MATH 6701, AUTUMN 2013

Distributions and the Frobenius Theorem

By a p-dimensional *distribution* on a manifold M we mean a smooth subbundle η of fibre dimension p in the tangent bundle TM. Given a p-dimensional distribution $\eta \subset TM$, a submanifold N of M is said to be an *integral manifold* of η if $T_xN = \eta_x$ for every $x \in N$ (thus, $\dim N = p$), and a distribution η on M is called *integrable* if every point of M lies in an integral manifold of η.

Let η be a distribution on a manifold M, and let ζ be the *normal bundle* of η, that is, the quotient vector bundle TM/η. The *curvature* of η is the smooth section Ω of $\mathrm{Hom}(\eta^{\wedge 2}, \zeta)$ such that, for any smooth local sections v, w of η, we have $\Omega(v,w) = \pi[v,w]$, where $\pi : TM \to \zeta$ is the quotient projection.

The Frobenius Theorem. *A distribution is integrable if and only if its curvature is identically zero.*

In other words, integrability of a distribution $\eta \subset TM$ means precisely that the set of smooth local sections of η is closed under the Lie-bracket operation.

Before proving the Frobenius theorem, let us note how it provides a solvability criterion for systems of first-order partial differential equations that are in normal form (solved for the partial derivatives), in the sense of being written as

$$\partial_j y^a \;=\; F_j^a(x^1, \dots, x^p, y^{p+1}, \dots, y^m). \tag{1}$$

Here x^j (with $1 \le j \le p$) are the independent variables, y^a (with $p+1 \le a \le m$) are the unknown functions, ∂_j stands for the partial derivative $\partial/\partial x^j$, and F_j^a are fixed smooth functions of m variables x^j, y^a, defined on some open set U in $\mathrm{I\!R}^m$.

One calls the system (1) *completely integrable* if, for every $z \in \mathrm{I\!R}^p$ and every $w \in \mathrm{I\!R}^{m-p}$ with $(z,w) \in U$, there exists a solution $y = (y^{p+1}, \dots, y^m)$ to (1), defined on a neighborhood of z in $\mathrm{I\!R}^p$ and such that $y(z) = w$.

One can naturally associate with the system (1) a p-dimensional distribution η on U, given by

$$dy^a \;=\; F_j^a\, dx^j, \tag{2}$$

and then complete integrability of (1) is equivalent to integrability of the distribution η with (2); more precisely, integral manifolds of η are, locally, the same as graphs of solutions to (1).

Furthermore, the infinitesimal condition that, according to the Frobenius theorem, is equivalent to integrability of η (namely, closedness of the set of smooth local sections of η under the Lie bracket) amounts to the following property of "intrinsic consistency" for (1): the system

$$\partial_k F_j^a + F_k^b \partial_b F_j^a = \partial_j F_k^a + F_j^b \partial_b F_k^a \tag{3}$$

of partial differential equations imposed on the functions F_j^a, arising when one rewrites the equalities $\partial_k \partial_j y^a = \partial_j \partial_k y^a$ with the aid of (1), is satisfied identically on U. In fact,

$$\Omega_{jk}^a \;=\; \partial_j F_k^a - \partial_k F_j^a + F_j^b \partial_b F_k^a - F_k^b \partial_b F_j^a \tag{4}$$

for the curvature Ω of η, in the local trivializations for η and $\zeta = TM/\eta$ given by $e_j = p_j + F_j^a p_a$ and $e_a = \pi p_a$, where p_j, p_a are the coordinate vector fields.

As $\dot{y}^a = \dot{x}^j \partial_j y^a$ by the chain rule, (1) gives

$$\dot{y}^a \;=\; \dot{x}^j F_j^a(x^1, \ldots, x^p, y^{p+1}, \ldots, y^m). \tag{5}$$

further text in preparation

Exercise. Verify (4) and the fact that the vector fields $e_j = p_j + F_j^a p_a$ form a local trivialization for η (and, in particular, are local sections of η).