangular matrices and V_2 consists of all the (3×3) upper-triangular matrices.
   a) Show that V_1 and V_2 are subspaces of V.
   b) Find bases for V_1 and V_2.
   c) Calculate $\dim(V)$, $\dim(V_1)$, and $\dim(V_2)$.
2. Suppose that V_1 and V_2 are subspaces of a vecto space V. Show that $V_1 \cap V_2$ is also a subspace of V It is not necessarily true that $V_1 \cup V_2$ is a subspace of V. Let $V = R^2$ and find two subspaces of R^2 whose union is not a subspace of R^2.
3. Let V, V_1, and V_2 be as in Exercise 1. By Exercise 2, $V_1 \cap V_2$ is a subspace of V. Describe $V_1 \cap V_2$ and calculate its dimension.
4. Let V be as in Exercise 1, and let W be the subset of all the (3×3) symmetric matrices in V. Clearly W is a subspace of V. What is $\dim(W)$?

5. Recall that a square matrix A is called skew symmetric if $A^T = -A$. Let V be as in Exercise 1 and let W be the subset of all the (3×3) skew-symmetric matrices in V. Calculate $\dim(W)$.

6. Let W be the subspace of $\mathcal{P}_2$ consisting of polynomials $p(x) = a_0 + a_1x + a_2x^2$ such that $2a_0 - a_1 + 3a_2 = 0$. Determine $\dim(W)$.

7. Let W be the subspace of $\mathcal{P}_4$ defined thus: $p(x)$ is in W if and only if $p(1) + p(-1) = 0$ and $p(2) + p(-2) = 0$. What is $\dim(W)$?

In Exercises 8–13, a subset S of a vector space V is given. In each case choose one of the statements i), ii), or iii) that holds for S and verify that this is the case.

i) S is a basis for V.
ii) S does not span V.
iii) S is linearly dependent.

8. $S = \{1 + x - x^2, x + x^3, -x^2 + x^3\}$; $V = \mathcal{P}_3$

9. $S = \{1 + x^2, x - x^2, 1 + x, 2 - x + x^2\}$; $V = \mathcal{P}_2$

10. $S = \{1 + x + x^2, x + x^2, x^2\}$; $V = \mathcal{P}_2$

11. $S = \left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$;
V is the set of all (2×2) real matrices.

12. $S = \left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right\}$;
V is the set of all (2×2) real matrices.

13. $S = \left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}\right\}$;
V is the set of all (2×2) real matrices.

14. Let W be the subspace of $C[-\pi, \pi]$ consisting of functions of the form $f(x) = a \sin x + b \cos x$. Determine the dimension of W.

15. Let V denote the set of all infinite sequences of real numbers:

$$V = \{\mathbf{x}: \mathbf{x} = \{x_i\}_{i=1}^{\infty}, \quad x_i \text{ in } R\}.$$

If $\mathbf{x} = \{x_i\}_{i=1}^{\infty}$ and $\mathbf{y} = \{y_i\}_{i=1}^{\infty}$ are in V, then $\mathbf{x} + \mathbf{y}$ is the sequence $\{x_i + y_i\}_{i=1}^{\infty}$. If c is a real number, then $c\mathbf{x}$ is the sequence $\{cx_i\}_{i=1}^{\infty}$.

a) Prove that V is a vector space.
b) Show that V has infinite dimension. [*Hint:* For each positive integer, k, let $\mathbf{s}_k$ denote the sequence $\mathbf{s}_k = \{e_{ki}\}_{i=1}^{\infty}$, where $e_{kk} = 1$, but $e_{ki} = 0$ for $i \neq k$. For each positive integer n, show that $\{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ is a linearly independent subset of V.]

16. Let V be a vector space, and let W be a subspace of V, where $\dim(W) = k$. Prove that if V is finite dimensional, then $\dim(V) \geq k$. [*Hint:* W must contain a set of k linearly independent vectors.]

17. Let W be a subspace of a finite-dimensional vector space V, where W contains at least one nonzero vector. Prove that W has a basis and that $\dim(W) \leq \dim(V)$. [*Hint:* Use Exercise 36 of Section 5.4 to show that W has a basis.]

18. Prove Theorem 6. [*Hint:* Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a subset of V, where $k \geq p + 1$. Consider the vectors $[\mathbf{u}_1]_B, [\mathbf{u}_2]_B, \ldots, [\mathbf{u}_k]_B$ in R^p and apply Theorem 5 of Section 5.4.]

19. Prove Theorem 7.

20. Prove Theorem 8.

21. (*Change of basis;* see also Section 5.10). Let V be a vector space, where $\dim(V) = n$, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for V. Let $\mathbf{w}$ be any vector in V, and suppose that $\mathbf{w}$ has these representations in terms of the bases B and C:

$$\mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_n\mathbf{v}_n$$
$$\mathbf{w} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

By considering Eq. (10) of Section 5.4, convince yourself that the coordinate vectors for $\mathbf{w}$ satisfy

$$[\mathbf{w}]_B = A[\mathbf{w}]_C,$$

where A is the $(n \times n)$ matrix whose ith column is equal to $[\mathbf{u}_i]_B$, $1 \leq i \leq n$. As an application, consider the two bases for $\mathcal{P}_2$: $C = \{1, x, x^2\}$ and $B = \{1, x + 1, (x + 1)^2\}$.

a) Calculate the (3×3) matrix A just described.
b) Using the identity $[p]_B = A[p]_C$, calculate the coordinate vector of $p(x) = x^2 + 4x + 8$ with respect to B.

22. The matrix A in Exercise 21 is called a transition matrix and shows how to transform a representation with respect to one basis into a representation with respect to another. Use the matrix in part a) of Exercise 21 to convert $p(x) = c_0 + c_1x + c_2x^2$ to the form $p(x) = a_0 + a_1(x + 1) + a_2(x + 1)^2$, where:

a) $p(x) = x^2 + 3x - 2$;
b) $p(x) = 2x^2 - 5x + 8$;
c) $p(x) = -x^2 - 2x + 3$;
d) $p(x) = x - 9$.

23. By Theorem 5 of Section 5.4, an $(n \times n)$ transition matrix (see Exercises 21 and 22) is always nonsingular. Thus if $[\mathbf{w}]_B = A[\mathbf{w}]_c$, then $[\mathbf{w}]_C = A^{-1}[\mathbf{w}]_B$. Calculate A^{-1} for the matrix in part a) of Exercise 21 and use the result to transform each of the following polynomials to the form $a_0 + a_1x + a_2x^2$.

a) $p(x) = 2 - 3(x + 1) + 7(x + 1)^2$
b) $p(x) = 1 + 4(x + 1) - (x + 1)^2$
c) $p(x) = 4 + (x + 1)$
d) $p(x) = 9 - (x + 1)^2$

24. Find a matrix A such that $[p]_B = A[p]_c$ for all $p(x)$ in $\mathcal{P}_3$, where $C = \{1, x, x^2, x^3\}$ and $B = \{1, x, x(x - 1), x(x - 1)(x - 2)\}$. Use A to convert each of the following to the form $p(x) = a_0 + a_1x + a_2x(x - 1) + a_3x(x - 1)(x - 2)$.

a) $p(x) = x^3 - 2x^2 + 5x - 9$
b) $p(x) = x^2 + 7x - 2$
c) $p(x) = x^3 + 1$
d) $p(x) = x^3 + 2x^2 + 2x + 3$