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MATH 6701, AUTUMN 2026

M-W-F 10:20 a.m., EC 330

A DAY-BY-DAY LIST OF TOPICS

Included are references to the following texts, accessible through the course homepage at <https://people.math.osu.edu/derdzinski.1/courses/6701/6701.html>

[DG]: *Differential Geometry*, [DF]: *Distributions and the Frobenius Theorem*,
[PS]: *Projective Spaces and Grassmannians*, [OD]: *Ordinary Differential Equations*

August 26: Charts, compatibility, the definition of an n -dimensional atlas [DG, p.1]. The resulting topology [DG, Problem 4 on p.4]. Maximal atlases. The existence of a unique maximal atlas containing a given one [DG, p.1]. Definition of a (Hausdorff) manifold [DG, pp.1–2] including the countability axiom [DG, p.53], stated as the existence of a countable subatlas of the maximal atlas. (A manifold is thus a nonempty set with a fixed maximal atlas satisfying the Hausdorff and countability axioms.) Geometric properties, including openness of sets [DG, p.4]. Examples of manifolds: zero-dimensional ones (that is, nonempty countable sets); vector spaces [DG, pp.2]; affine spaces [DG, the first paragraph on p.3 and pp.191-192]; open submanifolds [DG, Problem 11 on p.5]. Cartesian products of manifolds [DG, Problems 1-3 on p.4].

August 28: Chart mappings taking values in affine spaces, such as cosets of vector subspaces [DG, p.3]. Openness of a subset of a chart domain equivalent to openness of its chart-mapping image [DG, Problem 10 on p.2]. The fact that the Hausdorff property follows when any two points lie in a common chart domain. Continuous and differentiable mappings between manifolds, homeomorphisms, diffeomorphisms [DG, pp.5–6]. Continuity in terms of convergent sequences on the one hand, and of pre-images of open sets on the other [DG, Problem 2 on p.8]. Unit Euclidean spheres, with stereographic projections serving as chart mappings [DG, p.3]. The formula $x \mapsto z = [2x - (1 + \langle x, v \rangle)v] / (1 - \langle x, v \rangle)$ for the stereographic projection $\Sigma \setminus \{v\} \rightarrow v^\perp - v = \{z : \langle v, z \rangle = -1\}$ from the pole $v \in \Sigma$, and $z \mapsto x = [(|z|^2 - 1)v + 4z] / (|z|^2 + 3)$ for its inverse, Σ being the unit sphere centered at 0 in a Euclidean space with the inner product $\langle \cdot, \cdot \rangle$. Tori, defined to be products of circles [DG, p.3]. Real and complex projective spaces [DG, pp.3–4; [PS, p.1], with the Hausdorff property obvious since any two points lie in a common chart domain, and the countability axiom immediate from the existence of a finite subatlas, generated by a basis of the vector space in question.

August 31: Gluing of two n -dimensional manifolds along a diffeomorphism between their open submanifolds, the result being “almost” a manifold, except for the Hausdorff axiom which need not hold [DG, p.7]. Connected sums of manifolds [DG, p.7]. Closed orientable surfaces of any nonnegative genus [DG, p.8], and the nonorientable ones, defined to be connected sums of real projective planes. Closedness and compactness of subsets of manifold, the former meaning openness of the complement [DG, Problem 5 on p.2], the latter defined in terms of sequences [DG, p.4]. Closedness versus convergent sequences [DG, Problem 5 on p.2]. Closedness of compact sets and compactness of their closed subsets; the fact that a continuous image of a compact set is compact, while a bijective continuous mapping from a compact set is a homeomorphism [DG, Problem 4 on p.9]. Compactness in Euclidean spaces, equivalent to being closed and bounded [DG, Problem

5 on p.9]. The fact that Euclidean spheres and all projective spaces are compact manifolds [DG, Problem 15 on p.10], while compactness of a zero-dimensional manifold amounts to its finiteness [DG, Problem 1 on p.8]. Functions on, and curves in manifolds [DG, p.6].

September 2: (Pathwise) connectedness of sets in manifolds [DG, p.6]. Preservation of connectedness under continuous images and under Cartesian products [DG, Problems 4, 7, 8 on p.9]. Connectedness of spheres except in dimension zero, and of all projective spaces [DG, Problem 15 on p.10]. The fact that the union of any nonempty family of connected sets with a nonempty intersection is necessarily connected, while nonempty connected subsets of $\mathbb{R}$ are precisely the intervals. Connectedness of Cartesian products of connected subsets of manifolds [DG, p.9]. Connected components of a nonempty set K in a manifold, defined to be the equivalence classes of the relation in which two points of K are called equivalent if they are joined by a continuous curve in K . The observation that any nonempty connected subset K' of K is contained in a (unique) connected component of K , which is also the largest connected subset of K containing K' [DG, p.6]. Openness the connected components of an open set, and simultaneous openness and closedness of the connected components of a manifold [DG, Problem 10 on p.9]. The union of a nonempty countable family of pairwise disjoint n -dimensional manifolds (which is again an n -dimensional manifold) and the immediate conclusion that every manifold equals the union, in this sense, of the family of its connected components [DG, p.6]. Lie groups [DG, p.12]. Examples: countable groups, the additive group of a finite-dimensional real/complex vector space [DG, p.12]. Open subgroups of Lie groups G , constituting Lie groups in their own right [DG, Problem 2 on p.14], an example being the identity component G° of G [DG, p.43]. The group of invertible elements of a finite-dimensional real associative algebra $\mathcal{A}$ with unity [DG, p.12], and its openness in $\mathcal{A}$ due to the mutual equivalence of three conditions imposed on $x \in \mathcal{A}$, which are: invertibility of x , injectivity/surjectivity of L_x , and injectivity/surjectivity of R_x , so that invertibility of x amounts to $\det L_x \neq 0$ [DG, Problem 4 on p.15]. The specific implication from injectivity/surjectivity of L_x to invertibility of x , by first finding y with $xy = 1$ (using surjectivity), so that $L_x 1 = x1 = 1x = (xy)x = x(yx) = L_x(yx)$, and injectivity gives $yx = 1$. The automorphism group $\text{GL}(V)$ of any finite-dimensional real/complex vector space V , including $\text{GL}(n, \mathbb{K})$ for $V = \mathbb{K}^n$ [DG, p.12]. Connectedness of the automorphism group $\text{GL}(V)$ (and of the set $\mathcal{B}(V)$ of all ordered bases) of a finite-dimensional complex vector space V , based on the fact – established by induction on $\dim V$ with the aid of the fundamental theorem of algebra – that any endomorphism of V is, in a suitable basis, represented by an upper triangular matrix. Such a matrix A , if nonsingular, has the form $\text{diag}(e^{z_1}, \dots, e^{z_n}) + B$, where B is upper triangular with zeros on the diagonal, and the curve $[0, 1] \ni t \mapsto \text{diag}(e^{tz_1}, \dots, e^{tz_n}) + tB$ in $\text{GL}(n, \mathbb{C})$ joins the identity to A .

September 4: Orientations in a real vector space of a positive finite dimension [DG, pp.192–193]. The algebra $\mathbb{H}$ of quaternions [DG, p.13]. The Lie-group structures of $\mathbb{R} \setminus \{0\}$, $\mathbb{C} \setminus \{0\}$, $\mathbb{H} \setminus \{0\}$ and of the unit spheres S^0, S^1, S^3 in $\mathbb{R}, \mathbb{C}, \mathbb{H}$ [DG, p.14]. The index notation for manifolds [DG, p.17] and vector spaces.

September 9: The chart-dependent partial derivatives [DG, p.17]. The chain rule for ∂_j [DG, p.18]. Components of mappings [DG, p.19]. Curves and tangentiality, tangent vectors, velocity, vector components and the transformation rule [DG, p.18]. The tangent vector space of a manifold at a point [DG, p.18]. Directional derivatives as scalar-valued derivations of the algebra of germs of functions at a point [DG, p.19].

September 11: Tangent spaces in vector and affine spaces [DG, p.19, Examples 5.1–5.2]. Tangent spaces of open submanifolds [DG, p.20, Example 5.3]. The basis $p_j(z)$ of the tangent space $T_z M$ naturally distinguished by the given chart x^j [DG, p.20]. Differentials of differentiable mappings and their expression in coordinates [DG, p.20]. The chain rule for differentials [DG, p.20]. Invariance of the dimension under diffeomorphisms [DG, Problem 10 on p.21]. Differentials of linear and affine mappings [DG, Problems 8–9 on p.21]. Differentials of real-valued functions.

September 14: Differentials of vector-valued functions. Tangent vectors as derivations [DG, Problem 11 on p.22 and Problem 1 on p.24]. Dual bases in finite-dimensional vector spaces [DG, p.19]. Cotangent spaces and vectors [DG, p.20]. Bases of tangent and cotangent spaces naturally distinguished by a given chart [DG, p.20]. Tangent vector fields on manifolds, directional derivatives along vector fields [DG, p.22]. Derivations of an associative algebra (which need not be commutative), and the trivial observation that the commutator of two derivations is a derivation [DG, p.26, Example 7.5]. The Lie bracket and its interpretation as a commutator of directional differentiations [DG, p.23].

September 16: Abundance of cut-off functions and global extensibility of germs [DG, p.217]. The Jacobi identity [DG, Problem 5 on p.24]. Lie algebras [DG, pp.26–27]. Examples: Abelian Lie algebras with the zero multiplication, the Lie algebra $\mathcal{X}M$ of C^∞ vector fields on a manifold M , with the Lie-bracket operation, and the underlying Lie algebra of an associative algebra, with the commutator operation [DG, p.26, Examples 7.1–7.3]. Vector fields projectable under mappings [DG, p.23]. Projectability of Lie brackets [DG, p.24].

September 18: Projectability of vector fields under diffeomorphisms and diffeomorphic push-forwards [DG, p.24]. Arbitrary mappings $M \times N \rightarrow P$ of class C^1 written as multiplications $(x, y) \mapsto xy$, the vectors $xw, vy \in T_{xy}P$ arising from $v \in T_x M$ and $w \in T_y N$, with $(x, y) \in M \times N$, and the Leibniz rule $(xy)' = \dot{x}y + x\dot{y}$ [DG, pp.38–39]. More on Lie algebras: homomorphisms and examples [DG, pp.26–27]. The Lie algebra of (left-invariant vector fields on) a Lie group G and its identification with $T_1 G$ [DG, p.28]. The cases of the additive group of a vector space, where the Lie algebra is Abelian [DG, p.28, Example 8.1] and of the group G of invertible elements of a finite-dimensional real associative algebra $\mathcal{A}$ with unity, the Lie algebra $\mathfrak{g} = T_1 G = \mathcal{A}$ having the Lie bracket equal to the commutator in $\mathcal{A}$ [DG, p.28, Example 8.2], which includes $\text{GL}(V)$ for a finite-dimensional real/complex vector space [DG, p.28, Example 8.23]. Projectability of left-invariant fields under Lie-group homomorphisms [DG, p.29]. The Lie-algebra homomorphism induced by a Lie-group homomorphism [DG, p.29].

September 21: Volume-form characterizations of the determinant and trace [DG, Problem 12 p.31]. The derivative of the determinant for a C^1 curve of linear automorphisms

of a finite-dimensional real or complex vector space [DG, Problem 13 p.31]. The conclusion that the determinant, as a Lie-group homomorphism $GL(V) \rightarrow \mathbb{K} \setminus \{0\}$, induces the Lie-algebra homomorphism $\text{tr} : \mathfrak{gl}(V) \rightarrow \mathbb{K}$ equal to the trace functional [DG, p. 30, Example 8.9]. Metric spaces, open and closed metric balls, open sets, convergence of sequences and continuity of mappings, Cauchy sequences and completeness [IM, pp.1]. The Lipschitz condition, implying continuity. The “neat” version of Banach’s fixed-point theorem [IM, the Corollary on p.2].

September 23: Normed vector spaces as metric spaces [IM, p.1]. Mutual equivalence of all norms in a finite-dimensional real or complex vector space [DG, Problem 7–9 on p.196]. The operator norm [DG, p.198] and the Lipschitz estimate for C^1 mappings along intervals [IM, formula (2)] and [DG, formula (74.5) on p.199]. The “convoluted” or “messy” version of Banach’s fixed-point theorem [IM, p.1]. The regularity part (differentiability of the inverse) in the inverse mapping theorem [IM, the Lemma on p.2].

September 25: The regularity part, continued. The inverse mapping theorem, obtained as a consequence of the “convoluted” version of Banach’s fixed-point theorem [IM, pp.2–3]. The statement of the implicit mapping theorem [DG, p.199]. The case of polynomials P and simple roots: if P , of degree not exceeding n , has a root t with $P'(t) \neq 0$, then every Q near P has a unique root near t , depending smoothly on Q .

September 28: Tangent spaces of Cartesian products [DG, Problem 28 on p.38]. The proof of the implicit mapping theorem [DG, p.199]. The rank of a mapping at a point, openness of the maximum-rank subset, and the rank theorem [DG, p.33].

September 30: Submersions and their openness [DG, p.38]. Immersions and their local injectivity [DG, p.34]. Embeddings, submanifolds, with or without the subset topology [DG, p.34]. Examples: open submanifolds [DG, p.34], the figure eight in the plane. The subset topology as a consequence of compactness [DG, Problem 2 on p.55]. Continuity versus differentiability for submanifold-valued mappings [DG, p.34]. Uniqueness of submanifold structure with the subset topology [DG, p.35]. Critical and regular points and values of mappings, submanifolds defined by equations, their dimensions and tangent spaces [DG, pp.35–36]. Euclidean spheres as an example [DG, p.36].

October 2: Preimages of points under constant-rank mappings [DG, p.41]. Lie subgroups of Lie groups and their Lie algebras [DG, pp.43–44]. Lie-group actions. Isotropy groups [DG, p.46].

October 5: The uniform distance and supremum norm [OD, p.1]. Ordinary differential equations [OD, p.1]. Existence and uniqueness of solutions for ordinary differential equations with initial conditions, derived from the “neat” version of Banach’s fixed-point theorem. [OD, p.2].