Chapter 1. Basic Material

A. Introduction

1.1. In this first chapter, we collect the basic material which will be used throughout the book. In particular, we recall the definitions of the main notions of Riemannian (and pseudo-Riemannian) geometry. This is mainly intended to fix the definitions and notations that we will use in the book. As a consequence, many fundamental theorems will be quoted without proofs because these are available in classical textbooks on Riemannian geometry such as [Ch-Eb], [Hel 1], [Ko-No 1 and 2], [Spi].

We also give results which, though standard, are not available in the above-mentioned textbooks. For them we give complete proofs. This is the case in particular for the canonical irreducible decompositions of curvature tensors (§ G, H), the Weitzenböck formulas (§ I) and the first variations of curvature tensor fields under changes of metrics (§ K).

We emphasize that this chapter is for reference, and is not a complete course in Riemannian geometry. The whole chapter should be skipped by experts and used only when needed (non-experts may find it useful to read one of the above-mentioned textbooks).

1.2. We assume that the reader is familiar with all basic notions of differential topology, differential geometry, algebraic topology or Lie group theory such as manifolds and differentiable maps, fibre bundles (especially principal and vector bundles), tangent bundles, tensors and tensor fields, differential forms, spinors and spinor fields, Lie groups (and differentiable actions), Lie algebras and exponential maps (of Lie groups), differential operators and their symbols, singular homology and cohomology, Cartan-de Rham cohomology, homotopy groups, ...

1.3. Unless otherwise stated,

all manifolds are assumed to be smooth (i.e., C^∞), finite dimensional, Hausdorff, paracompact and (usually) connected; all maps are smooth (in particular, all actions of Lie groups are smooth).

In the particular case of differential operators, the smooth theory is often not the most convenient one, and we will have to work with Sobolev spaces. A theory of these spaces and some additional tools from analysis have been collected in an Appendix.

1.4 Notation. Given an n-dimensional manifold M, we *denote* by $T^{r,s}M$ the vector bundle of r-times covariant and s-times contravariant tensors on M, with two exceptions: the *tangent bundle* $T^{(0,1)}M$ will be denoted by TM and the *cotangent bundle* $T^{(1,0)}M$ by T^*M. Notice that $T^{(r,s)}M = \bigotimes^r T^*M \otimes \bigotimes^s TM$.

We *denote* by $\bigwedge^s M$ the vector bundle of s-forms on M. Hence $\bigwedge^s M = \bigwedge^s(T^*M)$. More generally, $\bigwedge$ will denote any alternating procedure and similarly S or $\odot$ will denote symmetrization.

It will sometimes be convenient to denote with the corresponding script letter the space of *sections* of a fibre bundle. For example, $\mathscr{T}^{(r,s)}M$ will be the vector space of (smooth) (r,s)-tensor fields on M (sections of $T^{(r,s)}M$). In particular $\mathscr{T}M$ is the space of *vector fields* on M. Also $\mathscr{C}M$ will be the space of C^∞-*functions* on M, and we denote by $\Omega^p M$ the space of *differential p-forms* on M.

In differential geometry, we often use quantities which are not tensor fields (such as connections or differential operators), from which we may deduce some tensor fields (like the Riemann curvature field for example). So, in order to characterize a tensor field, we will frequently use the following result.

1.5 Theorem (Fundamental lemma of differential geometry). *Let $p\colon E \to M$ be a vector bundle with finite rank over a manifold M. We* denote *by $\mathscr{E}$ the vector space of its sections. Then, given a linear map $A\colon \mathscr{E} \to \mathscr{C}M$, there exists a (necessarily unique) section α of the dual vector bundle $E^* \to M$ such that, for any point x in M and any element X in $\mathscr{E}$,*

$$A(X)(x) = \alpha(X(x))$$

if and only if A is $\mathscr{C}M$-linear, i.e., if and only if, for any function f in $\mathscr{C}M$, $A(fX) = fA(X)$.

This theorem will be mainly applied to $\mathscr{C}M$-multilinear maps $A\colon (\mathscr{T}M)^r \to (\mathscr{T}M)^s$. We then say that such a map A "defines" an (r,s)-tensor field. In order to illustrate this procedure, we recall two well-known examples:

(a) The *exterior differential* $d\alpha$ of a differential p-form α may be defined by its values on $p+1$ vector fields $X_0, X_1, \ldots, X_p$ through the formula

$$\textbf{(1.5a)}\quad d\alpha(X_0, X_1, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i \alpha(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \sum_{0 \leqslant i < j \leqslant p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p)$$

(where $\hat{X}_i$ means that X_i has to be deleted).

One easily sees that the right-hand side is $\mathscr{C}M$-multilinear.

(b) The *Lie derivative* $L_X A$ of a (r,s)-tensor field A along a vector field X may be defined by its values on r vector fields $X_1, \ldots, X_r$ through the following formulas

(1.5b) if A is a $(r, 0)$-tensor

$$L_X A(X_1, \ldots, X_r) = X \cdot A(X_1, \ldots, X_r) - \sum_{i=1}^{r} A(X_1, \ldots, [X, X_i], \ldots, X_r);$$

(1.5c) if A is a $(r, 1)$-tensor

$$L_X A(X_1, \ldots, X_r) = [X, A(X_1, \ldots, X_r)] - \sum_{i=1}^{r} A(X_1, \ldots, [X, X_i], \ldots, X_r);$$

and the rule

(1.5d) L_X is linear and a derivation with respect to the tensor product $\otimes$.

Notice that $L_X A$ is $\mathscr{C}M$-linear with respect to $X_1, \ldots, X_r$, but that it is *not* $\mathscr{C}M$-linear with respect to X.

(c) Finally, we recall that for a differential p-form α, these derivatives are related through the formula

(1.5e)
$$L_X \alpha = i_X \, d\alpha + d(i_X \alpha),$$

where $i_X \colon \Omega^p M \to \Omega^{p-1} M$ denotes the *interior product* (with a vector field X), given by the formula

(1.5f)
$$(i_X \beta)(X_2, \ldots, X_p) = \beta(X, X_2, \ldots, X_p) \quad \text{if } p > 0,$$
$$\text{and} \quad i_X = 0 \quad \text{if } p = 0.$$

B. Linear Connections

1.6. The notion of a connection, fundamental in differential geometry, has various aspects and many equivalent definitions (see the comment by M. Spivak [Spi] volume 5 p. 604). In Chapter 9, we need the general (geometric) notion of an Ehresmann-connection which we recall there, together with its links to the usual notions of principal and linear connections. In many textbooks (see for example [Ko-No 1]) principal connections are introduced first. Here we only recall the notion of a linear connection from the *covariant derivative point of view* (this approach is due to J.L. Koszul).

1.7 Definition. Let $p \colon E \to M$ be a vector bundle over a manifold M. A *linear connection* or a *covariant derivative* on E is a map

$$\nabla \colon \mathscr{T}M \times \mathscr{E} \to \mathscr{E}$$
$$(X, s) \mapsto \nabla_X s$$

which, for any vector fields X and Y in $\mathscr{T}M$, any sections s and t in $\mathscr{E}$, and any functions f and h in $\mathscr{C}M$, satisfies

(1.7)
$$\begin{aligned}\nabla_{fX+hY}s &= f\nabla_X s + h\nabla_Y s,\\ \nabla_X(s+t) &= \nabla_X s + \nabla_X t,\\ \nabla_X(fs) &= X(f)s + f\nabla_X s.\end{aligned}$$

1.8 Remarks. a) Note that, for any section s of E, the map $\nabla s\colon \mathscr{T}M \to \mathscr{E}$ defined by $(\nabla s)(X) = \nabla_X s$ is $\mathscr{C}M$-linear, so by Theorem 1.5, ∇s is a differential 1-form on M with values in E. In particular, we may define $\nabla_X s$ for any tangent vector X in TM. The map $s \mapsto \nabla_X s$ is on the other hand, not $\mathscr{C}M$-linear (in fact, ∇ is a first order differential operator, see Section I).

b) From Theorem 1.5, it follows easily that, given two connections ∇ and ∇' on the same vector bundle E, the difference

$$A(X,s) = \nabla'_X s - \nabla_X s$$

is $\mathscr{C}M$-linear, hence defines a section of $\bigwedge^1 M \otimes E^* \otimes E$.

Conversely, given any connection ∇ on E, and any section A of $\bigwedge^1 M \otimes E^* \otimes E$, the map $\nabla'_X s = \nabla_X s + A(X,s)$ is a connection on E.

c) One example of a connection is the following. Let $B \times F$ be a *trivialized* vector bundle over B. Then there exists a unique connection, called the *trivial connection*, such that the *constant* sections (i.e., sections s such that $s(b) = (b,\xi)$ with a constant ξ) satisfy $\nabla_X s = 0$ for any X.

As a corollary, let ∇ be any connection on a vector bundle E on B and let (U,φ) be a local trivialization of E, i.e., U is an open subset of B and φ a (C^∞)-fibered isomorphism

$$p^{-1}(U) \to U \times F,$$

then $U \times F$ admits the trivial connection ∇^F as defined above, and we may consider the connection ∇^φ on $p^{-1}(U)$ such that φ interchanges ∇^φ and ∇^F. The difference Γ between ∇ and ∇^φ is a section of $\bigwedge^1 U \otimes E^* \otimes E$. It is called the *Christoffel tensor of* ∇ *with respect to the trivialization* φ.

Moreover, if U is the domain of a chart on M with coordinates (x^i) and if we identify the fibre F with $\mathbb{R}^p$, then the components $\Gamma^\alpha_{i\beta}$ of Γ in the canonical basis $\left(\frac{\partial}{\partial x^i}\right)$ *of* TU and (ε_α) of $\mathbb{R}^p$ are the classical *Christoffel symbols* of ∇.

1.9. Let

$$\begin{array}{ccc} F & \xrightarrow{h} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f} & M \end{array}$$

be a vector bundle homomorphism (i.e., p and q are vector bundles, $p \circ h = f \circ q$ and h is linear when restricted to any fibre of q).

Given any linear connection ∇ on E, there exists a unique linear connection ∇' on F such that, for any tangent vector X of N, and any sections s of p and t of q satisfying $s \circ f = h \circ t$, we have

$$h(\nabla'_X t) = \nabla_{f_* X} s.$$

In particular, for any map $f: N \to M$, any linear connection ∇ on E (vector bundle over M) induces a linear connection ∇^f, called the *induced connection* on the induced vector bundle f^*E on N.

1.10. Let E_1, E_2 be two vector bundles over M. Then any linear connection ∇ on the direct sum vector bundle $E = E_1 \oplus E_2$ induces naturally two linear connections ∇^1 on E_1 and ∇^2 on E_2 in the following way. For any vector field X in $\mathcal{T}M$ and any section s_1 in $\mathcal{E}_1$, we define ∇^1 as

$$\nabla^1_X s_1 = pr_1(\nabla_X s_1)$$

where, on the right hand side, s_1 is viewed as a section of E and pr_1 is the projection onto the first factor in the direct sum $E = E_1 \oplus E_2$.

Note that an immediate consequence of Theorem 1.5 is the following. Keeping the same notations, the map $(X, s_1) \mapsto A_X s_1 = pr_2(\nabla_X s_1)$ defines a section of $\bigwedge^1 M \otimes E_1^* \otimes E_2$ (where of course pr_2 is the projection onto the second factor). Later we will encounter geometric situations where this construction is relevant (see for example Chapter 9).

1.11 Definition. The *curvature* R^∇ of a linear connection ∇ on a vector bundle E is the 2-form on M with values in $E^* \otimes E$ defined by

$$R^\nabla_{X,Y} s = \nabla_{[X,Y]} s - [\nabla_X, \nabla_Y] s \tag{1.11}$$

for any vector fields X, Y in $\mathcal{T}M$ and any section s in $\mathcal{E}$.

The fact that R^∇ is a tensor field follows easily from Theorem 1.5.

Since the curvature is the most important invariant attached to a connection, it is worth mentioning that there are many other interpretations of R (see for example (9.53b) for Ehresmann's point of view). One of them is the following.

1.12. We consider the differential p-forms on M with values in E, i.e., the sections of $\bigwedge^p M \otimes E$. We define the exterior differential d^∇ associated with ∇ by the following formula. For any section α of $\bigwedge^p M \otimes E$, $d^\nabla \alpha$ is the section of $\bigwedge^{p+1} M \otimes E$ such that for $X_0, \ldots, X_p$ in $T_x M$, extended to vector fields $\tilde{X}_0, \ldots, \tilde{X}_p$ in a neighborhood,

$$(d^\nabla \alpha)(X_0, \ldots, X_p) = \sum_i (-1)^i \nabla_{X_i}(\alpha(\tilde{X}_0, \ldots, \hat{\tilde{X}}_i, \ldots, \tilde{X}_p))$$
$$+ \sum_{i \neq j} (-1)^{i+j} \alpha([\tilde{X}_i, \tilde{X}_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p).$$

Then

$$R^\nabla_{X,Y} s = -d^\nabla(d^\nabla s)(X, Y). \tag{1.12}$$

Note that $d^\nabla \circ d^\nabla$ is not zero in general unlike the case of the ordinary exterior differential.

1.13 Examples. a) The curvature of the trivial connection of a trivialized bundle vanishes.

b) If $\tilde{\nabla} = \nabla + A$, with A a section of $\bigwedge^1 M \otimes E^* \otimes E$, then we obtain easily

$$R^{\tilde{\nabla}}_{X,Y}s - R^{\nabla}_{X,Y}s = -(d^{\nabla}A)_{X,Y}s - A_X(A_Y s) + A_Y(A_X s)$$

where A is considered as a 1-form with values in $E^* \otimes E$.

This shows that R^{∇} is not zero in general since any A gives a connection. Moreover this formula permits the computation of R^{∇} in a local trivialization by comparing ∇ with the trivial connection in terms of its Christoffel tensor.

1.14 Theorem (Differential Bianchi identity). *Let ∇ be a connection on a vector bundle E over M. Then*

$$d^{\nabla}R^{\nabla} = 0. \tag{1.14}$$

Proof. This follows from the second definition and the fact that $(d^{\nabla} \circ d^{\nabla}) \circ d^{\nabla} = d^{\nabla} \circ (d^{\nabla} \circ d^{\nabla})$. □

1.15 Some more definitions. A section s of a vector bundle E equipped with a connection ∇ is called *parallel* if $\nabla s = 0$. Note that, given a point x in M, there always exists a neighbourhood U and a finite family $(s_i)_{i \in I}$ of sections of E under U such that $(s_i(y))_{i \in I}$ is a basis of the fibre E_y for each y in U and $\nabla s_i(x) = 0$. (Take a local trivialization of E.)

If the vector bundle E is equipped with some additional structure (such as a Euclidean fibre metric h, an imaginary map J, a symplectic form ω or a Hermitian triple (g, J, ω)), then a linear connection ∇ on E is called respectively *Euclidean* (or metric), *complex*, *symplectic* or *Hermitian*, if respectively g, J, ω or (g, J, ω) are parallel. In such a case, the curvature R^{∇} of ∇ satisfies additional properties, namely, for each X, Y in T_xM the linear map $R^{\nabla}_{X,Y}$ on E_x is respectively skewsymmetric, complex, skewsymplectic or skew-Hermitian.

In the particular case where E is the tangent bundle of the base manifold M, further considerations can be developed.

1.16 Definition. A *linear connection D on a manifold M* is a linear connection on the tangent bundle TM of M.

1.17. Let ∇ be a linear connection on a vector bundle E over M. For any section s in $\mathscr{E}$, ∇s is a section of $T^*M \otimes E$ (see Remark (1.8a)). Now let D be a linear connection on M. Then D and ∇ induce a linear connection (that we still denote by ∇) on $T^*M \otimes E$, so we may define $\nabla(\nabla s)$, and we denote it by $\nabla^2 s$. It is the section of $T^{(2,0)}M \otimes E$ defined by

$$(\nabla^2 s)_{X,Y} = \nabla_X(\nabla_Y s) - \nabla_{(D_X Y)}s.$$

Instead of $(\nabla^2 s)_{X,Y}$ we may write $(\nabla\nabla s)(X, Y)$, or $\nabla^2_{X,Y}s$, or even $(\nabla^2 s)(X, Y)$.

Now, using an obvious induction, we may define the *iterated covariant derivative* $\nabla^p s$ as the section of $T^{(p,0)}M \otimes E$ such that

$$(\nabla^p s)_{X_1,\dots,X_p} = (\nabla_{X_1}(\nabla^{p-1}s))_{X_2,\dots,X_p}.$$

The main point to notice here is that the order in which $X_1, \ldots, X_p$ are written is important, because the covariant derivatives (unlike usual derivatives) do not commute in general.

1.18 Definition. The *torsion tensor* T of a linear connection D on a manifold M is the (2, 1)-tensor field defined by

$$T_{X,Y} = D_X Y - D_Y X - [X, Y]. \tag{1.18}$$

The fact that T is a tensor field follows easily from Theorem 1.5.

1.19 Remarks. a) Note that the torsion is, by definition, skew-symmetric in its covariant variables.

b) For any (1, 2)-tensor field A on M, then the connection $\tilde{D}$ defined by $\tilde{D}_X Y = D_X Y + A_{X,Y}$ is also a connection. Its torsion tensor $\tilde{T}$ is given by

$$\tilde{T}_{X,Y} = T_{X,Y} + A_{X,Y} - A_{Y,X}.$$

In particular, for any connection D on M with torsion tensor T, the connection $(X, Y) \mapsto D_X Y - \frac{1}{2} T_{X,Y}$ is *torsion-free* (i.e., its torsion tensor vanishes).

c) The assumption that the connection D be *torsion-free* enables one to express brackets of vector fields in terms of D:

$$[X, Y] = D_X Y - D_Y X$$

Dually, the exterior differential of differential forms may also be expressed in terms of D. For $\alpha \in \Omega^p M$, and vectors $X_0, \ldots, X_p$,

$$d\alpha(X_0, \ldots, X_p) = \sum_{i=0}^{p} (-1)^i (D_{X_i}\alpha)(X_0, \ldots, \hat{X}_i, \ldots, X_p).$$

1.20. We now come back to the relations between the iterated covariant derivatives. The simplest one is the following so-called *Ricci formula*.

$$\nabla^2_{X,Y} s - \nabla^2_{Y,X} s = -R^\nabla_{X,Y} s - \nabla_{T_{X,Y}} s \tag{1.21}$$

where T is the torsion field of D and R^∇ the curvature of ∇.

Note that if D is torsion-free (i.e. *symmetric*), then the right hand side does not involve ∇s.

Note that, if $\tilde{\nabla}_X s = \nabla_X s + B_X s$, then

$$\tilde{\nabla}^2_{X,Y} s = \nabla^2_{X,Y} s + (\nabla_X B)_Y s + B_Y(\nabla_X s) + B_X(\nabla_Y s) + B_X(B_Y s)$$

and in this way we may obtain the formula for $R^{\tilde{\nabla}}$.

At order 3 we obtain many formulas of which we give only two.

1.22 Corollary. *Further Ricci formulas read*

$$\nabla^3_{X,Y,Z} s - \nabla^3_{Y,X,Z} s = -R^\nabla_{X,Y}(\nabla_Z s) + \nabla_{R_{X,Y}Z} s - \nabla^2_{T_{X,Y},Z} s \tag{1.22a}$$

$$\nabla^3_{X,Y,Z} s - \nabla^3_{X,Z,Y} s = -(\nabla_X R^\nabla)_{Y,Z} s - R^\nabla_{Y,Z}(\nabla_X s) - \nabla^2_{X,T_{Y,Z}} s - \nabla_{(D_X T)_{Y,Z}} s. \tag{1.22b}$$

Proof. a) Apply 1.21 to ∇s.
b) Apply ∇_X to 1.21. □

1.23. We recall that a linear connection D on a manifold M induces a connection on any tensor bundle and we may apply the preceding formulas in this case. In particular, since the torsion T and the curvature R of D are tensor fields on M, we may define their covariant derivatives of any order D^pT or D^pR. These are still tensor fields on M. There are many relations between these tensor fields. The simplest ones involve T, R and their first covariant derivatives. They are called Bianchi identities. We have already met one of them under the name of the differential Bianchi identity. This one is valid for a general bundle over a manifold equipped with any connection.

1.24 Theorem. *Let D be a linear connection on M. Then its torsion field T and curvature field R satisfy*

$$\mathfrak{S}_{X,Y,Z}(R_{X,Y}Z + T_{T_{X,Y}}Z + (D_X T)_{Y,Z}) = 0$$

Proof. We compute in a local chart φ. Then the trivial connection D^φ has neither torsion nor curvature. Hence the Christoffel tensor $\Gamma = \nabla - D^\varphi$ satisfies

$$T_{X,Y} = \Gamma_{X,Y} - \Gamma_{Y,X}.$$

And the formula computing R from R^φ gives the result after one has taken the cyclic sum. □

1.25 Proposition. *The differential Bianchi identity for a connection ∇ on a general vector bundle E can be given the following expression (if M is equipped with a linear connection D with torsion T)*

(1.25) $$\mathfrak{S}_{X,Y,Z}(\nabla_X R^\nabla)_{Y,Z} + R^\nabla_{T_{X,Y},Z} = 0.$$

In particular, if D is torsion-free, we have

(1.25′) $$\mathfrak{S}_{X,Y,Z}(\nabla_X R^\nabla)_{Y,Z} = 0.$$

Proof. It follows directly from expressing 1.14 in terms of iterated covariant derivatives. □

1.26 Definition. Given a linear connection D on a manifold, a *geodesic* (for D) is a smooth curve $c: I \to M$ such that

$$D_{\dot c}\dot c = 0$$

$\left(\text{i.e., the vector field } \frac{d}{dt} \text{ on the interval } I \text{ is parallel for the induced connection } c^*D\right.$ on the induced bundle $\left.c^*TM \text{ on } I\right)$.

We recall that the general existence theorem for solutions of a differential equation implies that, for any tangent vector X in T_xM, there exists a unique geodesic $c_X\colon I \to M$ such that $c_X(0) = x$ and $\dot{c}_X(0) = T_0 c\left(\dfrac{d}{dt}\right) = X$ and I is a maximal open interval.

For any x in M, we denote by $\mathscr{D}_x$ the set of tangent vectors X in T_xM such that 1 belongs to the interval I of definition for c_X. And we denote by $\mathscr{D}$ the union of all $\mathscr{D}_x$ for all x in M. Then $\mathscr{D}_x$ is open in T_xM and $\mathscr{D}$ is open in TM.

1.27 Definition. The *exponential map* of D is the map $\exp\colon \mathscr{D} \to M$ defined by $\exp(X) = c_X(1)$.

We *denote by* $\exp_x$ the restriction of exp to $\mathscr{D}_x = \mathscr{D} \cap T_xM$.

1.28 Theorem. *The tangent map to* $\exp_x$ *at the origin* 0_x *of* T_xM *is the identity of* T_xM *(if we identify* $T_{0_x}T_xM$ *with* T_xM*).*

In particular the implicit function theorem implies that there exists a neighbourhood U_x of 0_x in T_xM and a neighbourhood V_x of x in M such that the restriction $\exp_x \restriction U_x$ is a diffeomorphism from U_x onto V_x. Such a diffeomorphism gives in particular a set of local coordinates around x in M.

1.29 Definition. A linear connection D on a manifold M is called *complete* if the domain $\mathscr{D}$ of its exponential map is all of TM.

The tangent map to $\exp_x$ at other points of its domain $\mathscr{D}_x$ is described by deforming infinitesimally a geodesic by geodesics. This gives rise to special "vector fields" along the geodesic.

1.30 Definition. Given a connection D on a manifold M and c a geodesic of D, a *Jacobi field along* c is a vector field J along c (i.e., the image by Tc of a section of c^*TM) satisfying

(1.30) $$D_{\dot{c}}D_{\dot{c}}J + R_{\dot{c},J}\dot{c} + D_{T_{\dot{c},\dot{c}}}J = 0.$$

Note that (1.30) is a second order differential equation along c, hence J is well defined as soon as, at one point $c(t)$, we know both $J_{c(t)}$ and $(D_{\dot{c}}J)_{c(t)}$.

1.31 Proposition. *For any* Y *in* T_xM *and* X *in* $\mathscr{D}_x$, $(T_X\exp_x)(Y)$ *is the value at* $c_X(1)$ *of the unique Jacobi field along* c_X *with initial data* $J(0) = 0$ *and* $(D_{\dot{c}}J)(0) = Y$.

It follows from Proposition 1.31 that the differential of $\exp_x$ is singular precisely when a Jacobi field vanishing at the origin vanishes again for some time t.

1.32 Definition. We say that $c(0)$ and $c(t)$ are *conjugate points* along the geodesic c if and only if there exists a non-zero Jacobi field along c such that $J(0) = 0$ and $J(t) = 0$.

The study of conjugate points plays an important role in differential geometry. See for example [Ch-Eb] for the Riemannian case.

C. Riemannian and Pseudo-Riemannian Manifolds

1.33 Definition. a) A *pseudo-Riemannian metric* of signature (p, q) on a smooth manifold M of dimension $n = p + q$ is a smooth symmetric differential 2-form g on M such that, at each point x of M, g_x is non-degenerate on T_xM, with signature (p, q). We call (M, g) a *pseudo-Riemannian manifold.*

b) In the particular case where $q = 0$ (i.e., where g_x is positive definite), we call g a *Riemannian metric* and (M, g) a *Riemannian manifold;*

c) in the particular case where $p = 1$ (and $q > 0$), we call g a *Lorentz metric* and (M, g) a *Lorentz manifold.*

1.34. *The First Examples are the Flat Model Spaces.* Let g_0 be a non-degenerate symmetric linear 2-form on $\mathbb{R}^n$ with signature (p, q) with $p + q = n$. The vector space structure of $\mathbb{R}^n$ induces a canonical trivialization of $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ (using the translations). We define the canonical pseudo-Riemannian metric g on $\mathbb{R}^n$ (associated to g_0) to be such that, for each x in $\mathbb{R}^n$, g_x is identified with g_0 when we identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$.

1.35. We obtain many more examples through the following general construction.

Let i: $N \to M$ be an immersion, and g a pseudo-Riemannian metric on M. We *assume* that, for any x in N, $(T_xi)(T_xN)$ is a non-isotropic subspace of $T_{i(x)}M$ (i.e., the induced form i^*g is non-degenerate). Then i^*g is a pseudo-Riemannian metric on N.

Notice that if g is a Riemannian metric, then i^*g is always non-degenerate and so is a Riemannian metric on N.

1.36. *Other Model Spaces.* By way of example, let g_p be the canonical 2-form with signature $(p, n + 1 - p)$ on $\mathbb{R}^{n+1}$ (i.e., $g_p = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{n+1}^2$). Then g_p induces a pseudo-Riemannian metric on $\mathbb{R}^{n+1}$ as in 1.34. We consider the imbedded submanifolds

$$S_p^n = \{x \in \mathbb{R}^{n+1}; g_{p+1}(x, x) = +1\}$$

and

$$H_p^n = \{x \in \mathbb{R}^{n+1}; g_p(x, x) = -1\}.$$

Then these imbeddings i satisfy the assumption made in 1.35, so i^*g is a pseudo-Riemannian metric and (S_p^n, i^*g_p) and (H_p^n, i^*g_{p+1}) are two pseudo-Riemannian manifolds with signatures $(p, n - p)$.

In the particular case when $n = p$, the Riemannian manifolds S_n^n and the connected component of $(0, \ldots, 0, 1)$ in H_n^n (which has two connected components corresponding to $x_{n+1} > 0$ and $x_{n+1} < 0$) are called respectively the *canonical sphere* S^n and the *canonical hyperbolic space* H^n.

For more details, see for example [Wol 4] p. 67.

1.37. Let (M, g) and (M', g') be two pseudo-Riemannian manifolds with signatures (p, q) and (p', q'). The *product manifold* $M \times M'$ admits a canonical splitting

$T(M \times M') = TM \oplus TM'$ of its tangent space. For each (x, x') in $M \times M'$, we define the symmetric 2-form $g \oplus g'$ on $T_{(x,x')}(M \times M') = T_xM \oplus T_{x'}M'$ as the direct sum of g_x on T_xM and $g_{x'}$ on $T_{x'}M'$. Then $g \oplus g'$ is obviously a pseudo-Riemannian metric on $M \times M'$ with signature $(p + p', q + q')$. It is called the *product metric.*

1.38. Let (M, g) be a pseudo-Riemannian manifold. Then at each point x of M, the non-degenerate quadratic form g_x induces a canonical isomorphism $T_xM \to T_x^*M$ and more generally, a canonical isomorphism between any $T_x^{(p,q)}M$ and $T_x^{(p+1,q-1)}M$ (hence onto any $T^{(r,s)}M$ with $r + s = p + q$). This isomorphism is often denoted by $\flat$ ("flat") and its inverse by $\sharp$ ("sharp") since in classical tensor notation, they correspond to lowering (resp. raising) indices, see below 1.42.

By composition of the isomorphism $T^{(p,q)}M \to T^{(q,p)}M$ with the "evaluation map" (pairing any vector space with its dual), we get a non-degenerate quadratic form (still denoted by g_x) on any $T^{(p,q)}M$ and consequently on any subspace of $T^{(p,q)}M$ such as $\bigwedge^p M$ or $S^p(T^*M)$. Note that if g is positive definite on TM, it is positive definite on any $T^{(p,q)}M$.

1.39 Theorem (Fundamental Theorem of (Pseudo-) Riemannian Geometry). *Given a pseudo-Riemannian manifold* (M, g), *there exists a* unique *linear connection* D *on* M, *called the* Levi-Civita connection (*of* g), *such that*

a) D *is metric* (*i.e.*, $Dg = 0$);

b) D *is torsion-free* (*i.e.*, $T = 0$).

1.40 Definition. On a pseudo-Riemannian manifold (M, g), the curvature tensor field R of the Levi-Civita connection is called the *Riemann curvature tensor* of (M, g).

Note that, since D is torsion free, the Bianchi identities 1.24, (1.25′) and the Ricci formula (1.21) take their simplified forms.

1.41. For future use, we compute the arguments of all these tensors in local coordinates.

Let $\varphi: U \to V$ be a chart on M, i.e., let φ be a diffeomorphism from some open subset U of $\mathbb{R}^n$ onto some open subset V of M. Using the coordinate functions x^i on $\mathbb{R}^n$, we get coordinate functions $x^i \circ \varphi^{-1}$ on V, which we still denote by x^i. Then the differential 1-forms (dx^i) give a basis of T^*V, and we denote by (∂_i) the dual basis of TV. Also the tensor fields $dx^i \otimes dx^j$ are a basis of $T^{(2,0)}V$ at each point of the chart, so we may write the restriction of g to V as

$$g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j,$$

where the g_{ij}'s are functions on V satisfying $g_{ij} = g_{ji}$.

Now we may characterize the Levi-Civita connection D through its values on the basis (∂_i). We get

$$D_{\partial_i}\partial_j = \sum_{k=1}^{n} \Gamma_{ij}^k \partial_k,$$

where the "*Christoffel symbols*" (Γ_{ij}^k) are given by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

(at each point x of V, (g^{kl}) is the inverse matrix of (g_{ij})). These Γ_{ij}^k's are the components of the difference tensor between D and the trivial connection on V (compare Remark 1.8c).

Finally, the curvature R has components R_{ijk}^l given by

$$R(\partial_i, \partial_j)\partial_k = \sum_{l=1}^{n} R_{ijk}^l \partial_l,$$

where

$$R_{ijk}^l = \partial_j(\Gamma_{ik}^l) - \partial_i(\Gamma_{jk}^l) + \sum_{m=1}^{n} (\Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l).$$

1.42. In classical tensor calculus, a convention is used to avoid too many summation signs $(\sum)$. Any index which is repeated has to be summed (usually from 1 to $n = \dim M$). For example, we write

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

and
$$R_{ijk}^l = \partial_j \Gamma_{ik}^l - \partial_i \Gamma_{jk}^l + \Gamma_{ik}^m \Gamma_{jm}^l - \Gamma_{jk}^m \Gamma_{im}^l.$$

Another convention avoids some g_{ij} and g^{kl}; this is the convention of "raising and lowering indices". Given any tensor A in $T^{(r,s)}M$ whose components in a local basis are $A_{i_1 \dots i_r}^{j_1 \dots j_s}$, we

denote by
$$A_{i_1 \dots i_r l}^{j_2 \dots j_s} = g_{lk} A_{i_1 \dots i_r}^{k j_2 \dots j_s},$$

and
$$A_{i_1 \dots i_{r-1}}^{l j_1 \dots j_s} = g^{lk} A_{i_1 \dots i_{r-1} k}^{j_1 \dots j_s}$$

(with the "summation of repeated indices" convention), and so on.

There are numerous other conventions, more or less widely used. For example, δ_i^j is the Kronecker symbol, defined as

$$\delta_i^i = 1 \text{ for any } i, \quad \text{and} \quad \delta_i^j = 0 \text{ for any } i \neq j.$$

Also a bracket [ ] around two indices means alternating them, or { } summing cyclicly, but if we use these conventions, we will remind the reader of what is meant.

1.43. Given a pseudo-Riemannian manifold (M, g), the geodesics, the exponential map and the Jacobi fields of its Levi-Civita connection D are called the *geodesics*, the *exponential map* and the *Jacobi fields* of (M, g). Furthermore, (M, g) is called *complete* if and only if D is complete.

Note that in local coordinates (x^i) a geodesic $c(t) = (x^i(t))_{i=1,\dots,n}$ satisfies the following system of n second order differential equations (for $i = 1, \dots, n$)

(1.43)
$$\ddot{x}^i + \sum_{j,k=1}^{n} \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0,$$

where the dot denotes the usual derivative in the variable t.

1.44. Using the exponential map, we may construct some special types of local coordinates around each point x of M in the following way. We choose some orthonormal frame $(X_1, \ldots, X_n)$ of T_xM, which induces a linear isomorphism α: $\mathbb{R}^n \to T_xM$. Let U be a neighborhood of 0 in T_xM and V a neighborhood of x in M such that $\exp_x$ is a diffeomorphism from U onto V (see 1.28). Now

$$\exp_x \circ \alpha \colon \alpha^{-1}(U) \to \mathrm{V}$$

is a local chart for M around x. The corresponding coordinates are called *normal coordinates* (*centered at* x).

Note that (∂_i) coincides with (X_i) *at* x, but that the basis (∂_i) is not necessarily orthonormal at other points of V.

Since $\exp_x$ maps a radial curve $(t \to tX)$ of T_xM onto a geodesic c_X, the geodesics issued from x become the radial curves in normal coordinates (centered at x).

We may characterize normal coordinates as follows.

1.45 Theorem (Folklore; see D.B.A. Epstein [Eps]). *Local coordinates (x^i) on a pseudo-Riemannian manifold (M, g), defined in an open disk centered at the origin, are normal coordinates (centered at x) if and only if the expression (g_{ij}) of g in these coordinates satisfies*

$$\sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n)x^j = x^i.$$

In fact, this theorem is no more than the classical "*Gauss Lemma*". It is usually stated more intrinsically as follows.

1.46 Theorem ("Gauss Lemma", see for example [Ch-Eb]). *Let (M, g) be a pseudo-Riemannian manifold, x a point in M and $X \in \mathscr{D}_x \subset T_xM$. Then*

a) $g_{c_X(1)}((T_X \exp_x)(X), (T_X \exp_x)(X)) = g_x(X, X)$,

b) *For any Y in T_xM such that $g_x(X, Y) = 0$, we have*

$$g_{c_X(1)}((T_X \exp_x)(X), (T_X \exp_x)(Y)) = 0.$$

1.47 Definition. The *volume element* μ_g of a pseudo-Riemannian manifold (M, g) is the unique density (i.e., locally the absolute value of an n-form) such that, for any orthonormal basis (X_i) of T_xM,

$$\mu_g(X_1, \ldots, X_n) = 1.$$

Obviously, in local coordinates (x^i), we have

$$\mu_g = \sqrt{|\det(g_{ij})|}\, |dx^1 \wedge \cdots \wedge dx^n|,$$

so that μ_g is locally "equivalent" to the Lebesgue measure in any set of coordinates.

In normal coordinates, there is a formula for μ_g involving the values of Jacobi fields.

1.48 Definition. Given normal coordinates (x^i) on M, we *define* the function $\theta = \sqrt{|\det(g_{ij})|}$.

Note that θ does *not* depend on the particular basis (X_i) that we choose to define the normal coordinates (because they are all orthonormal at the center).

For any *non-isotropic* vector X in $\mathscr{D}_x \subset T_xM$, we consider an orthonormal basis (X_i) such that X is proportional to X_n, and then the $n-1$ Jacobi fields $J_1, \ldots, J_{n-1}$ along c_X with initial data $J_i(0) = 0$, $(D_{\dot{c}_X} J_i)(0) = X_i$.

1.49 Proposition. *If c_X lies in the domain of normal coordinates,*

$$\theta(\exp_x X) = |\det(J_1, \ldots, J_{n-1})|(\exp_x X).$$

This follows directly from 1.31 and 1.46. The determinant is taken with respect to an orthonormal basis.

1.50. When the manifold (M, g) is *oriented*, we denote by ω_g the canonical n-form, called the *volume form* of (M, g), such that $\mu_g = |\omega_g|$ and ω_g is in the class of the given orientation. Note that $g(\omega_g, \omega_g) = (-1)^s$ where s is the number of -1 in the signature of g (i.e., g has signature $(n-s, s)$). The following definition gives a generalization.

1.51 Definition. For any p with $0 \leqslant p \leqslant n$, we define the *Hodge* operator $*$ to be the unique vector-bundle *isomorphism*

$$*\colon \textstyle\bigwedge^p M \to \bigwedge^{n-p} M$$

such that
$$\alpha \wedge (*\beta) = g(\alpha, \beta)\omega_g$$

for any α and β in $\bigwedge_x^p M$, and any x in M.

This operator $*$ satisfies the following properties.

1.52 Proposition. a) $*1 = \omega_g$ *and* $*\omega_g = (-1)^s$;

b) *for any α in $\bigwedge^p M$ and β in $\bigwedge^{n-p} M$, we have*

$$g(\alpha, *\beta) = (-1)^{p(n-p)} g(*\alpha, \beta);$$

c) *on $\bigwedge^p M$, we have*

$$*^2 = (-1)^{p(n-p)+s} Id_{\bigwedge^p M}.$$

1.53 Remark. In even dimensions $n = 2m$, $*$ will induce an automorphism of $\bigwedge^m M$. In the Riemannian case ($s = 0$), this automorphism is

—an involution if m is even,

—a complex structure if m is odd.

These facts have strong geometric consequences. For example, for a 4-dimensional Riemannian manifold, the splitting of $\bigwedge^2 M$ into two eigenspaces relative to $*$ gives rise to the notion of self-duality, which is developed in Chapter 13.

This contrasts with the fact that, for a 4-dimensional Lorentz manifold (with $s = 1$), $*$ induces a complex structure on $\bigwedge^2 M$. For an application to the classification of curvature tensors of space-times, see 3.14.

1.54. *Some more notation.* We recall that a pseudo-Riemannian metric induces canonical isomorphisms ($\flat$ and $\sharp$) between tensor spaces. But for some very useful objects, we prefer not to use these isomorphisms and we introduce a special notation.

Given a smooth function f on M,

a) the *gradient* of f is the vector field $Df = \sharp df$ (or $df^\sharp$), i.e., Df satisfies $g(Df, X) = X(f) = df(X)$ for any X in TM;

b) the *Hessian* of f is the covariant derivative of df, i.e., Ddf (we also *denote* it by $D^2 f$); it satisfies $Ddf(X, Y) = X \cdot Y \cdot f - (D_X Y)f$ (notice that Ddf is symmetric);

c) the *Laplacian* of f is the opposite of the trace of its Hessian with respect to g, i.e., $\Delta f = -\mathrm{tr}_g(Ddf) = -g(g, Ddf)$.

Note that Δ is an elliptic operator if and only if g is *Riemannian.*

1.55. Since g *induces a pseudo-Euclidean structure on each tensor bundle, any differential operator* A from tensor fields to tensor fields admits a canonical formal adjoint A^*. For example, the covariant derivative

$$D\colon \mathcal{T}^{(r,s)}M \to \Omega^1 M \otimes \mathcal{T}^{(r,s)}M$$

admits a formal adjoint

$$D^*\colon \Omega^1 M \otimes \mathcal{T}^{(r,s)}M \to \mathcal{T}^{(r,s)}M.$$

For vector fields $X_1, \ldots, X_r$ and α in $\Omega^1 M \otimes \mathcal{T}^{(r,s)}M$, $(D^*\alpha)(X_1, \ldots, X_r)$ is the opposite of the trace (with respect to g), of the $\bigotimes^s TM$-valued 2-form

$$(X, Y) \to (D_X \alpha)(Y, X_1, \ldots, X_r).$$

This also holds for natural subbundles of $T^{(r,s)}M$. In the Riemannian case, with an orthonormal basis $(Y_i)_{i=1,\ldots,n}$, we have

$$(D^*\alpha)(X_1, \ldots, X_r) = -\sum_{i=1}^n (D_{Y_i}\alpha)(Y_i, X_1, \ldots, X_r).$$

For the most useful cases, we use some special notation.

1.56 Definition. Let $d\colon \Omega^p M \to \Omega^{p+1} M$ denote the exterior differential on p-forms on M. We *denote* by δ its formal adjoint, and we call it the *codifferential.*

We may compute δ in a number of ways.

a) Take a local orientation of M and the corresponding Hodge operator $*_g$; then

$$\delta = -*_g \circ d \circ *_g;$$

b) we may consider $\bigwedge^{p+1} M$ as a subspace of $\bigwedge^1 M \otimes \bigwedge^p M$; then δ is simply the restriction of D^* to $\bigwedge^{p+1} M$;

c) in the Riemannian case, if $(Y_i)_{i=1,\ldots,n}$ is a local orthonormal basis of vector fields,

$$(\delta\alpha)(X_1, \ldots, X_p) = -\sum_{i=1}^n (D_{Y_i}\alpha)(Y_i, X_1, \ldots, X_p).$$

1.57. The operator $\Delta = d\delta + \delta d\colon \Omega^p M \to \Omega^p M$ is the *Hodge-de Rham Laplacian* on p-forms.

1.58. For any vector field X on M, its *divergence* $\operatorname{div} X$ (or δX) is the codifferential of the dual 1-form, i.e., $\operatorname{div} X = \delta(X^\flat)$. In the Riemannian case, we get

$$\operatorname{div} X = -\sum_{i=1}^{n} g(D_{Y_i} X, Y_i).$$

1.59. Instead of forms, we may also consider symmetric tensors. If we consider the covariant derivative

$$D\colon \mathscr{S}^p M \to \Omega^1 M \otimes \mathscr{S}^p M = \mathscr{S}^1 M \otimes \mathscr{S}^p M,$$

and compose with the symmetrization

$$\mathscr{S}^1 M \otimes \mathscr{S}^p M \to \mathscr{S}^{p+1} M,$$

we obtain a differential operator, *denoted* by δ^*,

$$\delta^*\colon \mathscr{S}^p M \to \mathscr{S}^{p+1} M,$$

whose formal adjoint is called the *divergence*, and denoted by δ,

$$\delta\colon \mathscr{S}^{p+1} \to \mathscr{S}^p M.$$

Notice that δ is nothing but the $\bigotimes^{p+1} TM$ restriction of D^* to $\mathscr{S}^{p+1}M$ included in $\mathscr{S}^1 M \otimes \mathscr{S}^p M$.

1.60 Lemma. *On* 1*-forms, the operator* $\delta^*\colon \Omega^1 M \to \mathscr{S}^2 M$ *satisfies*

$$\delta^* \alpha = -\tfrac{1}{2} L_{\alpha^\#} g,$$

where $L_{\alpha^\#}$ *denotes the Lie derivative of the vector field* $\alpha^\#$ *(dual of the* 1*-form* α*).*

In particular, $\delta^*\alpha = 0$ if and only if $\alpha^\#$ is a Killing vector field.

Proof.
$$\begin{aligned}
\delta^*\alpha(X, Y) &= \tfrac{1}{2}((D_X\alpha)(Y) + (D_Y\alpha)(X)) \\
&= \tfrac{1}{2}\{X\alpha(Y) - \alpha(D_X Y) + Y\alpha(X) - \alpha(D_Y X)\} \\
&= \tfrac{1}{2}\{X \cdot g(\alpha^\#, Y) - g(\alpha^\#, D_X Y) + Y \cdot g(\alpha^\#, X) - g(\alpha^\#, D_Y X)\} \\
&= \tfrac{1}{2}\{g(D_X\alpha^\#, Y) + g(D_Y\alpha^\#, X)\} \\
&= -\tfrac{1}{2}(L_{\alpha^\#} g)(X, Y)
\end{aligned}$$

(compare the proof of Theorem 1.81). □

D. Riemannian Manifolds as Metric Spaces

In the particular case of a Riemannian manifold, there is another very important invariant, the *distance*, which is defined in the following way.

Throughout section D, (M, g) is assumed to be Riemannian.

1.61 Definitions. Let (M, g) be a Riemannian manifold.

(a) Given a piecewise smooth curve $c: [a, b] \to M$, the *length* of c is

$$L(c) = \int_a^b \sqrt{g(\dot{c}, \dot{c})}\, dt.$$

(b) For each pair of points x and y in M, we *denote* by $d(x, y)$ the infimum of the lengths of all piecewise smooth curves starting from x and ending at y.

Note that in (a), if c is a geodesic, then $g(\dot{c}, \dot{c})$ is constant and $L(c) = (b - a)\sqrt{g(\dot{c}, \dot{c})}$; in (b), the infimum $d(x, y)$ may or may not be realized by a curve.

1.62 Theorem. *Given a Riemannian manifold (M, g), the function d is a distance on M, and the topology of the metric space (M, d) is the same as the manifold topology of M.*

A corollary of the Gauss Lemma 1.46 is that the distance is realized (at least locally) by geodesics. More precisely

1.63 Theorem. *For each x in M, there is a neighborhood U_x in M such that, for any y in U_x, the distance $d(x, y)$ is the length of the unique geodesic from x to y in U_x.*

1.64 Corollary. *Any geodesic minimizes the length between any pair of sufficiently near points on it; conversely, any curve having this property is (up to reparameterization) a geodesic.*

Also there is a notion of completeness of a metric space. Fortunately, these notions are as compatible as they can be. This is the content of the following theorem.

1.65 Theorem (H. Hopf-W. Rinow). *For a Riemannian manifold (M, g), the following conditions are equivalent.*

a) *(M, g) is complete for the Levi-Civita connection;*

b) *(M, d) is a complete metric space;*

c) *the bounded subsets of M are relatively compact.*

And these properties imply that

d) *for any two points x, y in M, there exists at least a geodesic starting at x and ending at y.*

Note however that there may be many more than one geodesic connecting x and y and that property d) does not imply the completeness of (M, g).

1.66 Corollary. *A compact Riemannian manifold is complete.*

(This need not be true for pseudo-Riemannian manifolds, even in the homogeneous case, see Chapter 7).

Note that a geodesic connecting two points does not necessarily realize the distance between them. The study of "minimizing" geodesics is a very important tool in Riemannian geometry. A key point is the fact that a limit of minimizing geodesics is a minimizing geodesic.

1.67 Lemma. *Let (c_k) be a sequence of geodesics and (t_k) a sequence of real numbers such that, for each k,*

$$d(c_k(0), c_k(t_k)) = t_k.$$

Assume that the vectors $(\dot{c}_k(0))$ converge in TM towards some vector X and (t_k) converge to t when k goes to infinity. Then the geodesic c such that $\dot{c}(0) = X$ satisfies

$$d(c(0), c(t)) = t.$$

Here are a few elementary applications. The *diameter* of a Riemannian manifold (M, g) is the supremum of the distances of any two points in M. A ray (respectively, a *line*) is an infinite geodesic c: $[0, +\infty[\to M$ (respectively c: $]-\infty, +\infty[\to M$) such that, for any two points x, y on c, the distance $d(x, y)$ is exactly the length of c between x and y (i.e., c minimizes the length between any two of its points).

1.68 Theorem. *If (M, g) is compact, the diameter of (M, g) is finite, and there exists x and y in M such that $d(x, y)$ is the diameter.*

If (M, g) is complete, non-compact, the diameter is infinite. For any x in M, there exists a ray c with $c(0) = x$.

Note that there is not always a line on a non-compact Riemannian manifold. But as soon as M has two "ends", there exists a line connecting them.

Finally, we want to mention that the notion of a conjugate point (see Definition 1.32) enters into the problem of distance through the following.

1.69 Theorem. *Let c be a geodesic on a Riemannian manifold (M, g); and let t_0 be such that $c(0)$ and $c(t_0)$ are* conjugate points *along c. Then, for any $t > t_0$, the geodesic c does not minimize the distance between $c(0)$ and $c(t)$.*

Note that the "*first cut point along c*" (i.e., the point $c(t_1)$ such that c fails to minimize the distance between $c(0)$ and $c(t)$ for any $t > t_1$) may appear before the first conjugate point to $c(0)$ along c.

E. Riemannian Immersions, Isometries and Killing Vector Fields

1.70 Definition. Let (M, g) and (N, h) be two pseudo-Riemannian manifolds. A smooth map f: $M \to N$ is a *pseudo-Riemannian immersion* if it satisfies $f^*h = g$ or, equivalently, if, for any x in M, the tangent map $T_x f$ satisfies

$$h((T_x f)X, (T_x f)Y) = g(X, Y)$$

for any X, Y in $T_x M$.

Note that such an f is obviously an immersion, and that the restriction of h to $(T_x f)(T_x M)$ is non-degenerate.

Conversely, given a smooth immersion $f\colon M \to N$ and a pseudo-Riemannian metric h on M, which is *non-degenerate* on $(T_x f)(T_x M)$ for *each* x in M, the map f is a pseudo-Riemannian immersion from (M, f^*h) into (N, h). Note that $f(M)$ does not need to be a submanifold of N; this happens only if f is an imbedding.

1.71. Let $f\colon (M,g) \to (N,h)$ be a pseudo-Riemannian immersion. Then we may consider the tangent bundle TM to M as a subbundle of the induced vector bundle $f^*(TN)$, which we endow with the pseudo-Euclidean structure induced from h, and the linear connection $\bar{D}$ induced from the Levi-Civita connection of h (see 1.9).

Let NM be the orthogonal complement of TM in $f^*(TN)$. We call it the *normal bundle* (of the immersion). Using 1.10, $\bar{D}$ induces a connection on TM and NM, together with a tensor. Obviously, the induced connection on TM is nothing but the Levi-Civita connection of g. We *denote* by ∇ the connection induced on NM, and we define the *second fundamental form* of f to be the unique "tensor"

$$\mathrm{II}\colon TM \otimes TM \to NM$$

such that, for two vector fields U and V on M,

$$\mathrm{II}(U, V) = \mathcal{N}(\bar{D}_U V)$$

where $\mathcal{N}$ is the orthogonal projection onto NM. We *define* also the tensor B: $TM \otimes NM \to TM$ such that for U, V in $T_x M$ and X in $N_x M$, $g(B_U X, V) = -g(\mathrm{II}(U, V), X)$. Then one easily proves

1.72 Theorem. *Let $f\colon (M,g) \to (N,h)$ be a pseudo-Riemannian immersion. Let U, V, W be vector fields on M, and X, Y be sections of NM. Then*

a) $\bar{D}_U V = D_U V + \mathrm{II}(U, V)$ *(Gauss Formula),*

b) $\bar{D}_U X = B_U X + \nabla_U X$ *(Weingarten Equation),*

c) $(\bar{R}(U, V)U, V) = (R(U, V)U, V) + |\mathrm{II}(U, V)|^2 - (\mathrm{II}(U, U), \mathrm{II}(V, V))$ *(Gauss Equation),*

d) $(\bar{R}(U, V)W, X) = -((\nabla_U \mathrm{II})(V, W), X) + ((\nabla_V \mathrm{II})(U, W), X)$ *(Codazzi-Mainardi Equation),*

e) $(\bar{R}(U, V)X, Y) = (R^\nabla(U, V)X, Y) - (B_U X, B_V Y) + (B_V X, B_U Y)$ *(Ricci Equation),*

where $\bar{R}$, R, R^∇ are the curvatures of $\bar{D}$, D and ∇ respectively, $\nabla\mathrm{II}$ is the (covariant) derivative of II with respect to ∇, and we have omitted g.

1.73 Definitions. Let $f\colon (M,g) \to (N,h)$ be a pseudo-Riemannian immersion.

a) The *mean curvature vector* of f at $x \in M$ is the normal vector

$$H_x = \operatorname{tr}\mathrm{II} = \sum_{i=1}^{n} \mathrm{II}(X_i, X_i),$$

where $X_1, \ldots, X_n$ is an orthonormal basis of $T_x M$.

b) A point x M is said to be *umbilic* if there exists a normal vector $v \in N_x M$ such that $\mathrm{II}(U, V) = g_x(U, V)v$ for any U, V in $T_x M$.

c) f has *constant mean curvature* if the normal vector field H is parallel, i.e., $\nabla H \equiv 0$;

d) f is *totally umbilic* if every point of M is umbilic;

e) f is *minimal* if $H \equiv 0$;

f) f is *totally geodesic* if $\mathrm{II} \equiv 0$.

Note that d) does not imply c) in general.

In the special case where the dimensions of M and N are equal, a pseudo-Riemannian immersion is *locally* a diffeomorphism (but not necessarily globally) and the signatures of (M, g) and (N, h) are the same.

1.74 Definition. An *isometry* is a pseudo-Riemannian immersion which is also a diffeomorphism.

In the special case of *Riemannian* manifolds, there is a characterization of isometries which involves distances.

1.75 Theorem. *A* surjective *smooth map $f\colon (M, g) \to (N, h)$ between two* Riemannian *manifolds is an isometry if and only if it preserves the distance, i.e., $d_h(f(x), f(y)) = d_g(x, y)$ for any x, y in M.*

Obviously, the composition of two isometries is an isometry and, for any isometry f, the inverse diffeomorphism f^{-1} is an isometry. As a consequence, the set of all isometries from one pseudo-Riemannian manifold (M, g) into itself is a group. We call it the *isometry group* of (M, g) and *denote* it by $I(M, g)$. As a subgroup of diffeomorphisms of M, it has a natural topology ("compact-open" topology).

1.76 Examples. a) Given any pseudo-Riemannian manifold (M, g) and any diffeomorphism $\alpha\colon N \to M$, then α is an isometry of $(N, \alpha^* g)$ onto (M, g).

b) On the flat model space $(\mathbb{R}^n, g_0)$ of 1.34, any translation is an isometry. More generally, the isometry group is exactly the semidirect product $\mathbb{R}^n \rtimes O(g_0)$ of the group of translations $\mathbb{R}^n$ by the orthogonal group $O(g_0)$ of g_0.

c) The groups $O(g_p)$ (respectively $O(g_{p+1})$) acting on $\mathbb{R}^{n+1}$ as in 1.36 preserve the submanifolds S^n_p (respectively H^n_p) and induce on them isometries of the induced metric. One may show that they induce in fact the whole isometry group.

d) More generally, if (M, g) is a pseudo-Riemannian submanifold of (N, h) (as in 1.35), any isometry of (N, h) which preserves M (i.e., such that $f(M) = M$) induces an isometry of (M, g).

The basic result on $I(M, g)$ is the following theorem.

1.77 Theorem (S.B. Myers-N. Steenrod [My-St]). *Let (M, g) be a pseudo-Riemannian manifold.*

a) *The group $I(M, g)$ of all isometries of (M, g) is a Lie group and acts differentiably on (M, g);*

b) *for any x in M, the* isotropy subgroup

$$I_x(M, g) = \{f \in I(M, g); f(x) = x\}$$

is a closed subgroup of $I(M,g)$. *Moreover, if we* denote *by*

$$\rho\colon I_x(M,g) \to Gl(T_xM), f \to \rho(f) = T_xf$$

the isotropy representation, *then* ρ *defines an isomorphism of* $I_x(M,g)$ *onto a closed subgroup of* $O(T_xM, g_x) \subset Gl(T_xM)$.

1.78 Corollary. *If* (M,g) *is a* Riemannian *manifold,* $I_x(M,g)$ *is a compact subgroup of* $I(M,g)$. *Moreover, if* (M,g) *is compact,* $I(M,g)$ *is compact.*

1.79 Remarks. a) More generally, $I(M,g)$ acts *properly* on any *Riemannian* manifold (M,g) (see S.T. Yau [Yau 6]).

b) Note that $I(M,g)$ may be compact (e.g., trivial), even if (M,g) is non-compact or non-Riemannian.

c) One may show that $\dim(I(M,g)) \leqslant \dfrac{n(n+1)}{2}$ with equality only if (M,g) has constant sectional curvature.

Of course, since an isometry preserves g, it preserves the Levi-Civita connection, the geodesics, the volume element and the different types of curvature (defined in § F). We now examine the corresponding infinitesimal notion.

1.80 Definition. Let (M,g) be a pseudo-Riemannian manifold. A vector field X on M is called a *Killing vector field* if the (local) 1-parameter group of diffeomorphisms associated to X consists in (local) isometries.

1.81 Theorem. *For a vector field* X, *the following properties are equivalent.*

a) X *is a Killing vector field;*

b) *the Lie derivative of* g *by* X *vanishes, i.e.,* $L_X g = 0$;

c) *the covariant derivative* DX *is skewsymmetric with respect to* g, *i.e.,* $g(D_YX, Z) + g(D_ZX, Y) = 0$;

Moreover, any Killing vector field satisfies also

d) *the Lie derivative of* D *by* X *vanishes, i.e.,* $L_X D = 0$;

e) *the restriction of* X *along any geodesic is a Jacobi field;*

f) *the second covariant derivative* D^2X *satisfies* $D^2_{U,V}X = R(X,U)V$.

Proof. We recall that, for any tensor (or connection) A,

$$L_X A = \frac{d}{dt}(\varphi_t^* A)|_{t=0},$$

where φ_t is the (local) 1-parameter group of diffeomorphisms generated by X.

The equivalence of a) and b) follows easily, together with d). Now

$$\begin{aligned}(L_X g)(Y,Z) &= X\cdot g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) \\ &= g(D_XY,Z) + g(Y,D_XZ) - g(D_XY,Z) + g(D_YX,Z) \\ &\quad - g(Y,D_XZ) + g(Y,D_ZX) \\ &= g(D_YX,Z) + g(Y,D_ZX),\end{aligned}$$

hence c) is equivalent to b). Then e) follows easily from the definition of Jacobi fields, since isometries preserve geodesics. Finally f) follows from e) through polarization and the algebraic Bianchi identity. □

1.82 Remarks. a) Conditions d), e), f), are *not* characteristic of Killing vector fields.

b) The bracket of two Killing vector fields is a Killing vector field, so the space of all Killing vector fields of (M, g) is a Lie subalgebra of the Lie algebra of all vector fields.

1.83 Theorem. *If (M, g) is complete, then any Killing vector field of (M, g) is complete, i.e., generates a 1-parameter group of isometries. Consequently, the Lie algebra of Killing vector fields is the Lie algebra of the Lie group $I(M, g)$.*

We finish with a vanishing theorem, due to S. Bochner [Boc 1], which involves the Ricci curvature defined in 1.90 below.

1.84 Theorem. *Let (M, g) be a compact Riemannian manifold, with Ricci curvature r.*

a) *If r is negative, i.e., if $r(U, U) < 0$ for any non-zero tangent vector U, then there are no non-zero Killing vector fields and the isometry group $I(M, g)$ is finite.*

b) *If r is nonpositive, i.e., if $r(U, U) \leqslant 0$ for any tangent vector U, then any Killing vector field on M is parallel, and the connected component of the identity in $I(M, g)$ is a torus.*

c) *If r vanishes identically, then the space of Killing vector fields has dimension exactly the first Betti number $b_1(M, \mathbb{R})$.*

We only sketch the starting point of the proof, which follows the same lines as 1.155. The relevant Weitzenböck formula is the following consequence of (1.81f): $D^*DX = \mathrm{Ric}(X)$ for any Killing vector field X. By evaluating against X and integrating over M, we get

$$\int_M |DX|^2 \mu_g = \int_M r(X, X) \mu_g$$

and the theorem follows. □

F. Einstein Manifolds

We first collect various properties of the Riemann curvature tensor R that we have met before.

1.85 Proposition. *The curvature tensor field R of a pseudo-Riemannian manifold (M, g) satisfies the following properties:*

(1.85a) *R is a $(3, 1)$-tensor;*

(1.85b) *R is skewsymmetric with respect to its first two arguments, i.e.,*

$$R(X, Y) = -R(Y, X);$$

(1.85c) *$R(X, Y)$ is skewsymmetric with respect to g, i.e.,*

$$g(R(X, Y)Z, U) = -g(R(X, Y)U, Z);$$

(1.85d) (*algebraic Bianchi identity*)

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0;$$

(1.85e) $g(R(X, Y)Z, U) = g(R(Z, U)X, Y)$;

(1.85f) (*differential Bianchi identity*)

$$(DR)(X, Y, Z) + (DR)(Y, Z, X) + (DR)(Z, X, Y) = 0.$$

Property (1.85e) follows from repeated applications of b), c) and d) (exercise!).

1.86. Using the metric g, we may also consider the curvature as a $(4, 0)$-tensor, namely

$$(X, Y, Z, U) \to g(R(X, Y)Z, U).$$

We will also use the $(2, 2)$-tensor deduced from R, that we *denote* by $\mathscr{R}$. Due to the symmetries, $\mathscr{R}$ may be considered as a linear map from $\bigwedge^2 M$ to $\bigwedge^2 M$, satisfying

$$g(\mathscr{R}(X \wedge Y), Z \wedge U) = g(R(X, Y)Z, U)$$

for any vectors X, Y, Z, U.

Notice that e) now states that $\mathscr{R}$ is a symmetric map with respect to the pseudo-Euclidean structure induced by g on $\bigwedge^2 M$ (see also 1.106).

1.87 Definition ($n \geqslant 2$). Given a non-isotropic 2-plane σ in T_xM, the *sectional curvature* of σ is given by

$$K(\sigma) = g(R(X, Y)X, Y)/(g(X, X)g(Y, Y) - g(X, Y)^2)$$

for any basis $\{X, Y\}$ of σ.

1.88 Proposition. *The sectional curvature of (M, g) is a constant k for any tangent two-plane at x if and only if the curvature tensor at x satisfies*

(1.88) $$R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y).$$

Note that for the canonical example $\mathbb{R}^n$ then $R \equiv 0$, and that the examples S_p^n and H_p^n (see 1.36) have constant sectional curvature, respectively $+1$ and -1.

The differential Bianchi identity does not allow the curvature to vary too simply.

1.89 Theorem (F. Schur). *Let $n \geqslant 3$. If for each x in M, the sectional curvatures of the 2-planes through x are all equal (to a number that might a priori depend on x), then (M, g) has constant curvature.*

Because of the various algebraic symmetries of R only one contraction gives an interesting result:

1.90 Definition. The *Ricci curvature* tensor (or *Ricci tensor*) r of a pseudo-Riemannian manifold (M, g) is the 2-tensor

$$r(X, Y) = \operatorname{tr}(Z \to R(X, Z)Y),$$

where tr denotes the trace of the linear map $Z \to R(X, Z)Y$.

Note that the Ricci tensor is symmetric (this follows from the fact that the Levi-Civita connection has no torsion).

1.91 Remark. We will sometimes view r as a $(1, 1)$-tensor and then *denote* it by $\mathrm{Ric}\colon TM \to TM$; it satisfies

$$r(X, Y) = g(\mathrm{Ric}(X), Y).$$

Notice that Ric is a g-symmetric map. In particular, in the *Riemannian* case, its eigenvalues are real (this is not always true in the pseudo-Riemannian case).

Also, still in the Riemannian case, the Ricci tensor is related to the "local" function θ (defined in 1.48) through the following inequality.

1.92 Proposition. *Let (M, g) be a Riemannian manifold, c a geodesic with $c(0) = x$ and θ the function defined in* 1.48 *for normal coordinates centered at x. If we define φ by $\varphi(c(t)) = \dfrac{1}{t}(\theta(c(t)))^{1/(n-1)}$, then*

$$(n - 1)\dot{c}\dot{c}\varphi + r(\dot{c}, \dot{c})\varphi \leqslant 0. \tag{1.92}$$

One may deduce from proposition 1.92 that if M is complete and there exists $a > 0$ such that $r(X, X) \geqslant (n - 1)a^2 g(X, X)$ for any X, then any geodesic c has a conjugate point $c(t_0)$ to $c(0)$ for some $t_0 \leqslant \dfrac{\pi}{a}$; together with 1.69, this gives Myers' theorem 6.51. More generally, a lower bound for r gives an upper bound for the growth of the volume of "distance balls" $B(x, t) = \{y \in M \text{ such that } d(x, y) \leqslant t\}$, see 0.62. In the case of *Lorentz* manifolds, similar formulas give the so-called "incompleteness theorems", see [Ha-El].

Of course, the trace of r with respect to g is also an interesting invariant.

1.93 Definition. The *scalar curvature* of a pseudo-Riemannian manifold (M, g) is the function $s = \operatorname{tr}_g r$ on M.

The scalar curvature will be studied in more detail in Chapter 4. Note that, on any manifold M, there are metrics with constant scalar curvature, see Chapter 4 below.

The derivatives of r and s are related by the following formula

1.94 Proposition. *$\delta r = -\frac{1}{2}ds$, where the divergence δ of a symmetric 2-tensor field has been defined in* 1.59.

Proof. Take suitable traces of the differential Bianchi identity. □

1.95 Definition. A pseudo-Riemannian manifold (M, g) is *Einstein* if there exists a real constant λ such that

$$r(X, Y) = \lambda g(X, Y) \tag{1.95}$$

for each X, Y in $T_x M$ and each x in M.

1.96 Remarks. a) This notion is relevant only if $n \geqslant 4$. Indeed, if $n = 1$, $r = 0$. If $n = 2$, then at each x in M, we have $r(X, Y) = \frac{1}{2} s g(X, Y)$, so a 2-dimensional pseudo-Riemannian manifold is Einstein if and only if it has constant (sectional or scalar) curvature. If $n = 3$, then (M, g) is Einstein if and only if it has constant (sectional) curvature (exercise!).

b) If we replace g by $t^2 g$ for some positive constant t, then r does not change, hence for an Einstein manifold, the number λ in 1.95 changes to λt^{-2}.

1.97 Theorem. *Assume $n \geqslant 3$. Then an n-dimensional pseudo-Riemannian manifold is Einstein if and only if, for each x in M, there exists a constant λ_x such that*

$$r_x = \lambda_x g_x. \tag{1.97}$$

Proof. The "only if" part is trivial. In the other direction, applying the divergence δ to both sides of (1.97), we get

$$\delta r = -\tfrac{1}{2} ds = -d\lambda.$$

So $\lambda - \frac{1}{2} s$ is a constant. Taking the trace of (1.97) with respect to g, we get $n\lambda = s$. So finally λ (and s) are constant. □

1.98. Einstein manifolds (especially in the Riemannian case) are the subject of this book, so many examples will be described in the coming chapters (see in particular Chapter 7). Here we give only the obvious examples. Any manifold with constant sectional curvature is Einstein. In particular, $\mathbb{R}^n$, S_p^n, H_p^n with the metrics described in 1.34 and 1.36 are Einstein.

1.99 Proposition. *The product of two pseudo-Riemannian manifolds which are Einstein* with the same constant λ *is an Einstein manifold with the same constant λ.*

Note that the product of two Einstein manifolds with different constants is *not* Einstein. In fact, we may describe these products in the following way.

1.100 Theorem. *If the Ricci tensor r of a Riemannian manifold (M, g) is* parallel (*i.e.*, $Dr = 0$), *then at least locally, (M, g) is the product (as in 1.37) of a finite number of Einstein manifolds.*

This is a consequence of the de Rham decomposition theorem, see 10.43.

In 7.117, there is an example of a *Lorentz* manifold with $Dr = 0$, but which is not, even locally, a product, and is *not* Einstein.

G. Irreducible Decompositions of Algebraic Curvature Tensors

1.101. In this paragraph, we show that the curvature tensor of a Riemannian manifold splits naturally into three components, involving respectively its scalar curvature, the trace-free part of its Ricci tensor and its Weyl curvature tensor (if $n \geqslant 4$). This basic fact does not appear in most textbooks on Riemannian geometry, but it is classical and was well known in past times.

The key point here is that the bundle in which the curvature tensor lives naturally (according to the symmetry properties 1.85) is not irreducible under the action of the orthogonal group, and consequently has a natural decomposition into irreducible components. These give a canonical decomposition of the curvature tensor.

1.102. We begin with a review of pseudo-Euclidean geometry from the point of view of the representations of the orthogonal group on higher tensor spaces. We also consider the special orthogonal group in dimension 4, which gives an extra decomposition of the Weyl tensor for oriented four dimensional Riemannian manifolds.

These results give better insight into the role played by scalar and Ricci curvature, and provide very simple proofs of some classical results.

1.103. Let E be an n-dimensional real vector space ($n > 1$). Then each tensor space $T^{(k,l)}E = \bigotimes^k E^* \otimes \bigotimes^l E$ is a representation space for the linear group $Gl(E)$. For any γ in $Gl(E)$, $\xi_1, \ldots, \xi_k$ in E^* and $x_1, \ldots, x_l$ in E, the natural action of $Gl(E)$ satisfies

$$\gamma(\xi_1 \otimes \cdots \otimes \xi_k \otimes x_1 \otimes \cdots \otimes x_l) = ({}^t\gamma^{-1}\xi_1) \otimes \cdots \otimes ({}^t\gamma^{-1})\xi_k \otimes (\gamma x_1) \otimes \cdots \otimes (\gamma x_l).$$

Let q be a *non-degenerate* quadratic form on E. Then q induces a canonical identification between E and E^*. Moreover, if γ belongs to the orthogonal group $O(q)$ of q, we have ${}^t\gamma^{-1} = \gamma$, so E and E^* are isomorphic as $O(q)$-modules, and we may consider tensor products of E only.

We recall that we write S^kE for the k-th symmetric power of E, and $\bigwedge^l E$ for the l-th exterior power of E. Also, we denote by $\circ$ the symmetric product of two tensors, with the convention that $x \otimes x = x \circ x$.

1.104. Of course, the $O(q)$-module E is irreducible. It is well-known that $\bigotimes^2 E$ is not irreducible (even as a $Gl(E)$-module). We *denote* by S_0^2E the space of traceless symmetric 2-tensors. We recall that q induces a *trace*, which may be considered as a linear map $\mathrm{tr}_q\colon S^2E \to \mathbb{R}$.

1.105 Proposition. *The irreducible orthogonal decomposition of the $O(q)$-module* $\bigotimes^2 E$ is the following

$$\bigotimes^2 E = \bigwedge^2 E \oplus S_0^2 E \oplus \mathbb{R}q.$$

Proof. It is obvious that for any k in $\bigotimes^2 E$ $(= \bigotimes^2 E^*)$, we have

$$k = \Lambda^2 k + S_0^2 k + (\mathrm{tr}_q k/n)q,$$

where
$$A^2k(x,y) = \tfrac{1}{2}(k(x,y) - k(y,x)),$$
$$S^2k(x,y) = \tfrac{1}{2}(k(x,y) + k(y,x))$$
and
$$S_0^2k = S^2k - (\operatorname{tr}_q k/n)q.$$

The non-trivial point is that $\bigwedge^2 E$ and $S_0^2 E$ are irreducible. One proof of that follows from invariant theory; it suffices to check that the vector space of $O(q)$-invariant quadratic forms on $\bigotimes^2 E$ is 3-dimensional (cf. [Wey] II.9 and II.17, [Bes 2] Exposé IX or [Be-Ga-Ma] p. 82–83 for more details). □

1.106. We now study the tensors satisfying the same algebraic identities as the curvature tensor of a pseudo-Riemannian manifold at one point. The properties 1.85a), b), c) and e) mean that, if $E = T_x^*M$ and $g = g_x$, through the identification $\bigotimes^3 E \otimes E^* = \bigotimes^4 E$, the curvature tensor lies inside the subspace $S^2\bigwedge^2 E$.

Moreover, we know that $\bigotimes^k E$ is not irreducible, already as a $Gl(E)$-module. Indeed, the symmetric group $\mathfrak{S}_k$ (and its algebra $\mathbb{R}(\mathfrak{S}_k)$) provides natural $Gl(E)$-morphisms. The $Gl(E)$-irreducible components of $\bigotimes^k E$ appear as kernels of certain idempotents of $\mathbb{R}(\mathfrak{S}_k)$, the so-called *Young symmetrizers* (cf [Wey] Chap. IV or [Nai] II.3).

In particular, the algebraic Bianchi identity 1.85d) corresponds to the following Young symmetrizer.

1.107 Definition. We define the *Bianchi map* b to be the following idempotent of $\bigotimes^4 E$

$$b(R)(x,y,z,t) = \tfrac{1}{3}(R(x,y,z,t) + R(y,z,x,t) + R(z,x,y,t))$$

for any R in $\bigotimes^4 E$ and x, y, z, t in E^*.

Obviously, b is $Gl(E)$-equivariant, $b^2 = b$, and b maps $S^2\bigwedge^2 E$ into itself. So we have the $Gl(E)$-equivariant decomposition

$$S^2\textstyle\bigwedge^2 E = \operatorname{Ker} b \oplus \operatorname{Im} b.$$

Moreover, an easy calculation shows that $\operatorname{Im} b$ is precisely $\bigwedge^4 E$ (note that this fact implies that $b = 0$ on $S^2\bigwedge^2 E$ if $n = 2$ or 3. In other words, the Bianchi identity follows from the other ones in these dimensions).

1.108 Definition. We let $\mathscr{C}E = \operatorname{Ker} b$ (in $S^2\bigwedge^2 E$) and we call it the vector space of "*algebraic curvature tensors*".

Of course, the curvature tensor of a pseudo-Riemannian manifold lies inside $\mathscr{C}T_x^*M$ at each point x. But the key point of the whole paragraph is that this space $\mathscr{C}E$ may be decomposed as a $O(q)$-module. For algebraic curvature tensors, we introduce the notions equivalent to the Ricci and scalar curvatures for curvature tensors.

1.109 Definition. The *Ricci contraction* is the $O(q)$-equivariant map

$$c\colon S^2\textstyle\bigwedge^2 E \to S^2 E$$

defined, for any R in $S^2\bigwedge^2 E$ and any x, y in E^*, by

$$c(R)(x,y) = \operatorname{tr} R(x,.,y,.).$$

Conversely, there is a canonical way to build an element of $S^2\bigwedge^2 E$ from two elements of $S^2 E$.

1.110 Definition. The *Kulkarni-Nomizu product* of two symmetric 2-tensors h and k is the 4-tensor $h \owedge k$ given by

$$(h \owedge k)(x,y,z,t) = h(x,z)k(y,t) + h(y,t)k(x,z) - h(x,t)k(y,z) - h(y,z)k(x,t),$$

for any x, y, z, t in E^*.

An easy computation gives

1.111 Proposition.
a) $h \owedge k = k \owedge h$;
b) $h \owedge k$ *belongs to* $\mathscr{C}E$;
c) $q \owedge q$ *is twice the identity of* $\bigwedge^2 E$ *(through the identification* $\operatorname{End}(\bigwedge^2 E) = \bigotimes^2 \bigwedge^2 E$*)*.

Note that in the Riemannian case, the curvature tensor of a manifold with constant sectional curvature k is exactly $\frac{k}{2} g_x \owedge g_x$.

1.112 Remark. The Kulkarni-Nomizu product is a special case of the natural product of the graded algebra $\sum_{p=0}^{n} S^2(\bigwedge^p E)$ where

$$(\alpha \circ \beta)\cdot(\mu \circ \nu) = (\alpha \wedge \mu) \circ (\beta \wedge \nu).$$

We recall that q also induces quadratic forms (still denoted by q) on $\bigotimes^k E$, and on subspaces as $S^2 E$ and $S^2\bigwedge^2 E$. If we identify $S^2 E$ with $\operatorname{End} E$ and $S^2(\bigwedge^2 E)$ with $\operatorname{End}(\bigwedge^2 E)$, then, in both cases, $q(h,k) = \operatorname{tr}(h \circ k)$ where $\circ$ is here the composition of linear maps. Then a straightforward computation gives

1.113 Lemma. *If* $n > 2$, *the map* $q \owedge \cdot$ *from* $S^2 E$ *into* $\mathscr{C}E$ *defined by* $k \mapsto q \owedge k$ *is* injective *and its adjoint is precisely the restriction to* $\mathscr{C}E$ *of the Ricci contraction.*

Now we come to the fundamental result.

1.114 Theorem. *If* $n \geqslant 4$, *the* $O(q)$*-module* $\mathscr{C}E$ *has the following orthogonal decomposition into (unique) irreducible subspaces*

(1.114) $$\mathscr{C}E = \mathscr{U}E \oplus \mathscr{Z}E \oplus \mathscr{W}E$$

where

$$\mathscr{U}E = \mathbb{R}q \owedge q,$$

$$\mathscr{Z}E = q \owedge (S_0^2 E)$$

$$\mathscr{W}E = \operatorname{Ker}(c \restriction \mathscr{C}E) = \operatorname{Ker} c \cap \operatorname{Ker} b.$$

Proof. The existence of the decomposition is clear from Lemma 1.113 (with $S^2E = \mathbb{R}q \oplus S_0^2E$). Here again the key point is the irreducibility of the factor $\mathscr{W}E$. This follows from invariant theory, since the vector space of $O(q)$-invariant quadratic forms on $\mathscr{C}E$ is 3-dimensional. More precisely, this vector space is generated by $q(R,R)$, $q(c(R),c(R))$ and $(\operatorname{tr} c(R))^2$, cf. [Bes 2] Exposé IX and [Be-Ga-Ma] pp. 82–83. □

1.115 Remark. In terms of representation theory, one may show that S_0^2E and $\mathscr{W}E$ (or more precisely their complexifications) are the irreducible $O(q)$-modules whose highest weights are twice the highest weights of E and $\bigwedge^2 E$ respectively.

1.116 Definitions. a) $\mathscr{W}E$ is called the space of *Weyl tensors*;

b) for any algebraic curvature tensor R, we *denote* by W (or $W(R)$) its component in $\mathscr{W}E$ and we call W the *Weyl part* of R.

One can compute W explicitly, using the Ricci contraction and the trace. Indeed, for any h in S^2E, an easy computation gives

$$c(q \owedge h) = (n-2)h + (\operatorname{tr} h)q.$$

So, if we write $r = c(R)$ and $s = \operatorname{tr} r$, we get the formula

$$R = \frac{s}{2n(n-1)} q \owedge q + \frac{1}{n-2} z \owedge q + W, \tag{1.116}$$

where we have *denoted* $r - \frac{s}{n} q$ by z.

In some sense, W appears as a remainder after successive "divisions" by q.

H. Applications to Riemannian Geometry

Let (M,g) be a pseudo-Riemannian manifold. For any x in M the curvature tensor R_x (through the identification $T_xM = T_x^*M$) belongs to $\mathscr{C}T_x^*M$ (with $q = g_x^*$). Of course $r = c(R)$ is the Ricci curvature and $s = \operatorname{tr} r$ the scalar curvature. Formula (1.116) gives the following

1.117 Definition. The *Weyl tensor* W of an n-dimensional pseudo-Riemannian manifold (M,g) (with $n \geqslant 4$) is the Weyl part (considered as a $(3,1)$-tensor) of its curvature tensor.

Of course an explicit computation of W is given by Formula (1.116) (as a $(4,0)$-tensor).

In section J below, we will show that W depends only on the conformal structure defined by g; in particular, $W = 0$ if and only if (M,g) is conformally flat (if $n \geqslant 4$), see 1.164 below.

1.118. From Definition 1.95, we see that (M,g) is Einstein if and only if the component of R in $\mathscr{Z}(T_xM)$ is zero for each x in M. Note that a manifold has constant sectional curvature if and only if the components of R in $\mathscr{Z}(T_xM)$ and $\mathscr{W}(T_xM)$ both vanish.

These various components of R are not completely independent as fields, since there are relations among their derivatives. The first basic relation is $\delta r = -\frac{1}{2}ds$ (see 1.94). We shall encounter three other similar relations, see 4.72 and 16.3.

1.119. When n is 2 or 3, the situation is simpler.

a) *If* $n = 2$, we have $S^2\wedge^2 E = \mathbb{R}q \owedge q$ and $R = \frac{s}{4} q \owedge q$, $r = \frac{s}{2} q$. Moreover, $s = 2\kappa$ where κ is the *Gauss curvature* (i.e., the sectional curvature of the 2-plane T_xM).

b) *If* $n = 3$, we have $S^2\wedge^2 E = \mathbb{R}q \owedge q \oplus S_0^2 E \owedge q$ and $R = \frac{s}{12} q \owedge q + \left(r - \frac{s}{3} q\right) \owedge q$.

In particular, the Ricci curvature determines the full curvature tensor and we have

1.120 Proposition. *A* 2 *or* 3*-dimensional pseudo-Riemannian manifold is Einstein if and only if it has constant sectional curvature.*

1.121. If we consider subgroups of $O(q)$, then more refined decompositions may appear. In the geometric case, this happens when M has additional structures (Hermitian or quaternionic, for example) and the theory is particularly interesting when the holonomy group of (M,g) is smaller than $O(n)$ (see Chapter 10).

Here we consider only the case of the special orthogonal group $SO(q)$, which corresponds geometrically to the case where M is oriented. It is known that the decomposition (1.114) is also $SO(q)$-irreducible if $n \neq 4$ (see [Kir] or [Bes 2] n° 9). But new phenomena occur when $n = 4$. This is related to the non-simplicity of $SO(q)$ (or its complexification if the signature of q is $(1,3)$.).

1.122. Throughout this section, the space E is an oriented 4-dimensional vector space and q a *positive definite* quadratic form on E. We restrict ourselves to this case because the phenomena are different for other signatures (see below 1.130).

We consider the Hodge operator $*$ defined in 1.51. First note that $*$ induces an isomorphism from $\wedge^4 E$ to $\wedge^0 E = \mathbb{R}$ as $SO(4)$-modules (we recall that they are *not* isomorphic as $O(4)$-modules). Also $*$ induces an automorphism of $\wedge^2 E$, which is selfadjoint, so we may consider it as well as an element of $S^2\wedge^2 E$. The main point here is that, as an automorphism of $\wedge^2 E$, $*$ is an involution. Let

$$\wedge^+ E = \{\alpha \in \wedge^2 E;\ *\alpha = \alpha\}$$
$$\wedge^- E = \{\alpha \in \wedge^2 E;\ *\alpha = -\alpha\}.$$

Then we have of course $\wedge^2 E = \wedge^+ E \oplus \wedge^- E$ and more precisely

1.123 Proposition. *The $SO(4)$-modules $\bigwedge^+ E$ and $\bigwedge^- E$ are both irreducible and 3-dimensional but they are* not *$SO(4)$-isomorphic. The $SO(4)$-module $\bigotimes^2 E$ admits the decomposition into (unique) irreducible subspaces*

$$\bigotimes^2 E = \bigwedge^+ E \oplus \bigwedge^- E \oplus S_0^2 E \oplus \mathbb{R}q.$$

For a proof, of this statement and of the next one, see [Ast] or [Bes 2] $n°$ IX. □
Another important property of these modules is the following

1.124 Lemma. *The subspace $\bigwedge^+ E \otimes \bigwedge^- E$ of $\bigotimes^4 E$ is exactly $\mathscr{Z}E$, so the map q Ⓐ· induces a canonical isomorphism of $S_0^2 E$ with $\bigwedge^+ E \otimes \bigwedge^- E$.*

1.125 Remark. As an $SO(4)$-module, $\bigwedge^2 E$ is isomorphic to the Lie algebra $\mathfrak{so}(4)$ of $SO(4)$ (with the adjoint representation). This isomorphism maps $\bigwedge^+ E$ and $\bigwedge^- E$ onto the two 3-dimensional commuting ideals in $\mathfrak{so}(4)$, isomorphic to $\mathfrak{so}(3)$ (cf [Bes 2] appendix 1).

We now study the irreducible decomposition of $S^2 \bigwedge^2 E$ as an $SO(4)$-module. Of course $*$ generates $\bigwedge^4 E$ and we see easily that for any R in $S^2 \bigwedge^2 E$, $b(R) = q(R, *)$.
Now the decomposition of $\bigwedge^2 E$ immediately yields the following

1.126 Theorem. *The $SO(4)$-module $S^2 \bigwedge^2 E$ admits the orthogonal decomposition into irreducible subspaces*

$$S^2 \bigwedge^2 E = \mathbb{R} Id_{\bigwedge^+ E} \oplus \mathbb{R} Id_{\bigwedge^- E} \oplus (\bigwedge^+ E \otimes \bigwedge^- E) \oplus S_0(\bigwedge^+ E) \oplus S_0(\bigwedge^- E).$$

We easily see that $Id_{\bigwedge^+ E} + Id_{\bigwedge^- E} = Id_{\bigwedge^2 E} = \frac{1}{2} q \,Ⓐ\, q$,

and $$Id_{\bigwedge^+ E} - Id_{\bigwedge^- E} = *;$$

thus $$\mathbb{R} Id_{\bigwedge^+ E} \oplus \mathbb{R} Id_{\bigwedge^- E} = \mathbb{R} q \,Ⓐ\, q \oplus \mathbb{R}*,$$

and $$\mathbb{R}* = \bigwedge^4 E.$$

1.127. Let

$$\mathscr{W}^+ E = S_0(\bigwedge^+ E)$$

and $$\mathscr{W}^- E = S_0(\bigwedge^- E).$$

Since $\mathscr{Z}E = \bigwedge^+ E \otimes \bigwedge^- E$, we have $\mathscr{W}E = \mathscr{W}^+ E \oplus \mathscr{W}^- E$, so the irreducible decomposition of $\mathscr{C}E$ as an $SO(4)$-module is the following

$$\mathscr{C}E = \mathscr{U}E \oplus \mathscr{Z}E \oplus \mathscr{W}^+ E \oplus \mathscr{W}^- E.$$

The only difference between this and 1.114 is that the Weyl tensor may be split into two parts, which we have denoted by W^+ and W^-.

1.128. We may characterize the various subspaces by the behaviour of their elements under composition with $*$ as selfadjoint linear maps of $\bigwedge^2 E$. Namely, we have

$$\mathscr{Z}E = \{R \in S^2\wedge^2 E;\ *R = -*R\}$$

$$\mathscr{W}^+E = \{R \in S_0^2\wedge^2 E;\ *R = R* = R\}$$

$$\mathscr{W}^-E = \{R \in S_0^2\wedge^2 E;\ *R = R* = -R\}.$$

As a corollary, if we consider R in $S^2\wedge^2 E$ as a linear map of $\wedge^2 E$ and if we decompose $\wedge^2 E = \wedge^+ E \oplus \wedge^- E$, we get a matrix

$$R = \left(\begin{array}{c|c} W^+ + \dfrac{s}{12} Id & Z \\ \hline {}_tZ & W^- + \dfrac{s}{12} Id \end{array}\right).$$

In dimension 4, we obtain a characterization of Einstein manifolds.

1.129 Corollary. *For a 4-dimensional Riemannian manifold, the following properties are equivalent:*

a) (M, g) *is Einstein;*

b) *for any local orientation,* $R* = *R$;

c) *for any x in M and any 2-plane σ in T_xM, its sectional curvature is the same as the sectional curvature of its orthogonal 2-plane $\sigma^\perp$ in T_xM.*

(All these results from 1.126 to 1.129 are due to I.M. Singer and J. Thorpe [Si-Th]).

Note that, given a point x in M, there always exists an orientation in a neighborhood of x in M.

1.130 Remark. a) If q has signature $(2, 2)$, then the same proof gives an analogous decomposition, since in this case also $*^2 = 1$.

b) Now if q has signature $(1, 3)$, the decomposition is quite different since $*^2 = -1$ (so $\wedge^2 E$ has a natural complex structure). The decomposition in this case (due to A. Petrov) is studied in Chapter 3.

The difference in decomposition arises from the fact that $SO(3, 1)$ is simple, unlike $SO(4)$ or $SO(2, 2)$, (although they all have the same complexification).

As shown in 1.119, the curvature tensor has special properties in dimension $n = 2$ or 3 and its decomposition differs from the "general case" $n \geqslant 4$. In fact, there are also some special properties in dimension 4 (besides the preceding ones) and we will need them in Chapters 4 and 12. We begin with some general definitions.

1.131 Definitions. Let R be an algebraic curvature tensor.

a) We *denote* by $\check{R}$ the symmetric 2-tensor such that

$$\check{R}(x, y) = \sum_{i,j,k=1}^{n} \varepsilon_i\varepsilon_j\varepsilon_k R(x, e_i, e_j, e_k) R(y, e_i, e_j, e_k)$$

where x, y are in E^* and (e_i) is an orthonormal basis of E^* with $q(e_i, e_j) = \delta_{ij}\varepsilon_i$.

b) We *denote* by $\mathring{R}$ the linear map of S^2E into itself such that

$$(\mathring{R}h)(x, y) = \sum_{i=1}^{n} \varepsilon_i h(R(x, e_i)y, e_i)$$

for any h in S^2E and x, y, e_i as above, where R is then viewed as an element of $\bigotimes^2 E \otimes E^*$.

Easy calculations give

1.132 Proposition. a) $\operatorname{tr} \check{R} = q(R, R)$;

b) $\mathring{R}$ *is symmetric*;

c) R *is Einstein* $\left(\text{i.e., } r = \dfrac{s}{n} q\right)$ *if and only if* $\mathring{R}$ *maps* S_0^2E *into itself.*

See [Bo-Ka] for a further study of $\mathring{R}$. This operator appears naturally in the infinitesimal variation of the Ricci curvature, hence in the study of moduli of Einstein manifolds (see 12.30).

1.133 Remark. As a corollary, if R is Einstein and $n = 4$, then $\check{R} = \frac{1}{4} q(R, R)R$, but the converse is not true. For example, the curvature tensor of the Riemannian product $S^2 \times H^2$ (where the two spaces have opposite constant curvature) satisfies $\check{R} = \frac{1}{4}|R|^2 q$ since here $s = W = 0$ but $S^2 \times H^2$ is not Einstein.

I. Laplacians and Weitzenböck Formulas

1.134. On a Riemmanian manifold (M, g), the differential operators giving information on the geometry of M must be tied to the tensor field g. Among differential operators of order 1 acting on tensor fields, we have already seen that the Levi-Civita connection D plays a very special role. It maps a tensor field of order k to a tensor field of order $k + 1$ (via the metric we ignore the variance). Its principal symbol σ_D: $T^*M \otimes (\bigotimes^k TM) \to \bigotimes^{k+1} TM$ is the identity. (Here, we use the identification of T^*M with TM given by the metric.)

1.135. We will be interested in a special class of second order differential operators from which many geometric statements can be derived. We first identify this family.

The principal symbol σ_A of a second order differential operator A acting on sections of a vector bundle E maps $S^2T^*M \otimes E$ to E. We say for convenience that A is a *Laplace operator* if, for ξ in T^*M and e in E, $\sigma_A(\xi \otimes \xi \otimes e) = -g(\xi, \xi)e$. (We still denote the metric on covectors by g.) Other definitions would be possible.

Of special importance is the *rough Laplacian* $\nabla^*\nabla$ of a connection ∇ on the vector bundle E. From now on, *we suppose that E is a hermitian vector bundle.* We denote the formal adjoint of ∇ by ∇^*. Notice that $\nabla^*\nabla$ can be written as $-\operatorname{Trace} \nabla^1 \nabla$ where ∇^1 is the covariant derivative on the bundle $T^*M \otimes E$ defined by $\nabla^1 = \nabla \otimes 1_{T^*M} + 1_E \otimes D$.

1.136. The main property of an operator of the class we want to consider is the following. A Laplace operator is said to admit *a Weitzenböck decomposition* if

$$A = \nabla^*\nabla + K(R^\nabla)$$

where $K(R^\nabla)$ depends linearly on the curvature $R^\nabla \in \Omega^2(M, \mathrm{End}(E))$ of the covariant derivative ∇ on the bundle E.

1.137. Weitzenböck decompositions are important because of the following. There is a method, due to Bochner [Boc 1], of proving vanishing theorems for the null space of a Laplace operator admitting a Weitzenböck decomposition (and further of estimating its lowest eigenvalue). This method works mostly on *compact* manifolds. If $A = \nabla^*\nabla + K(R^\nabla)$,

$$\int_M (A\varphi, \varphi)v_g = \int_M \{(\nabla^*\nabla\varphi, \varphi) + (K(R^\nabla)\varphi, \varphi)\}v_g$$
$$= \int_M |\nabla\varphi|^2 v_g + \int_M (K(R^\nabla)\varphi, \varphi)v_g.$$

When $K(R^\nabla)$ is a non-negative endomorphism at each point, then the right hand side is the sum of two non-negative terms. Hence $A\varphi = 0$ implies both $\nabla\varphi = 0$ and $K(R^\nabla)\varphi = 0$.

If $K(R^\nabla)$ is strictly positive, then clearly $\varphi = 0$. This argument is by now classical in differential geometry and will be used several times in this book for various Laplace operators A. Part of the game is often to manufacture the appropriate second order operator. The Weitzenböck formula is then just a verification. To work with an auxiliary coefficient bundle turns out to be useful. This can be the cotangent bundle as in 12.69, 16.9 or 16.54.

1.138. A few comments are in order. When a vector bundle E comes equipped with a covariant derivative, a sequence (a_k) of relative symbols can be attached to a differential operator A of order k mapping sections of E to sections of another vector bundle F, so that $A = \sum_{i=0}^k a_i \nabla^i$ where $a_i \in C^\infty(S^iT^*M \otimes E, F)$. Of course a_k coincides with the principal symbol of A. Thanks to the covariant derivative, one can give a more algebraic description of differential operators.

In this context a Laplace operator admits a Weitzenböck decomposition if its relative symbol of order 1 vanishes and if its relative symbol of order 0 depends linearly on the curvature. These conditions are automatically satisfied by natural Riemannian differential operators on subbundles of the tangent bundle (cf [Eps], [Str] or [Ter] for a complete theory).

1.139. For *any* representation ρ of $O(n)$ (or $SO(n)$ or $\mathrm{Spin}(n)$ if the manifold is orientable or spin respectively), the associated vector bundle E_ρ and the representation on the Lie algebra defines a section c_ρ with $c_\rho \in \Omega^2(M, \mathrm{End}(E))$.

Here we have used the metric to identify the exterior power representation with the adjoint representation of $O(n)$ on its Lie algebra of skewadjoint matrices. We also use the Levi-Civita connection on E.

If $\mathscr{R} \in C^\infty(\bigwedge^2 T^*M \otimes \bigwedge^2 T^*M)$ is the Riemann curvature tensor, then we may form

$$c_\rho \otimes c_\rho(R) \in C^\infty(\mathrm{End}(E) \otimes \mathrm{End}(E)),$$

and, composing the endomorphisms, a section $c_\rho^2(R) \in C^\infty(\mathrm{End}(E))$. If $(\omega_\alpha)\left(1 \leqslant \alpha \leqslant \frac{n(n-1)}{2}\right)$ is an orthonormal basis for $\bigwedge^2 T^*M$ and $\mathscr{R} = \sum_{\alpha\beta} R_{\alpha\beta}\omega_\alpha \otimes \omega_\beta$, then

(1.140)
$$c_\rho^2(R) = \sum_{\alpha,\beta} R_{\alpha\beta} c_\rho(\omega_\alpha) c_\rho(\omega_\beta).$$

As we know (cf. 1.86), $R_{\alpha\beta} = R_{\beta\alpha}$.

Since $c_\rho(\omega_\alpha)$ is skew-adjoint, $c_\rho^2(R)$ is then self-adjoint.

1.141. An operator of the form

(1.142)
$$A = D^*D + \mu c_\rho^2(R)$$

(where μ is a constant) has the important property that when the holonomy group of the Levi-Civita connection reduces to $G \subseteq O(n)$, the operator A commutes with the orthogonal projection onto subbundles of E_ρ determined by the decomposition of the representation into irreducible components under the action of G.

1.143. An illustration of this construction is given by the so-called *Lichnerowicz Laplacian* on p-tensor fields. In [Lic 2], A. Lichnerowicz introduced the operator Δ_L defined for T in $C^\infty(\bigotimes^p T^*M)$ by

$$\Delta_L T = D^*DT + \Gamma T$$

where

$$(\Gamma T)_{i_1\ldots i_p} = \sum_k r_{i_k j} T_{i_1\ldots}{}^{j}{}_{\ldots i_p} - \sum_{k\neq l} R_{i_k j i_l h} T_{i_1\ldots}{}^{j}{}_{\ldots}{}^{h}{}_{\ldots i_p}.$$

It is a straightforward matter to check that

(1.144)
$$\Gamma R = -2c_\rho^2(R)$$

for the natural representation of $O(n)$ on p-tensors.

1.145. The second order differential operator which defines first order deformations of Einstein metrics is the Lichnerowicz Laplacian for symmetric 2-tensor fields (see [Be-Eb] and Chapter 12). The curvature term may of course be described in the invariant manner given above.

1.146. We now list the Weitzenböck decompositions for the following operators from the above viewpoint:

i) the Hodge Laplacian $\Delta_H = dd^* + d^*d$ acting on exterior differential p-forms;

ii) the complex Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on exterior differential forms of type $(0, p)$ on a Kähler manifold (see Chapter 2);

iii) the Dirac Laplacian $\mathscr{D}^2$ acting on the (complex) spinor bundle ΣM (see next alinea);

iv) the Dirac Laplacian $(\mathscr{D}^\nabla)^2$ acting on sections of $\Sigma M \otimes E$ for a coefficient bundle E with connection ∇.

1.147. The Dirac operator $\mathscr{D}$: $C^\infty(\Sigma M) \to C^\infty(\Sigma M)$ on a spin manifold is defined by

$$\mathscr{D} = \sigma \circ D,$$

where σ: $C^\infty(T^*M \otimes \Sigma M) \to C^\infty(\Sigma M)$ is the Clifford multiplication map

$$\sigma(\alpha \otimes \varphi) = \alpha \cdot \varphi.$$

Thus

$$\mathscr{D}^2\varphi = \sigma^2 D^2 \varphi$$

where $\mathscr{D}^2\varphi \in C^\infty(T^*M \otimes T^*M \otimes \Sigma M)$ and

(1.148) $$\sigma^2(\alpha \otimes \beta \otimes \varphi) = \alpha \cdot \beta \cdot \varphi.$$

But, under Clifford multiplication,

$$\alpha \cdot \beta + \beta \cdot \alpha = -2g(\alpha, \beta) Id,$$

so that

(1.149) $$\mathscr{D}^2\varphi = D^*D\varphi + c_\sigma^2(R)\varphi.$$

Now on the Lie algebra the spin representation δ is given [Hit 2] for $a = \sum a_{ij} e_i \wedge e_j$ by

$$c_\delta(a) = -\frac{1}{4}\sum a_{ij} e_i \cdot e_j$$

so

$$\begin{aligned}\sum \sigma^2(e_i \otimes e_j \otimes a\varphi) &= \sum a_{ij} e_i \cdot e_j \cdot \varphi \\ &= -4\delta(a)\varphi.\end{aligned}$$

Thus, from (1.149),

$$\mathscr{D}^2 = D^*D - 4c_\delta^2(R).$$

It is a straightforward computation involving the symmetries of the curvature tensor to show that

$$\sum R_{ijkl} e_i \cdot e_j \cdot e_k \cdot e_l = -2s Id \in C^\infty(\mathrm{End}(\Sigma M))$$

and hence one obtains the Lichnerowicz formula [Lic 4].

(1.150) $$\mathscr{D}^2 = D^*D + \tfrac{1}{4}s.$$

1.151. It is often convenient to work with a coefficient bundle.

The Dirac operator $\mathscr{D}^\nabla: C^\infty(\Sigma M \otimes E) \to C^\infty(\Sigma M \otimes E)$ for a coefficient bundle E with connection ∇ is defined by

$$\mathscr{D}^\nabla = (\sigma \otimes 1)(D \otimes 1 + 1 \otimes \nabla).$$

Thus

(1.151) $$(\mathscr{D}^\nabla)^2 = \nabla^{1*}\nabla^1 + c_\sigma^2(R \otimes 1 + 1 \otimes R^\nabla).$$

Hence, from I.147,

(1.152) $$(\mathscr{D}^\nabla)^2 = \nabla^{1*}\nabla^1 + \tfrac{1}{4}s + c_\sigma^2(1 \otimes R^\nabla)$$

where

$$c_\sigma^2(1 \otimes R^\nabla)(\varphi \otimes e) = \sum e_i \cdot e_j \cdot \varphi \otimes R_{ij}(e).$$

In the special case (which will be important in Chapter 13) where (E, ∇) is itself associated with the principal frame bundle with the Levi-Civita connection by a representation ρ, we have

$$c_\sigma^2(1 \otimes R^\nabla) = -4c_{\delta\otimes\rho}(R) \in C^\infty(\mathrm{End}(\Sigma M \otimes E)),$$

with

(1.153) $$c_{\delta\otimes\rho}(R) = \sum_{\alpha,\beta} R_{\alpha\beta}\delta(\omega_\alpha) \otimes \rho(\omega_\beta).$$

Let us take the coefficient bundle $E = \Sigma M$, the spinor bundle. Then (cf [At-Bo-Sh], [Hit 2]), if n is even,

$$\Sigma M \otimes \Sigma M \simeq \bigoplus_{p=0}^{n} \textstyle\bigwedge^p T_{\mathbb{C}}^* M,$$

and, if n is odd,

$$\Sigma M \otimes \Sigma M \simeq \bigoplus_{p=0}^{[\frac{n}{2}]} \textstyle\bigwedge^{2p} T_{\mathbb{C}}^* M \simeq \bigoplus_{p=0}^{[\frac{n}{2}]} \textstyle\bigwedge^{2p+1} T_{\mathbb{C}}^* M.$$

Furthermore, Clifford multiplication on the left is just, for n even, $e(\alpha) - i(\alpha)$, and, for n odd, $\pm * e(\alpha) \pm * i(\alpha)$, where $e(\alpha)$ denotes exterior multiplication by a 1-form α and $i(\alpha)$ its adjoint, interior multiplication.

Consequently,

$$\mathscr{D}^D = d + d^* \qquad \text{for } n \text{ even},$$

$$\mathscr{D}^D = \pm * d \pm d * \quad \text{for } n \text{ odd},$$

and in either case

$$(\mathscr{D}^D)^2 = dd^* + d^*d, \text{ the Hodge Laplacian } \Delta.$$

We may therefore use the result of 1.151 to write

$$\Delta = dd^* + d^*d = D^*D - 4(c_\delta^2 \otimes 1 + c_\delta \otimes c_\delta)(R).$$

Now for the tensor product representation ρ on $\Sigma M \otimes \Sigma M$, the section $c_\rho \in C^\infty(\mathrm{Hom}(\bigwedge^2 T^*M, \Sigma M \otimes \Sigma M)$ is

$$c_\rho = c_\delta \otimes 1 + 1 \otimes c_\delta$$

and then

$$c_\rho^2(R) = (c_\delta^2 \otimes 1 + 2c_\delta \otimes c_\delta + 1 \otimes c_\delta^2)(R).$$

But $c_\rho^2(R) = -\frac{1}{16}s$ is a scalar, hence

$$(c_\delta^2 \otimes 1 + c_\delta \otimes c_\delta)(R) = \tfrac{1}{2}c_\rho^2(R),$$

and we obtain

(1.154) $$dd^* + d^*d = D^*D - 2c_\rho^2(R),$$

which is a Lichnerowicz Laplacian (cf. 1.143).

1.155. An even more special case consists of the representation on 1-forms, the standard n-dimensional representation of $O(n)$. Here

$$\begin{aligned} dd^* + d^*d &= D^*D - 2\sum \tfrac{1}{2}R_{ijjl}e_i \otimes e_l \\ &= D^*D + r \end{aligned}$$

which yields Bochner's vanishing theorem [Boc 1] (cf. Chapter 6): if $r > 0$, then the first Betti number of a compact manifold vanishes.

1.156. A Kähler manifold can be characterized as a Riemannian manifold whose holonomy reduces to $U(m) \subseteq SO(2m)$. From [At-Bo-Sh], [Hit 2], the spinor bundle ΣM is given by

(1.157) $$\Sigma M \simeq \bigoplus_{p=0}^{n} \Lambda^{0,p}T_{\mathbb{C}}^*M \otimes K^{1/2}$$

where $K^{1/2}$ is the line bundle associated with the (2-valued) representation

$$(\det)^{-1/2}\colon U(m) \to U(1).$$

(Note that in computing Weitzenböck formulas, the global problems of orientation and spin play no part.)

Thus, from (1.157), we obtain

$$\bigoplus \Lambda^{0,p}T_{\mathbb{C}}^*M \simeq \Sigma M \otimes K^{-1/2}$$

where, moreover, the Dirac operator

$$\mathscr{D}^D = \sqrt{2}(\bar\partial + \bar\partial^*) \qquad \text{[Hit 2]}.$$

Thus

$$\bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \tfrac{1}{2}(\mathscr{D}^D)^2.$$

From (1.152), we find

$$\bar\partial\bar\partial^* + \bar\partial^*\bar\partial = D^*D + \tfrac{1}{4}s - 4(c_\delta \otimes c_\rho)(R).$$

In this case the curvature of the coefficient bundle is half the Ricci form and the eigenvalues of $-4(c_\delta \otimes c_\rho)(R)$ are $\frac{1}{2}\sum_{i=1}^{m}(\pm\lambda_i)$ where the Ricci form has eigenvalues λ_i $(1 \leqslant i \leqslant m)$. Since

$$s = 2\sum_{i=1}^{m}\lambda_i,$$

we obtain from the vanishing argument quoted as Bochner's method (cf. 1.137) and the Dolbeault isomorphism the theorem of Bochner and Yano [Bo-Ya]: if on a Kähler manifold we have $r > 0$, then $H^p(M, \mathcal{O}) = 0$ for $p > 0$ where $\mathcal{O}$ is the sheaf of holomorphic functions on M.

For further vanishing theorems deduced from a spinorial approach, see [Mic], [Hsi].

J. Conformal Changes of Riemannian Metrics

We consider here the simplest non-homothetic deformations of a (pseudo-) Riemannian metric, namely the conformal ones. They are obtained by changing at each point the lengths of all vectors by a scaling factor (depending on the point) without changing the "angles". More precisely, we set

1.158 Definition. Two pseudo-Riemannian metrics g and g^1 on a manifold M are said to be

a) (*pointwise*) *conformal* if there exists a C^∞ function f on M such that $g^1 = e^{2f}g$;

b) *conformally equivalent* if there exists a diffeomorphism α of M such that $\alpha^* g^1$ and g are pointwise conformal.

Note that, if g and g^1 are conformally equivalent, then α is an *isometry* from $e^{2f}g$ onto g^1. So we will only study below the case $g^1 = e^{2f}g$.

In the following, we compute the various invariants of g^1 in terms of those of g and the derivatives of f (with respect to the Levi-Civita connection D of g). We denote as before R, r, s, W the various curvature tensors of g.

We recall that, for a function f, Df is the gradient, Δf the Laplacian and Ddf the Hessian with respect to g, see 1.54.

1.159 Theorem. *Let (M, g) be a pseudo-Riemannian manifold and f a function on M. Then the pseudo-Riemannian metric $g^1 = e^{2f}g$ has the following invariants*:

a) *Levi-Civita connection*

$$D^1_X Y = D_X Y + df(X)Y + df(Y)X - g(X, Y)Df$$

b) $(4, 0)$ *curvature tensor*

$$R^1 = e^{2f}(R - g \owedge (Ddf - df \circ df + \tfrac{1}{2}|df|^2 g))$$

c) $(3, 1)$ *Weyl tensor*

$$W^1 = W$$

d) *Ricci tensor*

$$r^1 = r - (n-2)(Ddf - df \circ df) + (\Delta f - (n-2)|df|^2)g$$

e) *trace-free part of the Ricci tensor*

$$Z^1 = Z - (n-2)(Ddf - df \circ df) - \frac{n-2}{n}(\Delta f + |df|^2)g$$

f) *scalar curvature*

$$s^1 = e^{-2f}(s + 2(n-1)\Delta f - (n-2)(n-1)|df|^2)$$

g) *volume element*

$$\mu_{g^1} = e^{nf}\mu_g$$

h) *Hodge operator on p-forms* (*if M is oriented*)

$$*_{g^1} = e^{(n-2p)f} *_g$$

i) *codifferential on p-forms*

$$\delta^1\beta = e^{-2f}(\delta\beta - (n-2p)i_{Df}\beta)$$

j) (*pseudo-*) *Laplacian on p-forms*

$$\Delta^1\alpha = e^{-2f}(\Delta\alpha - (n-2p)d(i_{Df}\alpha) - (n-2p-2)i_{Df}d\alpha +$$
$$2(n-2p)df \wedge i_{Df}\alpha - 2df \wedge \delta\alpha). \qquad \square$$

1.160 Remarks. a) The canonical isomorphisms $\sharp$ and $\flat$ between TM and T^*M are not the same for g and g^1. In particular, viewed as a (4, 0)-tensor, the Weyl tensor would satisfy $W^1 = e^{2f}W$ and conversely, if we view R^1 as a (3, 1)-tensor, there is no factor e^{2f}.

b) If we are interested in only one of these invariants, it is often useful to choose the scaling factor in a different form in order to simplify the expression. In particular we have

1.161 Corollary. a) *If* $g^1 = \psi^{4/n-2}g$ $(n \geqslant 3, \psi > 0)$, *then*

$$\psi^{n+2/n-2}s^1 = 4\frac{n-1}{n-2}\Delta\psi + s\psi. \tag{1.161a}$$

b) *If* $g^1 = \varphi^{-2}g$ $(\varphi > 0)$, *then*

$$Z^1 = Z + \frac{n-2}{\varphi}\left(Dd\varphi + \frac{\Delta\varphi}{n}g\right). \tag{1.161b}$$

c) *If* $g^1 = \tau^{-1}g$ $(\tau > 0)$, *then*

$$\Delta^1\alpha = \tau\Delta\alpha + \frac{n-2p-2}{2}i_{D\tau}d\alpha + \frac{n-2p}{2}d(i_{D\tau}\alpha) + d\tau \wedge \delta\alpha. \tag{1.161c}$$

Formula (1.161a) yields the equation of the Yamabe conjecture (see Chapter 4 and references there). Formula (1.161b) was studied in particular by M. Obata [Oba].

We notice that some other invariants do not vary under conformal changes. For example, for an even-dimensional oriented manifold, we have $*_{g^1} = *_g$ on $\frac{n}{2}$-forms. We will use this fact in particular for 4-manifolds. As a consequence, we have the following

1.162 Corollary. *Let M be a compact even dimensional manifold. For any function f, the harmonic $\frac{n}{2}$-forms are the same for g and $g^1 = e^{2f}g$.*

1.163. Another example is the following. Let c be a *null* geodesic for a pseudo-Riemannian manifold (M, g), i.e., c satisfies $g(\dot{c}(t), \dot{c}(t)) = 0$ (this implies that g is not Riemannian). Then, up to some change of parametrization, c is still a null geodesic for the pseudo-Riemannian metrics $e^{2f}g$ on M, for any function f. Indeed,

$$D^1_{\dot{c}}\dot{c} = 2df(\dot{c})\dot{c}$$

so if we choose $c_1 = c \circ \varphi$ where the real function $\varphi(t)$ satisfies $\varphi'' + 2\varphi' df(\dot{c})(c(t)) = 0$, we get $D^1_{\dot{c}_1}\dot{c}_1 = 0$.

Finally, we see that the second fundamental form of an immersion does not vary too badly under a (point-wise) conformal change of the ambient metric. If II is the second fundamental form of the Riemannian immersion $i: (M, g) \to (N, h)$, then the second fundamental form II^1 of the Riemannian immersion

$$i: (M, e^{2(f \circ i)}g) \to (N, e^{2f}h)$$

is given by

$$\mathrm{II}^1(U, V) = \mathrm{II}(U, V) - g(U, V)\mathcal{N}(Df) \tag{1.163}$$

where $\mathcal{N}(Df)$ denotes the normal component of the gradient of f.

The main invariant under conformal changes is the Weyl tensor. In particular, since $W = 0$ for a flat metric, the Weyl tensor is seen to be an obstruction for a pseudo-Riemannian metric to be locally conformal to a flat metric.

1.164 Definition. A pseudo-Riemannian manifold (M, g) is *conformally flat* if, for any x in M, there exists a neighborhood V of x and a C^∞ function f on V such that $(V, e^{2f}g)$ is flat.

Note that we do not require f to be defined on the whole of M. It turns out that, for $n \geqslant 4$, the Weyl tensor is the only obstruction to conformal flatness.

1.165 Theorem. *If $n \geqslant 4$, an n-dimensional pseudo-Riemannian manifold (M, g) is conformally flat if and only if its Weyl tensor vanishes.*

For a detailed proof, see [Eis] p. 85 or [Grr] p. 188. Following Theorem 1.159, this amounts to solving locally the second order overdetermined system

$$Ddf - df \circ df + \left(|df|^2 - \frac{\Delta f}{n-2}\right)g = r.$$

The main point is that for $n \geqslant 4$, if $W = 0$, there are no integrability conditions on higher derivatives. □

1.166 Remark. A necessary but not sufficient condition for conformal flatness is that $\mathscr{R}$ has a basis of decomposable 2-forms as eigenvectors. This is clear since, if $W = 0$, $R = \left(Z - \frac{s}{2n-2}g\right) \owedge g$.

1.167 Examples. 1) Any manifold with constant sectional curvature is conformally flat.

2) The product of a manifold with constant sectional curvature with S^1 or $\mathbb{R}$ is conformally flat.

3) The product of two Riemannian manifolds, one with sectional curvature 1, and the other with curvature -1, is conformally flat. Here is a short proof of 3): we (abusively) write

$$g = g_M + g_N \quad \text{on} \quad T(M \times N) = TM \times TN.$$

Then
$$R_M = g_M \owedge g_M \quad \text{and} \quad R_N = -g_N \owedge g_N.$$

So
$$\begin{aligned} R = R_M + R_N &= g_M \owedge g_M - g_N \owedge g_N \\ &= (g_M + g_N) \owedge (g_M - g_N) \\ &= g \owedge (g_M - g_N) \quad \text{and so} \quad W = 0. \end{aligned}$$ □

Indeed, it can be checked that 2) and 3) provide the only cases where a Riemannian product is conformally flat.

1.168. The same formalism also gives a simple proof of the following result due to E. Cartan [Car 4]: a hypersurface of a conformally flat Riemannian manifold (with $n \geqslant 5$) is conformally flat if and only if its second fundamental form either has a unique eigenvalue, or has two eigenvalues, one of which has multiplicity one. (The "if" part is related to phenomena studied in Chapter 16, for Codazzi tensors).

Note that Theorem 1.165 does not apply to dimensions 2 and 3, since the Weyl tensor disappears from the decomposition of R. The results here are quite different.

1.169 Theorem. *Any 2-dimensional pseudo-Riemannian manifold is conformally flat.*

In fact, this theorem is usually known under an equivalent formulation, namely the existence of "isothermal coordinates" (i.e., local coordinates (x, y) such that $g = e^{2f}(dx^2 + dy^2)$), which was proved first by C.F. Gauss [Gau] in the analytic case and then by A. Korn [Kor] and L. Lichtenstein [Lit] in the C^∞ case (see also the proof by S.S. Chern [Chr 1] or the presentation [Spi] volume 4 p. 455. Using Theorem 1.159, this amounts to solving locally the equation $2\Delta f + s = 0$. Since Δ is elliptic if g is Riemannian, the local solvability in this case is an immediate consequence of Appendix Theorem 45. □

1.170. In dimension 3, there is another obstruction to conformal flatness, which lives in the decomposition of DR into irreducible components. This obstruction is called the *Weyl-Schouten tensor field* and vanishes if and only if (M, g) is conformally flat. Since R is determined by r ($n = 3$), this Weyl-Schouten tensor may be given by a component of the decomposition of Dr; in 16.4.(v)d) we will see that (M, g) is conformally flat if and only if Dr is *symmetric*.

Finally, let us mention that conformally flat manifolds are not classified in full generality, although we have

1.171 Theorem (N. Kuiper [Kui]). *A conformally flat compact simply connected Riemannian manifold is conformally equivalent to the canonical sphere* (*of the same dimension*).

K. First Variations of Curvature Tensor Fields

1.172. If M is a compact manifold, the set $\mathscr{M}$ of all Riemannian metrics is open in $\mathscr{S}^2M$, for the compact open topology or for the L_k^p-topologies as well. Recall that the definition of the L_k^p-norm (cf. Appendix. 3) requires the choice of a metric on M. But when M is compact, the topology defined by such a norm does not depend on the metric. (When studying geometric problems, one has to consider Riemannian structures, i.e., Riemannian metrics modulo the group of diffeomorphisms. We refer to Chapters 4 and 12 for this point of view. Here it appears only as a remark.)

1.173. The formulas for various curvature tensors in local coordinates are not very pleasing (see 1.42). However, they prove that the maps $g \mapsto R_g$ (resp. $g \mapsto r_g$, $g \mapsto s_g$) from $\mathscr{M}$ into $C^\infty(\bigwedge^2 M \otimes \bigwedge^2 M)$ (resp. $\mathscr{S}^2M$, $\mathscr{C}M$), are quasilinear second order differential operators. In particular, they are differentiable if $\mathscr{M}$ is equipped with some L_k^2-norm ($k \geqslant 2$) and the target space with the corresponding L_{k-2}^2-norm. Moreover, for a given g, the differentials at the point g are linear second order differential operators, which will be *denoted* by R'_g, r'_g and s'_g. They have been computed in local coordinates by many relativists (see [Bln], [Lic 2]). Here we give a synthetic treatment, close to [Ble].

1.174 Theorem. *Let (M, g) be a pseudo-Riemannian manifold and h be in $\mathscr{S}^2M$. Then the differentials at g, in the direction of h, of the Levi-Civita connection and the various curvature tensors are given by the following formulas*:

(a) *Levi-Civita connection*

$$g(D'_g h(X, Y), Z) = \tfrac{1}{2}\{D_X h(Y, Z) + D_Y h(X, Z) - D_Z h(X, Y)\};$$

(b) (3, 1)-*curvature tensor*

$$R'_g h(X, Y)Z = (D_Y D'_g h)(X, Z) - (D_X D'_g h)(Y, Z);$$

(c) (4, 0)-*curvature tensor*

$$R'_g h(X,Y,Z,U) = \tfrac{1}{2}\{D^2_{Y,Z}h(X,U) + D^2_{X,U}h(Y,Z) - D^2_{X,Z}h(Y,U) - D^2_{Y,U}h(X,Z) + h(R(X,Y)Z,U) - h(R(X,Y)U,Z)\};$$

(d) *Ricci tensor*

$$r'_g h = \tfrac{1}{2}\Delta_L h - \delta^*_g(\delta_g h) - \tfrac{1}{2}D_g d(\mathrm{tr}_g h);$$

(e) *scalar curvature*

$$s'_g h = \Delta_g(\mathrm{tr}_g h) + \delta_g(\delta_g h) - g(r_g, h).$$

Proof. Since the difference between two connections is a tensor field, the differential of the Levi-Civita connection is a symmetric (2, 1)-tensor field. Now the Definition 1.39 gives the formula

(1.175) $$g(D_X Y, Z) = \tfrac{1}{2}\{Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\},$$

for any vector fields X, Y and Z. And (1.174a) follows easily. □

1.176 Notation. It will sometimes be convenient to consider the $(3,0)$-tensor field

$$C_g h(X,Y,Z) = g(D'_g h(X,Y),Z),$$

instead of $D'_g h$.

1.177. In order to compute the differential of R, we differentiate the Formula (1.11), which gives

$$R'_g h(X,Y)Z = D'_g h(Y, D_X Z) + D_Y(D'_g h(X,Z)) - D'_g h(X, D_Y Z) - D_X(D'_g h(Y,Z)) + D'_g h([X,Y],Z)$$

and (1.174b) follows immediately.

1.178 Remark. We will not prove (1.174c), since it will not be used in the book. We have mentioned it mainly for the sake of completeness. Recall that differentiation does *not* commute with changing the type of a tensor field, nor with taking the trace of a $(2,0)$-tensor, since these operations do involve the metric. But we want to point out the following facts.

(a) Using the Ricci identity (1.21), it can be checked that this tensor field is symmetric with respect to the pairs (X,Z) and (Y,U).

(b) The symmetries of $R'_g(h)$ can be viewed concisely at the symbol level (see Appendix 15). We have

$$\sigma_t R'_g(h) = -\tfrac{1}{2}(t \circ t) \owedge h.$$

1.179. In order to compute r'_g, we first notice that taking the trace of a linear map does commute with differentiation. Hence, if (X_i) is any orthonormal frame, we get

$$r'_g h(X,Y) = \sum_{i=1}^{n} \varepsilon_i (D_{X_i}(C_g h)(X,Y,X_i) - D_X(C_g h)(X_i,Y,X_i)).$$

The second term is easily seen to be $\frac{1}{2}D_g d(\operatorname{tr}_g h)$, and the first term may be written

$$\frac{1}{2}\sum_{i=1}^{n}\varepsilon_i(D^2_{X_i,X}h(Y,X_i)+D^2_{X_i,Y}h(X,X_i)-D^2_{X_i,X_i}h(X,Y)).$$

Now the Ricci identity (1.21) gives

$$D^2_{X_i,X}h(Y,X_i)=D^2_{X,X_i}h(Y,X_i)+h(R(X_i,X)Y,X_i)+h(R(X_i,X)X_i,Y).$$

We put all these terms together and we note that $D^*Dh=-\sum_{i=1}^{n}D^2_{X_i,X_i}h$ (where D^* is the formal adjoint of D and has been defined in 1.55). We get

(1.180a) $$r'_g h=\tfrac{1}{2}(D^*Dh+r_g\circ h+h\circ r_g-2\mathring{R}_g h-2\delta^*_g\delta_g h-D_g d(\operatorname{tr}_g h)).$$

In this formula, δ_g denotes the divergence and δ^*_g its formal adjoint, both defined in 1.59; also $h\circ k$ is the (2,0)-tensor associated by the metric to the composition of h and k viewed as (1,1)-tensors, i.e., as linear maps of TM into itself; and finally $\mathring{R}_g$ denotes the action of the curvature on symmetric 2-tensors, and has been defined in 1.131.

Now we recognize the operator

(1.180b) $$\Delta_L h=D^*Dh+r_g\circ h+h\circ r_g-2\mathring{R}_g h.$$

It is nothing but the Lichnerowicz Laplacian (restricted to symmetric (2,0)-tensors) which we have already defined in 1.143. This gives (1.174d).

1.181. In order to compute s'_g, the simplest procedure is to differentiate the formula $s_g=\operatorname{tr}_g r_g$. We get

$$s'_g h=-g(h,r_g)+\operatorname{tr}_g(r'_g h).$$

We notice that $\Delta_g(\operatorname{tr}_g h)=\operatorname{tr}_g(\Delta_L h)$, that $\operatorname{tr}_g(\delta^*_g(\delta_g h))=-\delta_g(\delta_g h)$ and that $\Delta_g f=-\operatorname{tr}_g(D_g df)$. This gives immediately (1.174e). □

1.182 Remark. The symbol of the operator s'_g is given by

(1.182) $$\sigma_t s'_g(h)=-|t|^2\operatorname{tr}_g h-h(t^\#,t^\#).$$

It is clearly surjective when the covector t is not zero. Therefore, the formal adjoint $(s'_g)^*$ of s'_g is an overdetermined elliptic operator from $\mathscr{C}M$ to $\mathscr{S}^2M$. It is given by the following formula

(1.183) $$(s'_g)^*f=D_g df+(\Delta_g f)g-fr_g$$

for any function f on M.

For future use (see 4.33 below), we will need the following

1.184 Proposition. *The differential Δ'_g of the Laplacian acting on functions is given by the formula*

(1.184) $$\Delta'_g h(f)=g(D_g df,h)-g(df,\delta_g h+\tfrac{1}{2}d(\operatorname{tr}_g h)).$$

Proof. We first compute the differential δ'_g of the codifferential δ_g acting on differential 1-forms. Since $\delta_g\alpha=D^*_g\alpha=-\operatorname{tr}_g(D_g\alpha)$, we have immediately $\delta'_g h(\alpha)=g(h,D_g\alpha)-$

$\operatorname{tr}_g(D'_g h(\alpha))$, and formula (1.174a) gives

(1.185) $$\delta'_g h(\alpha) = g(h, D_g\alpha) - g(\alpha, \delta_g h + \tfrac{1}{2}d(\operatorname{tr}_g h)).$$

And we get (1.184) by applying (1.185) to $\alpha = df$. □

Finally, an easy computation gives the differential of the volume and the volume element.

1.186 Proposition. *Let (M, g) be a pseudo-Riemannian manifold.*

(a) *The differential μ'_g of the volume element is given by the formula*

$$\mu'_g h = -\tfrac{1}{2}(\operatorname{tr}_g h)\mu_g.$$

(b) *We assume furthermore that M is* compact *and we* denote $\int_M \mu_g$ *by* $\operatorname{Vol}(M)(g)$. *Then we have*

$$\operatorname{Vol}(M)'_g h = \int_M \mu'_g h = -\frac{1}{2}\int_M (\operatorname{tr}_g h)\mu_g.$$

Note that, given a volume element μ_g, any other volume element μ_{g_1} may be written $\mu_{g_1} = f\mu_g$ for some positive function f on M.

1.187. As an application of these computations, we give an easy consequence of formulas (1.174e) and (1.186b) for compact Einstein manifolds.

If (M, g) is Einstein, then $r_g = \dfrac{1}{n}s_g g$ (with a constant s_g), hence $g(r_g, h) = \dfrac{1}{n}s_g g(g, h) = \dfrac{1}{n}s_g \operatorname{tr}_g h$. Moreover, if M is compact, the integral over M of a divergence (and a Laplacian) is always zero, and formula (1.174e) gives (in the case where (M, g) is Einstein and compact)

$\displaystyle\int_M (s'_g h)\mu_g = -\frac{1}{n}s_g \int_M (\operatorname{tr}_g h)\mu_g = -\frac{2}{n}s_g \int_M \mu'_g h$, and we have proved

1.188 Proposition (M. Ville [Vie]). *Let (M, g) be a compact Einstein manifold, and h be in $\mathscr{S}^2 M$. Then*

$$\int_M (s'_g h)\mu_g = -\frac{2}{n}s_g \operatorname{Vol}(M)'_g h.$$

In particular, if $s_g = 0$ or if $\operatorname{Vol}(M)'_g h = 0$, *then s'_g cannot have a constant sign unless it is identically* 0.

Be careful to note that $\int_M (s'_g h)\mu_g$ is *not* the derivative (in h) of the total scalar curvature $\int_M (s_g)\mu_g$, which will be studied in greater detail in Chapter 4 below.