MATH 6701, AUTUMN 2024

Ordinary Differential Equations

By V and W we always denote finite-dimensional normed real vector spaces.

A mapping $f : K \to X$ from a set K into a metric space (X, d) is said to be *bounded* if its image $f(K)$ is a bounded subset of (X, d) in the sense that it lies in a ball $B_z(r)$ with some center $z \in X$ and some radius $r > 0$. Let us denote $\mathcal{X} = B(K, X)$ the set of all bounded mappings $f : K \to X$ and define the *uniform distance* function $\mathrm{d}_{\mathrm{sup}} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\text{(dsu)} \qquad \mathrm{d}_{\mathrm{sup}}(f, f') = \sup \{\mathrm{d}\,(f(x), f'(x)) : x \in K\}.$$

Endowed with $\mathrm{d}_{\mathrm{sup}}$, the set $\mathcal{X}$ becomes a metric space (Problem .1); the convergence in $(\mathcal{X}, \mathrm{d}_{\mathrm{sup}})$ is called the *uniform convergence* of bounded mappings $K \to X$.

In the case where (X, d) is the underlying metric space of a normed vector space $(X, |\ |)$ (see **Homework #, Appendix**), and K is any set, it is clear that $(\mathcal{X}, \mathrm{d}_{\mathrm{sup}}) = (B(K, X), \mathrm{d}_{\mathrm{sup}})$ is the underlying metric space of the normed vector space $(\mathcal{X}, \|\ \|_\infty$ with the valuewise operations on X-valued functions f on K and the *supremum norm* $\|f\|_\infty = \sup \{|f(x)| : x \in K\}$.

If, moreover, K happens to be a manifold or a metric space, the set $\mathcal{X} = B(K, X)$ contains the subset $C_B(K, X)$ formed by all bounded mappings $K \to X$ which are also *continuous*. (In both cases, a mapping $f : K \to N$ is said to be *continuous* if $f(x_k) \to f(x)$ in X as $k \to \infty$ whenever x_k, $k = 1, 2, \ldots$, is a sequence of points in K that converges to a point $x \in K$.) When K is compact, we write $C(K, X)$ rather than $C_B(K, X)$, deleting the subscript 'B' as boundedness then follows from continuity (Problem .13). With the restriction of the distance function $\mathrm{d}_{\mathrm{sup}}$, the set $C_B(K, X)$ constitutes a metric space which is *complete* whenever so is (X, d) (Problems .2, .3 and .15).

We say that a mapping $f : K \to X$ between metric spaces (with both distances denoted d) satisfies the *Lipschitz condition* if there exists a constant $C \ge 0$ such that $\mathrm{d}(F(x), F(y)) \le C\,\mathrm{d}(x, y)$ for all $x, y \in K$. For instance, Problem .12 states that any norm satisfies the Lipschitz condition with $C = 1$. Note that the Lipschitz condition implies continuity.

Let us now consider an open subset U of a finite-dimensional real vector space V. By an *ordinary differential equation* of order $k \ge 1$ in U we mean a mapping $F : I \times U \times V^{k-1} \to V$, where $I \subset \mathbb{R}$ is an open interval and $V^{k-1} = V \times \ldots \times V$ is the $(k-1)$st Cartesian power of V. A C^k-differentiable curve $\gamma : I' \to V$ defined on a subinterval I' of I (open or not) is called a *solution* to the equation if

$$\text{(.1)} \qquad \gamma^{(k)} = F(t, \gamma, \dot\gamma, \ldots, \gamma^{(k-1)})$$

in the sense that $\gamma^{(k)}(t) = F(t, \gamma(t), \dot\gamma(t), \ldots, \gamma^{(k-1)}(t))$ for all $t \in I'$, where $\gamma^{(k)} = d^k\gamma/dt^k$. Such a solution is said to satisfy the *initial condition* $(t_0, x_0, v_1, \ldots, v_{k-1})$ if $(t_0, x_0, v_1, \ldots, v_{k-1}) \in I \times U \times V^{k-1}$ and

$$\text{(.2)} \qquad \gamma(t_0) = x_0\,,\ \dot\gamma(t_0) = v_1\,, \ldots,\ \gamma^{(k-1)}(t_0) = v_{k-1}\,.$$

The mapping F is usually referred to as the *right-hand side* of the equation (rather than being called the equation itself). Together, (.1) and (.2) are said to form a *kth order initial value problem.* An initial value problem (.1), (.2) of any order $k > 1$ can always be reduced to a first-order problem $\dot{\chi} = \Phi(t,\chi)$, $\chi(t_0) = z_0$ in the open set $U \times V^{k-1}$ of the higher-dimensional space V^k, by setting $\chi(t) = (\gamma(t), \dot{\gamma}(t), \ldots, \gamma^{(k-1)}(t)) \in U \times V^{k-1}$, $\Phi(t, x, w_1, \ldots, w_{k-1}) = (w_1, \ldots, w_{k-1}, F(t, x, w_1, \ldots, w_{k-1}))$ for $(t, x, w_1, \ldots, w_{k-1}) \in I \times U \times V^{k-1}$, and $z_0 = (x_0, v_1, \ldots, v_{k-1})$. The theorem proved below for $k = 1$ can therefore be easily extended to initial value problems of any order k (Problem .7).

The Existence and Uniqueness Theorem. *Let $I \subset \mathrm{I\!R}$ be an open interval, and let U be an open subset of a finite-dimensional real vector space V. If $F : I \times U \to V$ is continuous and satisfies the Lipschitz condition in $x \in U$ uniformly in $t \in I$, i.e., $|F(t,x') - F(t,x)| \le C|x' - x|$ for some fixed norm $|\ |$ in V, some constant $C \ge 0$, and all $t \in I$, $x, x' \in U$, then, for any initial condition $(t_0, x_0) \in I \times U$ there is $\varepsilon > 0$ such that the equation $\dot{\gamma} = F(t,\gamma)$ has a unique C^1 solution $\gamma : [t_0 - \varepsilon, t_0 + \varepsilon] \to U$ with $\gamma(t_0) = x_0$.*

Remark .1. The condition imposed on ε is

$$\varepsilon s_\varepsilon < (1 - C\varepsilon)\delta\,, \tag{.3}$$

with C, $|\ |$ as above, $s_\varepsilon = \sup\{|F(t,x_0)| : |t - t_0| \le \varepsilon\}$ and $\delta = \inf\{|y - x_0| : y \in V \smallsetminus U\} \in (0,\infty]$ equal to the distance between x_0 and the complement (or boundary) of U, where $r = \infty$ if $U = V$. Since $s_\varepsilon \to |F(t_0,x_0)|$ as $\varepsilon \to 0$, (.3) holds for all sufficiently small $\varepsilon > 0$.

Proof. For $\gamma : [t_0 - \varepsilon, t_0 + \varepsilon] \to U$, the requirement that γ be C^1 and satisfy $\dot{\gamma} = F(t,\gamma)$ and $\gamma(t_0) = x_0$, is equivalent to continuity of γ along with

$$\gamma(t) = x_0 + \int_{t_0}^{t} F(\tau, \gamma(\tau))\, d\tau \tag{.4}$$

for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. Let $\mathcal{X}_\varepsilon$ be the Banach metric space $C([t_0 - \varepsilon, t_0 + \varepsilon], V)$ with the supremum norm $\|\ \|_{\mathrm{sup}}$ defined above using the norm $|\ |$ in V. The mapping $h_\varepsilon : \mathcal{K}_\varepsilon \to \mathcal{X}_\varepsilon$ from the subset $\mathcal{K}_\varepsilon = C([t_0 - \varepsilon, t_0 + \varepsilon], U)$ of $\mathcal{X}_\varepsilon$ into $\mathcal{X}_\varepsilon$, given by $[h_\varepsilon(\gamma)](t) = x_0 + \int_{t_0}^{t} F(\tau,\gamma(\tau))\, d\tau$ then satisfies $\|h_\varepsilon(\gamma') - h_\varepsilon(\gamma)\|_{\mathrm{sup}} \le C\varepsilon\|\gamma' - \gamma\|_{\mathrm{sup}}$, as the length of the integration interval is $|t - t_0| \le \varepsilon$ and C is a Lipschitz constant for F. Denoting z the constant curve $x_0 \in \mathcal{K}_\varepsilon$ and setting $r_\varepsilon = (1 - C\varepsilon)^{-1}\|z - h_\varepsilon(z)\|_{\mathrm{sup}}$, we obtain $r_\varepsilon \le (1 - C\varepsilon)^{-1}\varepsilon s_\varepsilon$. Thus, for ε chosen as in (.3), $r_\varepsilon < \delta$ and hence the ball $B_z(r_\varepsilon)$ in $\mathcal{X}_\varepsilon$ is contained in $\mathcal{K}_\varepsilon$. The assumptions of Banach's fixed-point theorem (**Homework #, Appendix**) thus will be satisfied if we replace $X, \mathrm{d}\,, K, h, C, z, r$ in the statement of that theorem by $\mathcal{X}_\varepsilon$, $\mathrm{d}_{\mathrm{sup}}$, $\mathcal{K}_\varepsilon$, h_ε, $C\varepsilon$, $z = x_0$, and, respectively, r_ε, for any ε with (.3). The resulting existence and uniqueness of $\gamma \in \mathcal{K}_\varepsilon$ with $h_\varepsilon(\gamma) = \gamma$, i.e., (.4), now proves our assertion.

Global Solutions to Linear Differential Equations

Given an interval $I \subset \mathrm{I\!R}$ containing more than one point and otherwise arbitrary (so that I may be open, closed, or half-open, bounded or unbounded), and a nonnegative continuous function $h : I \to [0,\infty)$, we set

$$\int_I h(t)\,dt = \sup_{a,b\in I} \int_a^b h(t)\,dt \in [0,\infty]\,. \tag{.1}$$

Note that $\int_I h(t)\,dt$ equals the limit of $\int_a^b h(t)\,dt$ as $a \to \inf I(+)$ and, simultaneously, $b \to \sup I(-)$. (The limit always exists for reasons of monotonicity.)

The following results concerning *differential inequalities* will later be applied to linear ordinary differential equations and the local regularity theorem.

Lemma .1. *Suppose that* $I \subset \mathrm{I\!R}$ *is an interval,* $h : I \to [0,\infty)$ *is a continuous function, and* $\gamma : I \to V$ *is a* C^1 *mapping of* I *into a finite-dimensional real vector space* V *with an inner product* $\langle\,,\rangle$*. If, in addition,*

$$\text{a)}\quad C = \int_I h(t)\,dt < \infty, \qquad \text{b)}\quad |\dot\gamma| \le h|\gamma| \tag{.2}$$

everywhere in I*,* $|\;|$ *being the norm in* V *determined by* $\langle\,,\rangle$*, then*

$$\sup_I |\gamma| \le e^C \inf_I |\gamma|\,. \tag{.4}$$

Remark. By (.3), whenever $\gamma, h, I, V, \langle\,,\rangle$ satisfy the hypotheses of the lemma, then either $\gamma = 0$ identically, or $\gamma \ne 0$ everywhere in I. This fact, however, will have to be established separately in the course the following proof.

Proof. We may assume that γ is not identically zero. By the Schwarz inequality and (.3), $\varphi = \langle\gamma,\gamma\rangle : I \to [0,\infty)$ satisfies $|\dot\varphi| = 2|\langle\gamma,\dot\gamma\rangle| \le 2h\varphi$. Thus, if $a, b \in I$ and $\gamma \ne 0$ everywhere in the closed interval $\overline{ab}$ connecting a and b, we have

$$|\gamma(b)| \le e^{C(a,b)}|\gamma(a)|, \quad \text{where} \quad C(a,b) = \int_{\overline{ab}} h(t)\,dt, \tag{.5}$$

as $2\log|\gamma(b)| - 2\log|\gamma(a)| = \log\varphi(b) - \log\varphi(a) = \int_a^b \varphi^{-1}\dot\varphi\,dt \le 2\int_{\overline{ab}} h(t)\,dt$. Consequently, $\gamma \ne 0$ everywhere in I. In fact, otherwise, we could select a maximal open subinterval I' of I with $\gamma \ne 0$ everywhere in I', so that $\gamma(c) = 0$ for at least one endpoint $c \in I$ of I'; fixing $b \in I'$ and letting $a \in I'$ vary, we would obtain the contradiction $0 < |\gamma(b)| \le 0$ by taking the limit of (.5) as $a \to c$ and noting that $\int_{\overline{cb}} h(t)\,dt < \infty$. Therefore, by (.5), $|\gamma(b)| \le e^C|\gamma(a)|$ for all $a, b \in I$, with C as in (.4), and we can take the supremum over b and infimum over a.

Corollary .3. *Let* $\gamma, h, I, V, \langle\,,\rangle$ *satisfy the hypotheses of* **Lemma .1**. *Then* γ *has a limit at each finite endpoint of* I*, while the endpoint itself does not have to belong to* I*.*

In fact, by (.3), (.4) we have $\int_I |\dot\gamma(t)|\,dt \le Ce^C \inf_I |\gamma|$, and so we can use Problem .2.

Let U now be an open subset of V, and let $I \subset \mathrm{I\!R}$ be an open interval. A kth order ordinary differential equation (.1) in U, i.e.,

$$\gamma^{(k)} = F(t, \gamma, \dot\gamma, \ldots, \gamma^{(k-1)})\,,$$

is called *linear* if its right-hand side

$$F : I \times U \times V^{k-1} \to V$$

has the form

$$F(t, x, w_1, \dots, w_{k-1}) \;=\; B_0(t)x + B_1(t)w_1 + \dots + B_{k-1}(t)w_{k-1}$$

with some *coefficient functions* (curves) $B_0, B_1, \dots, B_{k-1} : I \to \mathrm{Hom}(V, V)$ valued in the vector space of all linear mappings $V \to V$. In other words, a linear kth order equation reads

$$\gamma^{(k)} \;=\; B_0(t)\gamma + B_1(t)\dot\gamma + \dots + B_{k-1}(t)\gamma^{(k-1)}\,. \tag{.6}$$

Note that, due to linearity of the $B_0(t), B_1(t), \dots, B_{k-1}(t)$, we may always assume that $U = V$.

A linear equation (.6) of any order $k > 1$ can always be reduced to a first-order linear equation $\dot\chi = A(t)\chi$ in a higher-dimensional space, with the coefficient curve $t \mapsto A(t)$ of the same regularity as the original $B_0, B_1, \dots, B_{k-1}$ (Problem .6).

Proposition .4. *Suppose that V is a finite-dimensional real vector space and $I \subset \mathrm{I\!R}$ is an open interval. If $F : I \times V \to V$ is continuous and locally Lipschitz in $x \in U$, locally uniformly in $t \in I$* (Problem .6*, and satisfies the inequality*

$$|F(t, x)| \le h(t)|x| \tag{.7}$$

for all $(t, x) \in I \times V$, where $h : I \to [0, \infty)$ is a continuous function and $|\ |$ is a fixed norm in V, then for any $(t_0, x_0) \in I \times V$ the initial value problem

$$\dot\gamma = F(t, \gamma)\,, \qquad \gamma(t_0) = x_0 \tag{.8}$$

has a unique solution $\gamma : I \to V$ defined everywhere *in I.*

Proof. We may assume that $|\ |$ is the norm determined by an inner product $\langle\,,\rangle$ in V (Problem .18). Let $\gamma : (a, b) \to V$ be the (unique) solution to (.8) defined on the largest possible interval $(a, b) \subset I$ with $t_0 \in (a, b)$ (Problem .5). To show that $(a, b) = I$, suppose on the contrary that, for instance, $b \in I$. Applying **Corollary .3** to $[t_0, b]$ instead of I, we see that $\gamma(t)$ has a limit y_0 as $t \to b(-)$, and so from the existence theorem (see, e.g., Problem .6), there is $\varepsilon > 0$ with $b + \varepsilon \in I$ and a C^1 curve $\gamma_1 : [b, b + \varepsilon) \to V$ with $\dot\gamma_1 = F(t, \gamma_1)$ and $\gamma_1(b) = y_0$. Combining γ with γ_1 as in Problem .1, we obtain a C^1 solution to (.8) defined on $(a, b + \varepsilon)$, which contradicts maximality of (a, b) and thus completes the proof.

The Global Existence Theorem for Linear Ordinary Differential Equations. *Every linear initial value problem*

$$\begin{aligned} \gamma^{(k)} &= B_0(t)\gamma + B_1(t)\dot\gamma + \dots + B_{k-1}(t)\gamma^{(k-1)}\,, \\ \gamma(t_0) &= x_0\,,\ \dot\gamma(t_0) = v_1\,,\ \dots\,,\ \gamma^{(k-1)}(t_0) = v_{k-1} \end{aligned} \tag{.9}$$

of order $k \ge 1$ in a finite-dimensional real vector space V, with continuous coefficient functions $B_0, B_1, \dots, B_{k-1} : I \to \mathrm{Hom}(V, V)$, where $I \subset \mathrm{I\!R}$ is an open interval, has a unique solution $\gamma : I \to V$ defined on the whole *interval I.*

Proof. Fix a norm $|\ |$ in V. We may assume that $k = 1$ (Problem .6), so that (.9) becomes $\dot\gamma = B(t)\gamma$ with $\gamma(t_0) = x_0$. Thus, (.7) is satisfied by $F(t,x) = B(t)x$ and $h(t) = |B(t)|$ (the operator norm; see **Homework #**, **Appendix**), and $h : I \to [0,\infty)$ is continuous according to Problem .18 (or .20). The assertion is now immediate from **Proposition .4**.

Differential Equations with Parameters

Again, V, W are always finite-dimensional normed real vector spaces.

$$\|\gamma\|_\infty \le |\gamma(a)| + L\|\dot\gamma\|_\infty .$$

In fact, $\gamma(t) = \gamma(a) + \int_a^t \dot\gamma(\tau)\,d\tau$, whenever $t \in I$, and so $|\gamma(t)| \le |\gamma(a)| + \int_a^t |\dot\gamma(\tau)|\,d\tau$, while the last term clearly does not exceed $L\|\dot\gamma\|_\infty$.

Lemma 2. *Let γ_i, $i = 1,2,\ldots$ be a sequence of V-valued C^1 functions on a closed interval I of length L such that $\gamma_i(a) \to z$ as $i \to \infty$ for some $a \in I$ and $z \in V$, while the derivatives $\dot\gamma_i$ converge uniformly on I to a function $\phi : I \to V$. Then $\gamma_i \to \gamma$ uniformly on I with a C^1 limit function $\gamma : I \to \mathrm{I\!R}$ having the derivative $\dot\gamma = \phi$.*

Proof. Define γ by $\gamma(t) = z + \int_a^t \phi(\tau)\,d\tau$. As $\gamma_i(t) = \gamma_i(a) + \int_a^t \dot\gamma_i(\tau)\,d\tau$, we get

$$|\gamma_i(t) - \gamma(t)| \le |\gamma_i(a) - z| + \int_a^t |\dot\gamma_i(\tau) - \phi(\tau)|\,d\tau \le |\gamma_i(a) - z| + L\|\dot\gamma_i - \phi\|_\infty ,$$

and it follows that $\|\gamma_i - \gamma\|_\infty \le |\gamma_i(a) - z| + L\|\dot\gamma_i - \phi\|_\infty$. ■

Lemma 3. *If mappings $\gamma_i : I \to V$ between metric spaces converge uniformly to a continuous mapping $\gamma : I \to V$ and $t_i \to a$ in I as $i \to \infty$, then $\gamma_i(t_i) \to \gamma(a)$.*

Proof. Obviously, $\mathrm{d}(\gamma_i(t_i), \gamma(a)) \le \mathrm{d}(\gamma_i(t_i), \gamma(t_i)) + \mathrm{d}(\gamma(t_i), \gamma(a))$, which is in turn less than or equal to $\mathrm{d}_{\sup}(\gamma_i, \gamma)) + \mathrm{d}(\gamma(t_i), \gamma(a))$, ■

According to the "neat" version of Banach's fixed-point theorem [**IM**, the corollary on p.2], any contraction h of a complete metric space (X, d), meaning: a mapping $h : X \to X$ with

(ctr) $\quad \mathrm{d}(h(x), h(x')) \le C\,\mathrm{d}(x, x')$ for all $x, x' \in X$ and some $C \in [0,1)$,

has a unique fixed point $x \in X$. In addition, $h^i(y) \to x$ as $i \to \infty$ for every $y \in X$. The next lemma states that this x depends on h continuously, relative to the supremum distance.

Lemma 4. *Given contractions h, h' of a complete metric space (X, d) with the unique fixed points x, x', one has $\mathrm{d}(x, x') \le (1 - C)^{-1}\mathrm{d}_{\sup}(h, h')$ for $\mathrm{d}_{\sup}(h, h') \in [0,\infty]$ defined by* (dsu) *and the constant $C < 1$ in* (ctr).

Proof. This is immediate from the following inequality, with $\eta = h'$ and $q = \mathrm{d}_{\sup}(h, h')$, for any $i \ge 1$ and $x \in X$, easily established by induction:

$$\mathrm{d}(h^i(x), \eta^i(x)) \le (1 + C + \cdots + C^{i-1})q.$$

The induction steps follows since

$$\mathrm{d}(h^{i+1}(x),\eta^{i+1}(x)) \le \mathrm{d}(h(h^i(x)),h(\eta^i(x))) + \mathrm{d}(h(\eta^i(x)),\eta(\eta^i(x))),$$

while the right-hand side does not exceed $(1+C+\cdots+C^{i-1})Cq + q$ due to the induction hypothesis. ■

Given an open set $U \subseteq V \times W$ and a C^∞ mapping $F : U \to V$, we will write points of U as pairs of vectors from V and W. Consider the following initial value problem with parameters:

$$\text{(ivp)} \qquad \dot\gamma = F(\gamma,\xi), \qquad \gamma(a) = z.$$

It includes the family of ordinary differential equations, parametrized by ξ, depending on $a \in \mathrm{I\!R}$ and $z \in V$ with $(z,\xi) \in U$, and having as solutions those differentiable functions $\gamma : I \to V$ defined on intervals $I \subseteq \mathrm{I\!R}$ for which $(\gamma(t),\xi) \in U$ and $\dot\gamma(t) = F(\gamma(t),\xi)$ whenever $t \in I$, while $\gamma(a) = z$. With any $(a,z,\xi) \in \mathrm{I\!R}\times U$ we now associate the maximal open interval $I_{a,z,\xi} \subseteq \mathrm{I\!R}$ on which (ivp) has a solution, and declare the set $Y \subseteq \mathbf{R}^2 \times U$ to be

$$Y = \{(t,a,z,\xi) \in \mathbf{R}^2 \times U : t \in I_{a,z,\xi}\}.$$

The Regular-Dependence Theorem. *The above set Y is open in $\mathbf{R}^2 \times V \times W$ and the mapping $Y \ni (t,a,z,\xi) \mapsto \gamma(t)$, with γ characterized by* (ivp)*, is of class C^∞.*

The proof proceeds by several steps, the first of which – openness of Y – is straightforward:

To simplify the remaining steps of the proof, we "fold" the initial-data pair (a,z) into the parameters, assuming from now on that $(a,z) = (0,0)$. This is achieved by replacing (t,F,γ) with $(s,G,\delta) = (t-a,G,\gamma-z)$, for G given by $G(\delta,\xi,z) = F(z+\delta,\xi)$, which turns (ivp) into $d\delta/ds = G(\delta,\xi,z)$ and $\delta(0) = 0$. Writing from now on (t,F,γ) instead of (s,G,δ), and using the notation $\gamma(t,\xi)$ to emphasize the dependence of the solution on the parameter ξ, we rephrase (ivp) as the autonomous initial value problem

$$\text{(par)} \qquad \dot\gamma(\,\cdot\,,\xi) = F(\gamma,\xi), \qquad \gamma(0,\xi) = 0.$$

with $(\ )^{\boldsymbol\cdot} = d/dt$. Now $F : U \to V$ is a C^∞ mapping from an open set $U \subseteq V \times W$, and $(t,\xi) \mapsto \gamma(t,\xi) \in V$ is defined on the set $\{(t,\xi) \in \mathrm{I\!R} \times W : t \in I_{0,0,\xi}\}$ (which, as we already know, is open in $\mathrm{I\!R} \times V$).

By (formally) applying $\partial/\partial\xi^\lambda$ to (par), and using the chain rule, we obtain

$$\text{(fop)} \qquad \dot\gamma(\,\cdot\,,\xi) = F(\gamma,\xi), \quad \gamma(0,\xi) = 0, \quad \dot\gamma_\lambda = \gamma_\lambda^j F_j + F_\lambda\,, \quad \gamma_\lambda(0,\xi) = 0.$$

where (par) is included as well, This time, $(\ )^{\boldsymbol\cdot} = \partial/\partial t$, the symbols involving γ, or F, are functions of (t,ξ) or, respectively, (γ,ξ), and the other partial derivatives are represented by subscripts:

$$\gamma_\lambda = \partial\gamma/\partial\xi^\lambda, \quad \gamma_\lambda^j = \partial\gamma^j/\partial\xi^\lambda, \quad F_j = \partial F/\partial\gamma^j, \quad F_\lambda = \partial F/\partial\xi^\lambda.$$

Of course, our derivation of (fop) is heuristic, rather than rigorous; the next two lemmas provide a proof of (fop).

Lemma 5. *The mapping* $(t,\xi) \mapsto \gamma(t,\xi)$ *is continuous.*

Proof.

Lemma 6. *Our* $\gamma(t,\xi)$ *is a* C^1 *function of* (t,ξ) *which, along with its first-order partial derivative* $\gamma_\lambda = \partial\gamma/\partial\xi^\lambda$ *satisfies the initial value problem* (fop).

Proof.

Proof of the Regular-Dependence Theorem. We use induction on $k \geq 1$ to show that $\gamma(t,\xi)$ is a C^k function of (t,ξ) for *every initial value problem* (par) *involving a* C^∞ *mapping* F. Lemma 6 settles the case $k = 1$. Assuming our assertion for some given $k \geq 1$, and fixing an initial value problem (par), we see – using Lemma 6 and the inductive assumption for (fop) rather than (par) – that the partial derivatives $\gamma_\lambda = \partial\gamma/\partial\xi^\lambda$ are of class C^k and so is, by (fop), $\partial\gamma/\partial t$. The domain of these first-order partial derivatives is the same as that of γ (since, fixing γ in (fop), we obtain a *linear* equation imposed on γ_λ, and we can invoke the Global Existence Theorem for Linear Ordinary Differential Equations). Thus, γ itself is of class C^{k+1}, on the same domain. ■