MATH 7711, AUTUMN 2019

Algebraic Curvature Tensors

In these notes, $\mathcal{T}$ is always a real vector space of dimension m, with $1 \le m < \infty$. For $\xi, \eta \in \mathcal{T}^*$, we denote by $\xi \otimes \eta$ the bilinear form $\mathcal{T} \times \mathcal{T} \to \mathrm{I\!R}$ given by $(\xi \otimes \eta)(u, v) = \xi(u)\eta(v)$. Then

(1) $\xi \otimes \eta$ is symmetric if and only if ξ, η are linearly dependent.

In fact, we may assume that $\xi \ne 0 \ne \eta$. The 'if' part is obvious. Now let $\xi \otimes \eta$ be symmetric. Fixing $u \in \mathcal{T}$ with $\xi(u) \ne 0$, we have $\xi(u)\eta(\,\cdot\,) = \xi(\,\cdot\,)\eta(u)$, that is, $\eta = \eta(u)\xi/\xi(u)$.

The space $\mathcal{R}(\mathcal{T})$ of *algebraic curvature tensors* in $\mathcal{T}$ consists of all real-valued quadrilinear forms R on $\mathcal{T}$ with $R(u, u', v, v') = -R(u', u, v, v') = -R(u, u', v', v)$ and $R(u, v, w, \,\cdot\,) + R(v, w, u, \,\cdot\,) + R(w, u, v, \,\cdot\,) = 0$ for $u, v, w, u', v' \in \mathcal{T}$. Then

(2) $\mathcal{R}(\mathcal{T}) = \{0\}$ if $m = 1$ and $\dim \mathcal{R}(\mathcal{T}) = 1$ when $m = 2$.

In the former case, this follows from the skew-symmetry requirement, while in the latter $R \mapsto R(u, v, u, v)$ is obviously, for any fixed basis u, v of $\mathcal{T}$, an isomorphism $\mathcal{R}(\mathcal{T}) \to \mathrm{I\!R}$. In any dimension m, Milnor's octahedron argument gives

(3) $R(u, u', v, v') = R(v, v', u, u')$ if $R \in \mathcal{R}(\mathcal{T})$ and $u, v, u', v' \in \mathcal{T}$.

We denote by $\mathcal{S}(\mathcal{T})$ the space of *symmetric 2-tensors* in $\mathcal{T}$, that is, symmetric bilinear forms $\mathcal{T} \times \mathcal{T} \to \mathrm{I\!R}$. The Kulkarni-Nomizu product $b \wedge d \in \mathcal{R}(\mathcal{T})$ of symmetric 2-tensors $b, d \in \mathcal{S}(\mathcal{T})$ is defined by $2(b \wedge d)(u, u', v, v') = b(u, v)d(u', v') + b(u', v')d(u, v) - b(u', v)d(u, v') - b(u, v')d(u', v)$. From now on we fix a pseudo-Euclidean inner product g in $\mathcal{T}$, so that $g \in \mathcal{S}(\mathcal{T})$. The *g-trace* functional $\mathrm{tr} : \mathcal{S}(\mathcal{T}) \to \mathrm{I\!R}$ assigns to b the trace of

(4) the linear operator $A : \mathcal{T} \to \mathcal{T}$ with $g(Au, \,\cdot\,) = b(u, \,\cdot\,)$ for all $u \in \mathcal{T}$.

The *Ricci contraction* $\mathrm{Ric} : \mathcal{R}(\mathcal{T}) \to \mathcal{S}(\mathcal{T})$ is the operator given by $[\mathrm{Ric}(R)](u, v) = \mathrm{tr}\, R(u, \,\cdot\,, v, \,\cdot\,)$, symmetry of $\mathrm{Ric}(R)$ being immediate from (3). We write

$$r = \mathrm{Ric}(R), \quad s = \mathrm{tr}\, r, \quad e = r - \frac{s}{m} g, \quad h = r - \frac{s}{2(m-1)} g, \quad W = R - \frac{2}{m-2} g \wedge h$$

whenever $R \in \mathcal{R}(\mathcal{T})$, and refer to s, r, e, h, W as the *scalar curvature* of R and its *Ricci, Einstein, Schouten* and *Weyl* tensors, with h (or, W) defined only if $m \ge 2$ (or, respectively, $m \ge 3$). It is a trivial exercise to verify that, for $m \ge 3$,

(5) $R = S + E + W$, where $S = \dfrac{s}{m(m-1)} g \wedge g$ and $E = \dfrac{2}{m-2} g \wedge e$.

The *anticommutator* $\{b, b'\}$ of $b, b' \in \mathcal{S}(\mathcal{T})$ is the element d of $\mathcal{S}(\mathcal{T})$ corresponding as in (4) to $AA' + A'A$, for A, A' related via (4) to b, b'. We have the formula

(6) $2\,\mathrm{Ric}(b \wedge d) = (\mathrm{tr}\, b)d + (\mathrm{tr}\, d)b - \{b, d\}$

whenever $b, d \in \mathcal{S}(\mathcal{T})$, and its consequences (obvious since $\{g, d\} = 2d$):

(7) $$2\,\mathrm{Ric}(g \wedge d) = (m-2)d + (\mathrm{tr}\, d)g, \qquad \mathrm{tr}\,[\mathrm{Ric}(g \wedge d)] = (m-1)\,\mathrm{tr}\, d$$

which, combined with the definitions of s, r, h and W, give

(8) $$\mathrm{tr}\, h = (m-2)s/(2m-2), \quad 2\,\mathrm{Ric}(g \wedge h) = (m-2)r, \quad \mathrm{Ric}(W) = 0.$$

Let the subpaces $\mathcal{W}(\mathcal{T}), \mathcal{B}(\mathcal{T})$ and $\mathcal{E}(\mathcal{T})$ of $\mathcal{R}(\mathcal{T})$ be defined by $\mathcal{W}(\mathcal{T}) = \mathrm{Ker}\,\mathrm{Ric}$, $\mathcal{B}(\mathcal{T}) = \{g \wedge b : b \in \mathcal{S}(\mathcal{T})\}$ and $\mathcal{E}(\mathcal{T}) = \{g \wedge b : b \in \mathcal{S}(\mathcal{T})$ and $\mathrm{tr}\, b = 0\}$. We then have the direct-sum decompositions

(9) $$\mathcal{R}(\mathcal{T}) = [\mathrm{I\!R} g \wedge g] \oplus \mathcal{E}(\mathcal{T}) \oplus \mathcal{W}(\mathcal{T}) \quad \text{and} \quad \mathcal{B}(\mathcal{T}) = [\mathrm{I\!R} g \wedge g] \oplus \mathcal{E}(\mathcal{T}).$$

To prove (9), we first note that it holds, for obvious reasons, when $m \le 2$ (all spaces involved are trivial if $m = 1$, cf. (2) while, as formula (7) always gives $\mathrm{tr}\,[\mathrm{Ric}(g \wedge g)] \ne 0$, for $m = 2$ we have $\mathcal{R}(\mathcal{T}) = \mathrm{I\!R} g \wedge g$ by (2) and $\mathcal{E}(\mathcal{T}) = \mathcal{W}(\mathcal{T}) = \{0\}$). Next, with $m \ge 3$, in both cases, the summands span the space in question: for $\mathcal{B}(\mathcal{T})$ this is clear from its definition, for $\mathcal{R}(\mathcal{T})$ – immediate from (5) and the last equality in (8) (as $\mathrm{tr}\, b = 0$). Furthermore, the two linear operators

(10) $\mathcal{S}(\mathcal{T}) \ni b \mapsto g \wedge b \in \mathcal{B}(\mathcal{T})$ and $\mathrm{Ric} : \mathcal{B}(\mathcal{T}) \to \mathcal{S}(\mathcal{T})$ are isomorphisms,

since their composite $\mathcal{S}(\mathcal{T}) \to \mathcal{B}(\mathcal{T}) \to \mathcal{S}(\mathcal{T})$ is injective by (7), so that they both are injective, while the first operator is surjective by the definition of $\mathcal{B}(\mathcal{T})$. In view of the first part of (10), the summands of $\mathcal{B}(\mathcal{T})$ in (9) are *direct* summands. The same now follows for the summands of $\mathcal{R}(\mathcal{T})$, as the second part of (10) gives $\mathcal{B}(\mathcal{T}) \cap \mathcal{W}(\mathcal{T}) = \{0\}$. This completes the proof of (9).

Remark 1. As an immediate consequence of (9), for any $R, \widehat{R} \in \mathcal{R}(\mathcal{T})$,

R and $\widehat{R}$ have the same $\mathcal{W}(\mathcal{T})$-component if and only if $\widehat{R} - R$ is $\wedge$-divisible by g.

Remark 2. We have $\mathcal{W}(\mathcal{T}) = \{0\}$ if $m = 3$.

Proof. Let $W \in \mathcal{W}(\mathcal{T})$. In an orthonornal basis e_1, e_2, e_3, with $g(e_i, e_i) = \varepsilon_i = \pm 1$, setting $W_{ijkl} = W(e_i, e_j, e_k, e_l)$, we have the following consequences of the condition $r = 0$, for $r = \mathrm{Ric}(W)$. First, $W_{ijij} = 0$ (where we may assume that $i \ne j$), since the three numbers a_{12}, a_{13}, a_{23}, given by $a_{ij} = a_{ji} = \varepsilon_i \varepsilon_j W_{ijij}$, being pairwise opposite to each other (due to the equality $\varepsilon_i r_{ii} = a_{ij} + a_{ik}$ when $i \ne j \ne k \ne i$), must all equal 0. Second, $W_{ijkl} = 0$ if i, j, k are distinct, as $W_{ijki} = -W_{jiki} = \varepsilon_i r_{jk}$, and similarly $W_{ijkj} = \varepsilon_j r_{ik}$, while, obviously, $W_{ijkk} = 0$. Thus, $W_{ijkl} = 0$ whenever the set $\{i, j, k\}$ has $1, 2$ or 3 elements.

Remark 3. For $m = 2$, the Ricci tensor $\mathrm{Ric}(R)$ is always a multiple of g.

This follows from the first equality in (7), since (10) gives $g \wedge g \ne 0$ and so $\mathcal{R}(\mathcal{T}) = \mathrm{I\!R} g \wedge g$ as a consequence of (2).

Exercise 1. Given $b \in \mathcal{S}(\mathcal{T})$, prove that $b \wedge b = 0$ if and only if $\mathrm{rank}\, b \le 1$, where $\mathrm{rank}\, b$ is the rank (dimension of the image) of $\mathcal{T} \ni u \mapsto b(u, \cdot\,) \in \mathcal{T}^*$. Also, show that the condition $\mathrm{rank}\, b \le 1$ is equivalent to the existence of $\xi \in \mathcal{T}^*$ with $b = \pm \xi \otimes \xi$ for some sign $\pm$.

Exercise 2. Verify (5) – (8).