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MATH 766, SPRING 2012
M-W-F 1:30 p.m., CH 240
ISOMETRIC IMMERSIONS

For any Riemannian metric g, in any local coordinate system x^j,

$$\partial_j g_{kl} \,=\, \Gamma_{jkl} \,+\, \Gamma_{jlk}\,, \tag{1}$$

where Γ_{jkl} are the Christoffel symbols of the second kind for g. (See **[DG]**, p.107.) Next, for the reciprocal metric g^{-1}, with the components g^{jk},

$$g^{jk} \,=\, g^{-1}(dx^j, dx^k). \tag{2}$$

Also, the $(0,4)$ curvature tensor of a Riemannian metric g has the components

$$R_{jklp} = g_{pq}(\partial_k \Gamma^q_{jl} - \partial_j \Gamma^q_{kl} + \Gamma^q_{ks}\Gamma^s_{jl} - \Gamma^q_{js}\Gamma^s_{kl}), \tag{3}$$

due to the formula $R_{jkl}{}^q = \partial_k \Gamma^q_{jl} - \partial_j \Gamma^q_{kl} + \Gamma^q_{ks}\Gamma^s_{jl} - \Gamma^q_{js}\Gamma^s_{kl}$ in the topics for May 7.

Given an immersion $F : M \to N$ of a manifold M in a Riemannian manifold (N, h) and a point $x \in M$, we identify $T_x M$ with its isomorphic image $dF_x(T_x M)$, calling both the *tangent space* of M at x, and refer to the h_x-orthogonal complement $[dF_x(T_x M)]^\perp$ in $T_{F(x)}N$ as the *normal space* of F at x. One then also says that F is an *isometric immersion* of (M, g) in (N, h), where $g = F^*h$ denotes the pullback metric.

We denote by D the Levi-Civita connection of h, and by ∇ the Levi-Civita connection of the pullback Riemannian metric $g = F^*h$ on M.

Whenever $x \in M$, we have the orthogonal direct-sum decomposition $T_{F(x)}N = dF_x(T_x M) \oplus [dF_x(T_x M)]^\perp$. If w is a vector field along F, or a vector field along a curve $t \mapsto y(t) = F(x(t))$ (where $t \mapsto x(t)$ is a curve in M), we thus obtain the corresponding decomposition of w into its tangent and normal parts w^{tng} and w^{nrm}. We know that smoothness of w^{tng} and w^{nrm} follows when w is smooth.

From the rank theorem it easily follows that every point $z \in M$ has a neighborhood U in M such that $F : U \to N$ is a homeomorphic embedding and, for a suitable diffeomorphic identification of a neighborhood U' of $F(z)$ in N with a neighborhood of 0 in $\mathbb{R}^n$, the point z corresponds to 0 and

$$F(U) \,=\, \{(x^1, \ldots, x^n) \in U' : x^\lambda = 0 \ \text{ for } \ \lambda = m+1, \ldots, n\}. \tag{4}$$

Here and below, $n = \dim N$ and $m = \dim M$, while the ranges of indices are

$$j, k, l \,=\, 1, \ldots, m, \qquad \lambda, \mu \,=\, m+1, \ldots, n. \tag{5}$$

We may assume, in addition, that

$$h_{j\lambda} = 0 \ \text{ everywhere in } \ F(U). \tag{6}$$

To achieve this, let us replace our x^j with $y^j = x^j + A^j_\mu x^\mu$ for functions A^j_μ chosen so as to satisfy $A^j_\mu h^{\mu\lambda} = -h^{j\lambda}$. Such A^j_μ exist since the matrix $[h^{\mu\lambda}]$, being positive definite, is invertible.) Applying d to the equality $y^j = x^j + A^j_\mu x^\mu$, at points where $x^\mu = 0$, we obtain $dy^j = dx^j + A^j_\mu\, dx^\mu$ on $F(U)$, and so the condition $h^{j\lambda} = 0$, clearly equivalent to (6), follows from (2).

Since vector fields tangent to U' (or, respectively, to $F(U)$) may now be identified with functions $U' \to \mathbb{R}^n$ (or, respectively, $F(U) \to \mathbb{R}^m \times \{(0,\dots,0)\}$), there are abundant smooth vector fields w on U' such that

$$\text{the restriction of } \ w \ \text{ to } \ F(U) \,\text{is tangent to } \ F(U). \tag{7}$$

In other words,

$$w^\lambda = 0 \ \text{ for all } \ \lambda \,\text{ wherever } \ x^\lambda = 0 \ \text{ for all } \ \lambda. \tag{8}$$

We can now describe the Levi-Civita connection of ∇ in terms of the Levi-Civita connection D of h, and introduce a fundamental invariant of isometric immersion F, called its *second fundamental form*. For both purposes, we will use vector fields w with (7) – (8), as well as smooth vector fields $t \mapsto v(t) \in T_{x(t)}M$ along smooth curves $t \mapsto x(t)$ in M. For the latter, we denote by $(dF)v$ the smooth vector field along the image curve $t \mapsto y(t) = F(x(t))$ in N, given by

$$w(t) \,=\, dF_{x(t)}[v(t)], \ \ \text{where } \ w = (dF)v.. \tag{9}$$

First, let (7) – (8) hold both for w and u. Then, on $F(U)$,

$$\nabla_u w \,=\, [\mathrm{D}_u w]^{\mathrm{tng}}. \tag{10}$$

On the other hand, assuming (9), we obtain

$$\nabla_{\dot x} v \,=\, [\mathrm{D}_{\dot y}((dF)v)]^{\mathrm{tng}}. \tag{11}$$

In addition, denoting by Γ^l_{jk} the Christoffel symbols of g and using the letter G (rather than Γ) to represent the Christoffel symbols of h, we have

$$\Gamma^l_{jk} \,=\, G^l_{jk}. \tag{12}$$

To prove (10) – (12),

(further text in preparation)

The *second fundamental form* B of the isometric immersion F assigning to each $x \in M$ a bilinear symmetric mapping $B_x : T_xM \times T_xM \to [dF_x(T_xM)]^\perp$, in such a way that $B_x(u_x, w_x) = [\mathrm{D}_u w]^{\mathrm{nrm}}$ for $x \in U$ and u, w satisfy (7) – (8). (We identify U with $F(U)$ using F.) We now show that

$$B \text{ is well-defined and smooth,} \tag{13}$$

its smoothness meaning that $B(u, w)$ is a smooth vector field along F whenever $u, w \in \mathfrak{X}M$. Also, under the assumption (9),

$$B_x(\dot{x}, v) \;=\; [\mathrm{D}_{\dot{y}}((dF)v)]^{\mathrm{nrm}}. \tag{14}$$

Finally, since the coordinate fields ∂_λ are, by (6), normal to $F(U)$ at points of $F(U)$, and so B (if well-defined) should have some component functions B^λ_{jk} with $B(\partial_j, \partial_k) = B^\lambda_{jk}\partial_\lambda$. These B^λ_{jk} are given by

$$B^\lambda_{jk} \;=\; G^\lambda_{jk}. \tag{15}$$

(further text in preparation)

The *Gauss equation*: in the case where F is the inclusion mapping of a submanifold $M \subset N$, we have

$$R^{(g)}(u, u', w, w') = R^{(h)}(u, u', w, w') + \langle B(u, w), B(u', w')\rangle - \langle B(u, w'), B(u', w)\rangle \tag{16}$$

for vector fields u, u', w, w' satisfying (7) – (8). Equivalently, in coordinates with (4) – (6),

$$R^{(g)}_{jklp} = R^{(h)}_{jklp} + h_{\lambda\mu}(B^\lambda_{jl}B^\mu_{kp} - B^\lambda_{kl}B^\mu_{jp}). \tag{17}$$

In fact, in view of (6), $R^{(h)}_{jklp}$ is nothing else than the right-hand side of (3), applied to h, in which the index s runs from 1 to n, while all other indices range over $\{1, \ldots .m\}$. Thus, by (3) applied to g, the difference $R^{(g)}_{jklp} - R^{(h)}_{jklp}$ equals $-h_{pq}(G^q_{k\lambda}G^\lambda_{jl} - G^q_{j\lambda}G^\lambda_{kl}) = -G_{k\lambda p}G^\lambda_{jl} + G_{j\lambda p}G^\lambda_{kl}$. However, by (6) and (1), $0 = \partial_k h_{\lambda p} = G_{k\lambda p} + G_{kp\lambda}$. Hence $R^{(g)}_{jklp} - R^{(h)}_{jklp} = G_{kp\lambda}G^\lambda_{jl} - G_{jp\lambda}G^\lambda_{kl} = h_{\lambda\mu}(G^\mu_{kp}G^\lambda_{jl} - G^\mu_{jp}G^\lambda_{kl})$. Now (15) yields (17).