MATH 6702, SPRING 2024

Projective Spaces and Grassmannians
Last updated on January 28, 2022

[**DG**] stands for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`

Projective Spaces. Let V be a vector space of positive dimension $n < \infty$ over the scalar field $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and, in the last (quaternionic) case, we mean a *left* vector space. By the *projective space* of V one means the set

$$\mathrm{P}V \,=\, \{L : L \text{ is a 1-dimensional vector subspace of } V\},$$

and a surjective *projection mapping* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ is defined by

$$\pi(x) \,=\, \mathbb{K}x.$$

The set $\mathrm{P}V$ carries a natural manifold structure provided by the atlas

$$\{(U_f, \varphi_f) : f \in V^* \smallsetminus \{0\}\} \tag{1}$$

indexed by all nonzero linear functionals on V, where

$$U_f \,=\, \{L \in P(V) : L \text{ is not contained in } \mathrm{Ker}\, f\}$$

(instead of 'is not contained in $\mathrm{Ker}\, f$' one could also write '$f(L) = \mathbb{K}$' or, equivalently, 'f maps L isomorphically onto $\mathbb{K}$'), and $\varphi_f : U_f \to f^{-1}(1)$ sends each $L \in U_f$ onto its unique intersection point with $f^{-1}(1)$. Also, $f^{-1}(1)$ is a coset of $\mathrm{Ker}\, f$, which makes it an affine space with the translation vector space $\mathrm{Ker}\, f$, and

$$\begin{aligned} &\varphi_f : U_f \to f^{-1}(1) \quad \text{is a bijection with the inverse } \pi : f^{-1}(1) \to U_f \text{ and} \\ &\varphi_f(\mathbb{K}x) \,=\, x/f(x) \quad \text{whenever } L = \mathbb{K}x \in U_f \text{ (that is, } x \in V \smallsetminus \mathrm{Ker}\, f). \end{aligned} \tag{2}$$

Compatibility of any two charts in (1) now follows since, for $f, h \in V^* \smallsetminus \{0\}$, the set $\varphi_f U_f \cap U_h) = A_f \smallsetminus \mathrm{Ker}\, h$ is open in $f^{-1}(1)$ (due to closedness of $\mathrm{Ker}\, h$ in the ambient space V), while $(\varphi_f \circ \varphi_h^{-1})(x) = x/f(x)$ as a consequence of (2). (For the meaning of compatibility, see [**DG**, Section 1].)

Lemma 1. *The atlas* (1) *satisfies the Hausdorff and countability axioms, cf.* [**DG**, Sections 1 and 14]*, and so it actually turns* $\mathrm{P}V$ *into a smooth manifold which, in addition, is compact.*

Proof. See Exercise 1. □

Lemma 2. *Every linear automorphism of V, acting in an obvious manner on* $\mathrm{P}V$*, constitutes a smooth diffeomorphism. The projection* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ *is a smooth mapping as well.*

Proof. Let $A : V \to V$ be a linear automorphism. Using the same symbol for $A : \mathrm{P}V \to \mathrm{P}V$, we obtain, from (2), the rational (and hence smooth) chart representations $(\varphi_f \circ A \circ \varphi_h^{-1})(x) = Ax/f(Ax)$. On the other hand, the chart representations of π are identity mappings, cf. the first line of (2). □

Lemma 3. *If* $\mathbb{K} = \mathbb{C}$*, the projective space* $\mathrm{P}V$ *carries a unique structure of a complex manifold such that all chart mappings* φ_f *are biholomorphisms. In addition, the projection* $\pi : V \smallsetminus \{0\} \to \mathrm{P}V$ *is then also holomorphic.*

Proof. This is immediate since the transition mappings $\varphi_f \circ \varphi_h^{-1}$, being rational, are holomorphic. For the claim about π, see the proof of Lemma 2. $\square$

When $V = \mathbb{K}^n$, rather than $\mathrm{P}V$ one writes $\mathbb{K}\mathrm{P}^{n-1}$ and speaks of the *real, complex or quaternionic projective space* of dimension $n-1$ over the respective field, where the latter the real/complex dimension $n-1$ or (for $\mathbb{K} = \mathbb{H}$) the real dimension $4(n-1)$. The 1-dimensional subspace $L \in P(V)$ spanned by a nonzero vector $(x^1, \ldots, x^n)$ in $\mathbb{K}^n$ is then denoted by $[x^1, \ldots, x^n] \in P(V)$, and one refers to $x^1, \ldots, x^n$ as *homogeneous coordinates* of $L = [x^1, \ldots, x^n]$.

Generalization to Grassmannians. In addition to $V, n, \mathbb{K}$ as above, let us also fix an integer q with $0 \le q \le n$, set

$$\mathrm{Gr}_q V \;=\; \{W : W \text{ is a } q\text{-dimensional vector subspace of } V\},$$

and define a surjective *projection mapping* $\pi : \mathrm{St}_q V \to \mathrm{Gr}_q V$ by

$$\pi(\mathbf{x}) \;=\; \mathrm{span}\,\mathbf{x} \quad \text{for } \mathbf{x} = (x_1, \ldots, x_q) \in \mathrm{St}_q V,$$

where $\mathrm{St}_q V$ denotes the *Stiefel manifold* formed by all *q-frames* (that is, linearly independent ordered q-tuples of vectors) in V. (Thus, $\mathrm{St}_q V$ is an open subset of the qth Cartesian power V^q.) One calls $\mathrm{Gr}_q V$ the *Grassmannian of q-planes* in V. The set $\mathrm{Gr}_q V$ carries a natural manifold structure provided by the atlas

$$\text{(3)} \qquad \{(U_f, \varphi_f) : f \in V^* \smallsetminus \{0\}\}, \quad \text{with } U_f \;=\; \{W \in \mathrm{Gr}_q V : f(W) = \mathbb{K}^q\},$$

indexed by all surjective linear operators $f : V \to \mathbb{K}^q$. (Instead of '$f(W) = \mathbb{K}^q$' one may equivalently write 'f maps W isomorphically onto $\mathbb{K}^q$'). The chart mappings

$$\text{(4)} \qquad \varphi_f : U_f \to f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$$

with $e_1, \ldots, e_q$ denoting the standard basis of $\mathbb{K}^q$, are slightly more complicated: φ_f sends each $W \in U_f$ onto the unique ordered q-tuple $\mathbf{x} = (x_1, \ldots, x_q)$ of vectors in W such that $fx_a = e_a$ for $a = 1, \ldots, q$. In other words, using the inverse f_W^{-1} of the restriction isomorphism $f_W : W \to \mathbb{K}^q$, we have $\varphi_f(W) = (f_W^{-1}(e_1), \ldots, f_W^{-1}(e_q))$. Note that $f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$ a coset, in V^q, of the qth Cartesian power of $\mathrm{Ker}\, f$, and hence an affine subspace of V^q. If $\mathbf{x} = (x_1, \ldots, x_q)$ equals $\varphi_f(W)$, then, obviously, $W = \pi(\mathbf{x})$, the span of $x_1, \ldots, x_q$, which easily proves bijectivity of (4). Let there be now given two surjective linear operators $f, h : V \to \mathbb{K}^q$. The φ_h-image of $U_f \cap U_h$ is open in the affine space $f^{-1}(e_1) \times \ldots \times f^{-1}(e_q)$, being its intersection with the set of all $\mathbf{x} = (x_1, \ldots, x_q)$ in the Cartesian power V^q such that $hx_1, \ldots, hx_q$ are linearly independent or, equivalently, form a basis of $\mathbb{K}^q$. (The latter condition means that h restricted to $\mathrm{span}\,\mathbf{x}$ is a linear isomorphism onto $\mathbb{K}^q$.) For $\mathbf{x} = (x_1, \ldots, x_q) \in \varphi_h(U_f \cap U_h)$ we see that $(\varphi_f \circ \varphi_h^{-1})(\mathbf{x}) = \varphi_f(\mathrm{span}\,\mathbf{x})$ is the basis y_a, $a = 1, \ldots, q$, of $\mathrm{span}\,\mathbf{x}$ which f sends onto the standard basis e_a of $\mathbb{K}^q$, $a = 1, \ldots, q$, and so, with

$$\text{(5)} \qquad y_a \;=\; S_a^c x_c,$$

the unknown coefficients S_a^c are characterized by $e_a = S_a^c f x_c$. The entries S_a^c thus form the transition matrix between the bases $fx_1, \ldots, fx_q$ and $e_1, \ldots, e_q$ of $\mathbb{K}^q$ and, if one writes the former basis as a $q \times q$ matrix (having the ath column fx_a

for $a = 1, \ldots, q$), the matrix $\mathbf{S}$ with the entries S_a^c is its inverse. Thus, the chart transition mapping $\varphi_f \circ \varphi_h^{-1}$ must be smooth, being equal to the composite

$$(x_1, \ldots, x_q) \mapsto (fx_1, \ldots, fx_q) \mapsto \mathbf{S}$$

in which the constituents are smooth: one is in fact linear, the other (the matrix inverse) rational. The atlas (3) thus turns $\mathrm{Gr}_q V$ into a manifold, cf. Exercise 4.

More precisely, the atlas (3) satisfies the Hausdorff and countability axioms [**DG**, Sections 1 and 14]: the former, since

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the latter, as $\mathrm{Gr}_q V$ is covered by a finite subatlas of (3). Namely, for any basis $v_1, \ldots, v_n$ of V, with the dual basis $\xi^1, \ldots, \xi^n$ of V^*, the restrictions of the functionals ξ^a to any given $W \in \mathrm{Gr}_q V$ span W^* (or else they would span a subspace of W^* of some dimension $p < q$, and rearranging the v_a we might assume the restrictions of $\xi^1, \ldots, \xi^p$ to form a basis of this subspace; extending these p functionals to a basis $\xi^1, \ldots, \xi^p, \eta^{p+1}, \ldots, \eta^n$ of V^* such that the restrictions of $\xi^1, \ldots, \xi^p, \eta^{p+1}, \ldots, \eta^q$ form a basis of W^*,

all ξ^a would vanish on a nontrivial subspace of W, and a vector $w \neq 0$ from such a subspace would have all components $w^a = \xi^a w$ equal to 0).

Tautological Bundles. Given a real or complex vector space V of real/complex dimension $n < \infty$ and an integer q with $0 \le q \le n$, one defines the *tautological vector bundle* $\mathcal{T}$ over the Grassmannian $\mathrm{Gr}_q V$ by

$$\mathrm{Gr}_q V \ni W \mapsto \mathcal{T}_W = W.$$

The chart mappings (4) for $\mathrm{Gr}_q V$, when regarded as local trivializations of $\mathcal{T}$, form a smooth atlas, parametrized by the set of all surjective linear mappings $f : V \to \mathbb{K}^q$ (where $\mathbb{K}$ is the scalar field). This is clear from smoothness of the functions S_a^c in (5), and turns $\mathcal{T}$ into a smooth real/complex vector bundle of fibre dimension q.

Exercises.

Exercise 1. Prove Lemma 1.

Exercise 2. Generalize Lemma 2 to Grassmannians.

Exercise 3. Generalize Lemma 3 to Grassmannians.

Exercise 4. Generalize Lemma 1 to Grassmannians.

Exercise 5. Verify that, if $n = \dim V$, the dimension of $\mathrm{Gr}_q V$ equals $(n-q)q$, for $\mathbb{K} = \mathbb{R}$, or $2(n-q)q$, for $\mathbb{K} = \mathbb{C}$.

Exercise 6. Show that every linear functional $\xi \in V^*$ may be viewed as a smooth section of the dual $\mathcal{T}^*$ of the tautological vector bundle $\mathcal{T}$ over the Grassmannian $\mathrm{Gr}_q V$ with ξ_W equal to the restriction of ξ to W whenever $W \in \mathrm{Gr}_q V$.