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MATH 6701, AUTUMN 2022
M-W-F 1:50 p.m., EC 322
A DAY-BY-DAY LIST OF TOPICS

Included are references to the following texts, accessible through the course homepage at
`https://people.math.osu.edu/derdzinski.1/courses/6701/6701.html`

[**DG**]: *Differential Geometry*, [**DF**]: *Distributions and the Frobenius Theorem*,
[**PS**]: *Projective Spaces and Grassmannians*, [**LL**]: *Local Lie-group structures*

August 24: Topological spaces, open sets, neighborhoods, continuity of mappings, convergence of sequences. The Hausdorff and second-countability axioms. Metric spaces, including $\mathrm{I\!R}^n$. Charts and C^r compatibility, $r = 0, 1, \ldots, \infty, \omega$. Definition of an n-dimensional C^r atlas and the resulting topology. Maximal atlases. The existence of a unique maximal atlas containing a given one [**DG**, p.1]. Definition of a C^r (Hausdorff) manifold [**DG**, pp.1–2] including the second-countability axiom [**DG**, p.53]. Equivalence of the latter axiom to the existence of a countable subatlas of the maximal atlas. (A manifold is thus a set with a fixed maximal atlas satisfying the Hausdorff and countability axioms.) Geometric properties, including openness of sets [**DG**, p.4]. Examples of manifolds: zero-dimensional ones (that is, nonempty countable sets); vector spaces [**DG**, pp.2]; affine spaces [DG, pp. 3 and 191-192]; open submanifolds [**DG**, p.5].

August 24: Cartesian products of manifolds [**DG**, p.4]. The first-countability axiom (the existence of a countable basis of neighborhoods at each point of the topological space in question, which can clearly be replaced by a descending one, that is, a sequence U_j, $j \ge 1$, of neighborhoods of x with every neighborhood of x containing some U_j). The repeatedly used observation that, given such U_j and x, a sequence x_j, $j \ge 1$, having $x_j \in U_j$ for all j, necessary converges to x. Continuous and differentiable mappings between manifolds, homeomorphisms, diffeomorphisms [**DG**, pp.5–6]. Continuity in terms of convergent sequences on the one hand, and of pre-images of open sets on the other. Equivalence – immediate from the above observation — of both characterizations of continuity when the source topological space satisfies the first-countability axiom. Chart mappings taking values in affine spaces, such as cosets of vector subspaces [**DG**, p.3]. Real and complex projective spaces [**DG**, pp.3–4; [**PS**, p.1]. Unit Euclidean spheres, with stereographic projections serving as chart mappings [**DG**, p.3]. The formula $x \mapsto z = [2x - (1 + \langle x, v\rangle)v]/(1 - \langle x, v\rangle)$ for the stereographic projection $S \smallsetminus \{v\} \to v^\perp - v$ from the pole $v \in S$, and $z \mapsto x = [(|z|^2 - 1)v + 3z]/(|z|^2 + 2)$ for its inverse, S being the unit sphere centered at 0 in a Euclidean space with the inner product $\langle\,,\rangle$.

August 29: More on unit Euclidean spheres, including the case of just two stereographic projections from mutually opposite poles $v, -v$ which – when treated as valued in $v^\perp$ – have the transiton mapping $y \mapsto 4y/|y|^2$. Closedness and compactness of subsets in topological space, the former meaning openness of the complement [**DG**, p.2], the latter involving the open-covering test [**DG**, p.54]. Closedness in terms of convergent sequences when the first-countability axiom is assumed. Compactness of continuous images of compact sets [**DG**, p.9] and their finite unions. Some consequences of the first-countability axiom: closedness of compact sets and compactness of their closed subsets; the fact that a bijective continuous mapping from a compact topological space is a homeomorphism.

A family of sets subordinate to another family. Equivalence of compactness to sequential compactness [**DG**, p.4] in spaces satisfying the second-countability axiom: the Heine-Borel theorem [**DG**, p.54]. Compactness in Euclidean spaces, equivalent to being closed and bounded [**DG**, p.9]. The fact that Euclidean spheres and all projective spaces are compact manifolds, while compactness of a zero-dimensional manifold amounts to its finiteness [**DG**, p.8]. Gluing of two n-dimensional manifolds along a diffeomorphism between their open submanifolds, the result being "almost" a manifold, except for the Hausdorff axiom which need not hold [**DG**, p.7].

August 31: The non-Hausdorff result of "doubling a point" in a manifold M and, more generally, of gluing two copies of M along the identity self-diffeomorphism of an open submanifold which has a boundary point in M. Connected sums of manifolds [**DG**, p.7]. Compactness of Cartesian products of compact topological spaces. Compactness of Cartesian products of compact manifolds [**DG**, p.9], including tori, and of connected sums of compact manifolds. Closed orientable surfaces of any nonnegative genus [**DG**, p.8]. Functions on, and curves in, topological spaces, including manifolds [**DG**, p.6]. Connectedness and pathwise connectedness of sets in topological spaces, their preservation under continuous images, and the fact that the latter implies the former (since the union of any nonempty family of connected sets with a nonempty intersection is necessarily connected, while nonempty connected subsets of $\mathrm{I\!R}$ are easily seen to be precisely the intervals). Connectedness of the whole space being the same as its pathwise connectedness when the space is assumed locally pathwise connected (since one then has an obvious equivalence relation with open equivalence classes). Connectedness of Cartesian products of connected manifolds [**DG**, p.9]. Connected components of a topological space (the equivalence classes of the relation in which two points are called equivalent if they both lie in a connected set). The observation that the connected component of a point x is the union of all connected sets containing x, as well as the largest connected set containing x. Simultaneous openness and closedness of the connected components when the space is locally pathwise connected. The union of a nonempty family of pairwise disjoint n-dimensional manifolds (which is again an n-dimensional manifold) and the immediate conclusion that every manifold equals the union, in this sense, of the family of its connected components [**DG**, p.6].

September 2: Topological groups. The observation that, in a compact Hausdorff topological group satisfying the first-countability axiom, continuity of the inverse follows from continuity of the group operation. Lie groups [**DG**, p.12]. Examples: countable groups, the additive group V and the isomorphism group $\mathrm{GL}(V)$ of any finite-dimensional real/complex vector space V (including $\mathrm{GL}(n,\mathrm{I\!K})$ for $V=\mathrm{I\!K}^n$), the group of invertible elements of a finite-dimensional real associative algebra with unity [**DG**, p.12]. Open subgroups of Lie groups G, constituting Lie groups in their own right, an example being the identity component G^o of G [**DG**, p.43]. Connectedness of the automorphism group $\mathrm{GL}(V)$ (and of the set $\mathcal{B}(V)$ of all ordered bases) of a finite-dimensional complex vector space V [**DG**, p.193]. Orientations in a real vector space of a positive finite dimension [**DG**, pp.192–193].

September 7: The group $\mathrm{GL}^+(V)$ for a real vector space V with $0<\dim V<\infty$, including $\mathrm{GL}^+(n,\mathrm{I\!R})$ when $V=\mathrm{I\!R}^n$. The canonical orientation of the underlying real space of a complex vector space of a positive finite dimension [**DG**, p.50]. Lie-group homomorphisms and isomorphisms [**DG**, p.13]. The algebra $\mathrm{I\!H}$ of quaternions [**DG**, p. 13]. The Lie-group structures of the unit spheres S^0, S^1, S^3 in $\mathrm{I\!R}, \mathbb{C}, \mathrm{I\!H}$ [**DG**, p.14]. The index notation for manifolds [**DG**, p.17], vector spaces and affine spaces. The chart-dependent partial derivatives [**DG**, p.17], the chain rule and group property [**DG**, p.18].

September 9: Curves and tangentiality, tangent vectors, velocity, vector components and the transformation rule [**DG**, p.18]. The tangent vector space, tangent spaces in vector and affine spaces, directional derivative, germs of functions, components of mappings [**DG**, p.19]. Differentials of differentiable mappings and the chain rule for differentials, tangent spaces of open submanifolds, differentials of C^1 functions [**DG**, p.20]. Differentials of linear and affine mappings.

September 12: Invariance of the dimension under diffeomorphisms [**DG**, p.21]. Dual bases in finite-dimensional vector spaces. Cotangent spaces and vectors. Bases of tangent and cotangent spaces naturally distinguished by a given chart [**DG**, p.20]. Tangent vector fields on manifolds, directional derivatives along vector fields [**DG**, p.22]. The Lie bracket and its interpretation as a commutator of directional differentiations [**DG**, p.23].

September 14: Vector fields projectable under mappings [**DG**, p.23]. Push-forwards of vector fields under diffeomorphisms. Projectability of Lie brackets [**DG**, p.24]. Lie algebras [**DG**, pp.26–27]. Vector fields on open submanifolds of vector and affine spaces. Linear vector fields and their Lie bracket, corresponding to the opposite of the commutator uf the underlying linear endomorphisms [**DG**, p.28]. Tangent spaces of Cartesian products [**DG**, p.38]. Arbitrary C^1 mappings $M \times N \to P$ written as multiplications and the Leibniz rule [**DG**, pp.38–39].

September 16: More on Lie algebras: homomorphisms and examples [**DG**, pp.26–27]. The Lie algebra of (left-invariant vector fields on) a Lie group G and its identification with T_1G [**DG**, p.28]. The cases of the additive group of a vector space (where the Lie algebra is Abelian) and of the group G of invertible elements of a finite-dimensional real associative algebra $\mathcal{A}$ with unity, the Lie algebra $\mathfrak{g} = T_1G = \mathcal{A}$ having the Lie bracket equal to the commutator in $\mathcal{A}$ [**DG**, p.28]. Projectability of left-invariant fields under Lie-group homomorphisms [**DG**, p.29]. The Lie-algebra homomorphism induced by a Lie-group homomorphism. Examples: the determinant; inner automorphisms [**DG**, p.30].

September 19: Volume-form characterizations of the determinant and trace [**DG**, p. 31]. Banach's fixed-point theorem [**DG**, pp.195–196]. Equivalence of norms in finite-dimensional vector spaces [**DG**, p.196].

September 21: The operator norm [**DG**, p.198] and the Lipschitz estimate for C^1 mappings along intervals[**DG**, p.197]. The inverse mapping theorem, without differentiability of the inverse [**DG**, p.198].

September 23: Differentiability of the inverse in the inverse mapping theorem [**DG**, p.197]. The implicit mapping theorem [**DG**, p.199]. The rank of a mapping at a point, openness of the maximum-rank subset, and the rank theorem [**DG**, p.33].

September 26: Submersions and their openness [**DG**, p.38]. Immersions, embeddings, submanifolds, with or without the subset topology [**DG**, p.34]. The subset topology as a consequence of. compactness. Continuity versus differentiability for submanifold-valued mappings [**DG**, p.34]. Uniqueness of submanifold structure with the subset topology [**DG**, p.35]. Critical and regular points and values of mappings, submanifolds defined by equations, their dimensions and tangent spaces [**DG**, pp.35–36]. Euclidean spheres as an example [**DG**, p.36]. The tangent spaces of projective spaces [**DG**, p.40].

September 28: Preimages of points under constant-rank mappings [**DG**, p.41]. Lie subgroups of Lie groups and their Lie algebras [**DG**, pp.43–44]. Lie-group actions. Isotropy groups [**DG**, p.46].

September 30: Automorphism groups of bilinear and sesquilinear forms [**DG**, p.47]. The linear Lie groups of the $\mathrm{SL}, \mathrm{O}, \mathrm{SO}, \mathrm{U}, \mathrm{SU}$ series and their Lie algebras [**DG**, pp.44, 47]. Accidental isomorphism and two-to-one homomorphism between linear Lie groups in low dimensions: $\mathrm{U}(1) \to \mathrm{SO}(2)$ and $S^3 = \mathrm{SU}(2) \to \mathrm{SO}(3)$ [**DG**, p.48].

October 3: Further accidental isomorphism and finite-to-one homomorphism between linear Lie groups: $S^3 \times S^3 \to \mathrm{SO}(4)$ and $\mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$, as well as $S^1 \times \mathrm{SU}(n) \to \mathrm{U}(n)$ [**DG**, p.48]. The diffeomorphism $\mathrm{SO}(3) \to \mathrm{I\!RP}^3$ [**DG**, p.48]. Ordinary differential equations and the reduction of order [**DG**, pp.203-204].

October 5: Existence and uniqueness of solutions for ordinary differential equations [**DG**, pp.203-204]. Flows of vector fields [**DG**, pp.219-221].

October 7: Lie brackets and flows [**DG**, pp.222-223].

October 10: Completeness of a C^∞ vectors field w on a manifold M [**DG**, p.224]. Examples: when the domainn of the flow contains $M \times [0, \varepsilon]$ for some $\varepsilon > 0$, when w has a compact support, when M is compact, when w is preserved by a set of diffeomorphisms operating on M transitively, when w is left or right invariant on a Lie group [**DG**, p.224]. Abundance of cut-off functions and global extensibility of germs [**DG**, p.217]. "Half-completeness" of "half-integral curves" contained in a compact set [**DG**, p.225]. Completeness of bounded vector fields on a vector space [**DG**, p.225]. Whitney's embedding theorem, the immersion part [**DG**, pp.54-55].

October 12: The remainder of Whitney's embedding theorem [**DG**, pp.54-55]. Maximal integral curves $t \mapsto x(t)$ of left-invariant vector fields on a Lie group G with $x(0) = 1$ being the same as smooth homomorphisms $\mathrm{I\!R} \to G$ [**DG**, pp.231-232]. The exponential mapping of a Lie group [**DG**, p.232]. The relation $F \circ \exp = \exp \circ F_*$ for any Lie-group homomorphism F [**DG**, pp.232-233]. The differential of $\exp$ at 0, equal to Id, so that the inverse mapping theorem can be applied [**DG**, p.232] to treat $\exp^{-1}$, on a neighborhood of 1, as a $\mathfrak{g}$-valued chart in G. The observation that an open subgroup of a Lie group G contains the identity component G^o of G, while any subgroup of G containing an open set is open (and hence equals G is G is connected). The conclusions that a Lie group homomorphism F between connected Lie groups is uniquely determined by F_*, and that a connected Lie group is Abelian if and only if so is its Lie algebra.

October 17: A summary of some facts: given a Lie group G, every open subgroup of G contains G^o (and hence equals G if G is connected), every subgroup of G with a nonempty open subset (that is, nonempty interior) is open; for a Lie group homomorphism $F : G \to H$ and connected Lie subgroups $G' \subseteq G$ and $H' \subseteq H$, one has $F(G') \subseteq H'$ if and only if $F_*(\mathfrak{g}') \subseteq \mathfrak{h}'$. Tensor products of finite-dimensional real or complex vector spaces and the tensor multiplication of vectors [**DG**, p.143]. The tensor-product bases of tensor-product spaces [**DG**, p.146]. The universal factorization property [**DG**, p.147]. Canonical isomorphic identifications $\mathbb{K} \otimes V = V$, where $\mathbb{K}$ is the scalar field, $V^* \otimes W = \mathrm{Hom}(V, W)$ and $V_1 \otimes \ldots V_r = V_{\sigma(1)} \otimes \ldots V_{\sigma(r)}$ for any permutation σ of $\{1, \ldots, r\}$ [**DG**, p.148].

October 19: Tensor, symmetric and exterior powers, symmetric and exterior products of vectors [**DG**, pp.143-144]. Bases for symmetric and exterior powers [**DG**, pp.146-147]. The universal factorization properties [**DG**, p.147]. tensor, symmetric and exterior algebras [**DG**, p.148]. Exterior forms at a point of a manifold M and differential forms on M [**DG**, pp.149-150]. The exterior derivative [**DG**, p.151].

October 21: Restricting local operators to open sets. The local-coordinate formula $(d\omega)_{j_0 \ldots j_r} = \sum_{q=0}^{r} (-1)^q \, \partial_{j_q} \omega_{j_0 \ldots \widehat{j_q} \ldots j_r}$. The spaces $Z^r M$ and $B^r M$ of closed and exact smooth differential forms on a manifold M [**DG**, p.152]. The cohomology spaces $H^r M$ [**DG**, p.153]. The Betti numbers $b_r(M)$ and the Poincaré polynomial $\mathrm{P}[M]$ [**DG**, p.157]. the differential forms; the Poincaré lemma [**DG**, p.152].

October 24: Cup product and the cohomology algebra $H^* M$ [**DG**, p.157]. The cohomology functor [**DG**, p.158]. Smooth homotopies and the algebraic homotopy formula [**DG**, p.158].

October 26: The homotopy theorem [**DG**, p.158]. Homotopy equivalences, homotopy inverses, the homotopy type, and their effects on cohomology [**DG**, p.160]. Deformation retracts [**DG**, p.160]. Exact sequences [**DG**, p.162].

October 28: The Mayer-Vietoris sequence [**DG**, p.162]. Cohomology of spheres and projective spaces [**DG**, p.164].

October 31: Contractible manifolds. Tubular neighborhoods of compact submanifolds of Euclidean spaces. The existence of an open covering of any compact manifold M with the property that each nonempty intersection of a nonempty subfamily is contractible. The resulting finite dimensionality of $H^* M$ [**DG**, p.163]. Orientability and orientations of a manifold [**DG**, p.165].

November 2: Finite partitions of unity [**DG**, p.121]. Oriented integration of top degree compactly supported continuous differential form on an oriented manifold [**DG**, p.165]. Stokes's theorem [**DG**, p.166]. The oriented-integration functional $H^n M \to \mathbb{R}$ for a compact oriented manifold M of dimension n [**DG**, p.166].

November 4: The observation that the integration functional $H^n S^n \to \mathbb{R}$ is an isomorphism in the case of an oriented n-dimensional sphere and, as a consequence, given two different concentric closed balls K, K' with $K \subseteq K' \subseteq \mathbb{R}^n$, a smooth differential n-form on $\mathbb{R}^n$ having zero integral and a support in K equals $d\theta$ for some smooth differential

$(n-1)$-form on $\mathbb{R}^n$ supported in K' [**DG**, p.168]. The obvious generalization of the last observation to the case where $\mathbb{R}^n$ is replaced by any oriented n-dimensional manifold [**DG**, p.168]. The conclusion that the integration functional $H^nM \to \mathbb{R}$ is an isomorphism for every oriented n-dimensional manifold M [**DG**, pp.168-169]. Bundles, with total spaces E, bases B, model fibres F, bundle projections $\pi : E \to B$, and local trivializations $\pi^{-1}(U) \approx U \times F$ over open sets $U \subseteq M$. Surjectivity of π, and the fact that the fibres $E_y = \pi^{-1}(y)$ over all $y \in B$ are submanifolds of E with the subset topology, diffeomorphic to F. Special cases: trivial bundles (admitting a global trivialization, with $U = B$), such as a product bundle $E = B \times F$, and covering projections (bundles with zero-dimensional fibres). A preimage characterization of covering projections with connected bases. Bundles with various types of fibre geometry prescribed in the model fibre F, and preserved by local trivializations, one example being provided by vector bundles (here F is a vector space), another by G-principal bundles (G being a Lie groups acting simply transitiuvely on F). The observation that $\mathbb{Z}_2$-principal bundles are nothing else than two-fold coverings. The orientation bundle (two-fold covering) of a manifold.

November 7: The canonical orientation of the orientation bundle (covering) of a manifold. The push-forward of an orientation under a diffeomorphism. Orientation-preserving and orientation-reversing diffeomorphisms between oriented manifolds (assumed connected in the latter case), and their effect on oriented integrals of top degree compactly supported continuous differential forms. The standard involution of the total space of a two-fold covering, and its orientation-reversing property in the case of the orientation bundle (covering) of a manifold. The conclusion that $H^nM = \{0\}$ for any nonorientable compact connected manifold M of dimension n. The π-lifts of continuous curves in the base B of a covering projection $\pi : E \to B$. The observation that the total space of a two-fold covering projection with a connected base is itself disconnected if and only if the covering is trivial (as a $\mathbb{Z}_2$-principal bundle). The projective spaces PV as bases of $(\mathbb{K} \smallsetminus \{0\})$-principal and S^{d-1}-principal bundles with total spaces $V \smallsetminus \{0\}$ and $\varSigma$, where d and $\varSigma$ are the real dimension of $\mathbb{K}$ and the unit sphere around 0 for a fixed Euclidean or Hermitian inner product in V.

November 9: Vector bundles over sets, sections over subsets, local trivializations, transition functions [**DG**, p.57]. Atlases, compatibility, smooth vector bundles over manifolds, smooth local/global sections, product bundles [**DG**, p.58]. Tangent bundles and tautological line bundles over projective spaces [**DG**, p.59]. The total space of a vector bundle [**DG**, p.66], and its manifold structure [**DG**, p.67]. Vector-bundle morphisms [**DG**, p. 69]. Operations on vector bundles: direct sum, the dual, Hom, the conjugate, the pullback [**DG**, p.68]. Smooth subbundles of vector bundles [**DG**, p.71]. The image and kernel of a constant-rank morphism [**DG**, p.72].

November 14: The quotient vector bundle [**DG**, p.72]. The differential of a smooth mapping $F : M \to N$ treated as a bundle morphism $dF : TM \to F^*TN$ [**DG**, p.72]. The tangent and normal bundles of an immersion [**DG**, p.72]. Symmetric and exterior powers of vector bundles, and theire equivalent description as subbundles of tensor powers. Tensors of type (p, q) in a vector space. Tensor bundles over a manifold. Distribution on a manifold, their integral manifolds, and integrability, with examples provided by the vertical distribution of fibrations, in which the fibres serve as integral manifolds [**DF**, p.1]. Horizontal distributions for fibrations (or, more generally, submersions), and the approach to a distribution in which one treats it, locally, as a horizontal distribution, with the defining formula $dy^\lambda = H_j^\lambda dx^j$ [**DF**, p.2], so that integral manifolds are, locally, graphs of mappings $(x^1,\dots,x^p) \mapsto (y^{p+1},\dots,y^m)$ satisfying the system $\partial_j y^\lambda = H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)$ of first-order partial differential equations [**DF**, pp.3-4]. Complete integrability of this system vs. integrability of the distribution [**DF**, p.3].

November 16: Submanifolds and mappings, including curves, tangent to a given distribution. The existence and uniqueness of such a curve with a prescribed projection on the base and initial value in a local fibration setting (making the given distribution D horizontal) [**DF**, p.3]. The normal bundle and curvature form of a distribution [**DF**, pp. 1-2]. The Frobenius theorem [**DF**, p.1], and the first step in its proof by induction on the dimension of the domain [**DF**, p.4].

November 18: The induction step, concluding the proof of the first part Frobenius theorem [**DF**, p.4], and the second (local fibration) part [**DF**, p.5]. Intersections of integral manifolds [**DF**, p.5]. Unions of nonempty countable families of integral manifolds [**DF**, p. 5]. Maximal integral manifolds, or leaves, of a (possibly nonintegrable) distribution, and the leaf theorem [**DF**, p.6].

November 21: The leaf-mapping theorem [**DF**, p.6]. Uniqueness of the manifold structure of an integral manifold of an integrable distribution [**DF**, p.7]. Diffeomorphic images of distributions, left-invariant distributions on a Lie group, and the Lie-subgroup theorem [**DF**, pp.7-8]. The image-group theorem [**DF**, p.8].

November 28: Local Lie-group structures as diffeomorphic images of standard ones, arising in Liegroups [**LL**]. Stiefel manifolds and Grassmannians, with a natural atlas for the latter [**PS**, p.2].

November 30: Hausdorff and countability axioms for Grassmannians, their connectedness and compactness [**PS**, p.3]. The representation of a manifold as a union of an ascending sequence of open sets, each of them having a compact closure contained in the next one [**DG**, p.217]. Locally finite families of sets [**DG**, p.218].

December 2: Locally finite partitions of unity subordinate to open coverings. Existence of fibre metrics in vector bundles and nowhere-zero top degree differential forms on orientable manifolds. The derivative $(\partial\psi)_z : T_zM \to \eta_z$, defined only if $\psi_z = 0$, of a local section ψ of a vector bundle η over a manifold M, with $z \in M$. The component representation $\partial_j\psi^a$. The special case of $\eta = TM$ and a vector field w vanishing at z, where $(\partial w)_z \in \mathfrak{gl}(T_zM)$ is the infinitesimal generator of the local flow of w acting on T_zM (so that the latter action is given by $t \mapsto \exp[t(\partial w)_z]$).

December 5: The Hessian $\mathrm{Hess}_z F$ of a mapping $F : M \to N$ at a point $z \in M$ where $dF_z = 0$, defined by $\mathrm{Hess}_z F = (\partial dF)_z$, with dF treated as a section of the vector bundle $\mathrm{Hom}(TM, F^*TN)$, and its coordinate form $\partial_j\partial_k F^\lambda$, showing that $\mathrm{Hess}_z F$ is a *symmetric* bilinear mapping $T_zM \times T_zM \to T_{F(z)}N$. Monomials and homogeneous polynomial functions of degrees $r \ge 0$ on a vector space V, the latter forming the space $\mathcal{P}_r(V)$, and the polynomial algebra $\mathcal{P}(V)$ [**DG**, p.144]. The space $\mathcal{P}_r(V,W)$ of degree r homogeneous polynomial functions on V valued in a vector space W, and the canonical isomorphism $S_r(V,W) = \mathrm{Hom}([V^*]^{\odot r}, W) \to \mathcal{P}_r(V,W)$ [**DG**, p.144]. The set $j^r_z(M,N)$ of r-jets of mappings $M \to N$ at a point $z \in M$. Special cases $j^0_z(M,N) = N$ and $j^1_z(M,\mathrm{I\!R}) = \mathrm{I\!R} \times T^*_z M$, as well as $j^1_z(\mathrm{I\!R},N) = TN$ at any $z \in \mathrm{I\!R}$.