MATH 7711, AUTUMN 2019

Metrics of Constant Curvature

Given a pseudo-Riemannian metric g on a manifold M, the *g-gradient* of a smooth function $f : M \to \mathrm{I\!R}$ is the vector field ∇f characterized by $g(\nabla f, \cdot\,) = df$, that is, obtained from the differential df by g-index raising. Its components, denoted by $\nabla^j f$ or $f^{,j}$, are thus expressed as $f^{,j} = g^{jk} f_{,k}$ or, equivalently, $\nabla^j f = g^{jk}\partial_k f$. Then, for the Levi-Civita connection ∇ of g,

$$d[g(\nabla f, \nabla f)] \,=\, 2[\nabla df](\nabla f, \cdot\,) \tag{1}$$

or, in coordinates, $(f^{,k} f_{,k})_{,j} = 2f^{,k} f_{,kj}$ (which is immediate from the Leibniz rule, as both g and its reciprocal metric in T^*M are ∇-parallel). On the other hand, for any connection ∇ on a manifold M, any smooth curve $t \mapsto x(t) \in M$, and any smooth function $f : M \to \mathrm{I\!R}$, with $(\;)^{\boldsymbol{\cdot}} = d/dt$, at every t and $x = x(t)$ one has

$$[f(x)]^{\boldsymbol{\cdot\cdot}} = [\nabla df](\dot x, \dot x) \,+\, d_w f, \quad \text{where} \;\; w \,=\, \nabla_{\!\dot x}\dot x. \tag{2}$$

This is obvious: in local coordinates, $[f(x)]^{\boldsymbol{\cdot\cdot}} = [\dot x^k \partial_k f]^{\boldsymbol{\cdot}} = \ddot x^k \partial_k f + \dot x^j \dot x^k \partial_j \partial_k f$, while $[\nabla df]_{jk} = f_{,jk} = \partial_j \partial_k f - \Gamma^l_{kj}\partial_l f$ and $[\nabla_{\!\dot x}\dot x]^k = \ddot x^k + \Gamma^k_{lj}\dot x^l \dot x^j$.

Let $K \in \mathrm{I\!R}$. By a *space of constant curvature* K we mean any pseudo-Riemannian manifold (M, g) such that $R = Kg \wedge g$, where R denotes the modified curvature tensor of g, cf. [*Differential Geometry*, Section 28].

The Hessian-metric equation lemma. *A pseudo-Riemannian manifold* (M, g) *is a space of constant curvature* K *if and only if, for every* $x \in M$, *every* $a \in \mathrm{I\!R}$, *and every* $\xi \in T^*_x M$, *there exists a smooth function* $f : U \to \mathrm{I\!R}$ *defined on some connected neighborhood* U *of* x, *having* $(f(x), df_x) = (a, \xi)$ *and satisfying the equation*

$$\nabla df \,=\, -Kfg\,. \tag{3}$$

In addition, such f *is uniquely determined by* (a, ξ) *once* U *is fixed.*

Proof. In any pseudo-Riemannian manifold (M, g), with any constant K, the assignment $f \mapsto (f, df)$ obviously defines a bijective correspondence between smooth solutions f to (3), defined on open subsets of M, and $\overline{\nabla}$-parallel local sections (f, ξ) of the vector bundle $E = [M \times \mathrm{I\!R}] \oplus T^*M$ over M, where $\overline{\nabla}$ is the connection in E given by $\overline{\nabla}_{\!v}(f, \xi) = (d_v f - \xi(v),\, \nabla_{\!v}\xi + Kfg(v, \cdot\,))$ whenever v is a vector field tangent to M. To evaluate $\overline{R}(u, v)(f, \xi)$ at $x \in M$, for the curvature $\overline{R}$ of $\overline{\nabla}$, we may assume that u, v, ξ are ∇-parallel at x and $df_x = 0$. One then easily verifies that, in general, $\overline{R}(u, v)(f, \xi) = (0, \eta)$, where $\eta(w) = -\xi[A(u, v, w)]$ with $A(u, v, w) = R(u, v)w - K[g(u, w)v - g(v, w)u]$. Thus, $\overline{\nabla}$ is flat if and only if g is a metric of constant curvature K, as required. (The final clause of the lemma is obvious: in view of connectedness of U, a $\overline{\nabla}$-parallel section of E defined on U is uniquely determined by its value at one point; see *Distributions and the Frobenius Theorem*, **Exercise 12**.)

The pseudosphere-geodesics lemma. *In a pseudo-Euclidean vector space* $\mathcal{E}$ *with the inner product* $\langle\,,\rangle$, *every nonempty pseudosphere* $M = \{x \in \mathcal{E} : \langle x, x\rangle = c\}$, *where* $c \in \mathrm{I\!R} \smallsetminus \{0\}$, *is a nondegenerate submanifold. For the Levi-Civita connection* ∇ *of the submanifold metric* g *of* M, *any* $z \in M$, *and* $v \in T_z M = z^\perp$, *the formula*

$$(4)\qquad \begin{aligned} x(t) &= z + tv \quad \text{if} \quad \langle v, v\rangle = 0, \\ x(t) &= z\cos t + v\sin t \quad \text{if } \langle v, v\rangle = c, \\ x(t) &= z\cosh t + v\sinh t \text{ if } \langle v, v\rangle = -c, \end{aligned}$$

then defines a ∇*-geodesic* $\mathrm{I\!R} \ni t \mapsto x(t) \in M$ *such that* $x(0) = z$ *and* $\dot{x}(0) = v$.

In fact, $\nabla_{\dot x}\dot x = [\mathrm{D}_{\dot x}\dot x]^{\mathrm{tng}}$ for the Levi-Civita connection D of the flat constant metric on $\mathcal{E}$ arising from the inner product $\langle\,,\rangle$, and $\mathrm{D}_{\dot x}\dot x = \ddot x$, so that our claim is obvious since $\ddot x = \varepsilon x$, with $\varepsilon = \in \{0, 1, -1\}$, is normal to M.

Corollary. *With assumptions and notations as in the preceding lemma, equation* (3) *holds on the pseudosphere* (M, g) *for* $K = 1/c$ *and* f *obtained as the restriction to* M *of any linear homogeneous function* $\mathcal{E} \to \mathrm{I\!R}$.

When $K = 0$, (3) *is satisfied by every affine function* f *on* (M, g) *which is defined to be a finite-dimensional real affine space with a flat constant metric arising from a pseudo-Euclidean inner product.*

Proof. If $K \neq 0$, due to bilinearity and symmetry of both sides, it suffices to verify (3), at any $z \in M$, evaluated on (v, v), where $v \in T_z M = z^\perp$ and $\langle v, v\rangle$ equals 0, c or $-c$. This is in turn obvious from linearity of f, combined with (2) applied to the geodesics $t \mapsto x(t)$ given by (4).

Now let $K = 0$. Then constant vector fields on M, including gradients of affine functions, are parallel, which proves our claim.

The following classification result is a special case of a theorem due to Cartan. (See Élie Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54**, 1926, 214–264.)

The Constant-Curvature Theorem. *Let* $c \in \mathrm{I\!R} \smallsetminus \{0\}$. *Then any nonempty pseudosphere* $\{x \in \mathcal{E} : \langle x, x\rangle = c\}$ *in a pseudo-Euclidean vector space* $\mathcal{E}$, *with its submanifold metric, is a space of constant curvature* $K = 1/c$.

Conversely, every pseudo-Riemannian space (M, g) *of constant curvature* K *is locally isometric to*

(i) *a pseudosphere with a metric obtained as above, if* $K \neq 0$, *or*
(ii) *a pseudo-Euclidean vector space with the constant metric, if* $K = 0$.

Proof. The first assertion is obvious from the Hessian-metric equation lemma along with the above corollary.

Conversely, let (M, g) be of constant curvature K, with $m = \dim M$. Every point of $z \in M$ then has a connected neighborhood U such that $\dim \mathcal{E} = m + 1$ for the vector space $\mathcal{E}$ of all smooth solutions $f : U \to \mathrm{I\!R}$ of the linear equation (3), and $f \mapsto (f(z), df_z)$ is a linear isomorphism $\mathcal{E} \to \mathcal{Z}$ onto the vector space $\mathcal{Z} = \mathrm{I\!R} \times T_z^* M$. (Note that $\mathcal{Z}$ is the space of the initial data (a, ξ) in the Hessian-metric equation lemma, and we may choose U to be contained in the intersection of the domains of solutions of (3) corresponding to pairs (a, ξ) ranging over a basis of $\mathcal{Z}$.) Fixing such U from now on, we see that, for every $x \in U$,

(5) $\quad \mathcal{E} \ni f \mapsto (f(x), [\nabla f]_x)$ is a linear isomorphism $\mathcal{E} \to \mathrm{I\!R} \oplus T_x M$.

Injectivity in (5) is obvious from the final clause of the Hessian-metric equation lemma, and surjectivity follows as both spaces have the same dimension $m+1$.

Next, for any $f, \hat{f} \in \mathcal{E}$, the function

$$\langle f, \hat{f} \rangle \;=\; g(\nabla f, \nabla \hat{f}) \;+\; K f \hat{f} \tag{6}$$

is constant, so that $\langle\,,\rangle$ defined by (6) is a symmetric bilinear form on $\mathcal{E}$. To see this, we first let $\hat{f} = f$, and then, from (1) and (3), $d\langle f, f\rangle = d[g(\nabla f, \nabla f)] + 2Kf\,df = 2[\nabla df](\nabla f, \,\cdot\,) + 2Kf\,df \,=\, 2[\nabla df + Kfg](\nabla f, \,\cdot\,) \,=\, 0$. The general case follows as $4\langle f, \hat{f}\rangle = \langle f + \hat{f}, f + \hat{f}\rangle - \langle f - \hat{f}, f - \hat{f}\rangle$ in view of bilinearity and symmetry.

In the case where $K \neq 0$, we use (5) to define a mapping $\varphi : U \to \mathcal{E}$ by

(7) $\quad \varphi(x) = f$ for the unique $f \in \mathcal{E}$ with $f(x) = 1/K$ and $[\nabla f]_x = 0$.

Our $\langle\,,\rangle$ is now nondegenerate, which makes $\mathcal{E}$ a pseudo-Euclidean space. (In fact, if $f \neq 0$, we find $\hat{f}$ with $\langle f, \hat{f}\rangle \neq 0$ by letting $\hat{f}$ be nonzero and $\nabla \hat{f}$ zero at a fixed point $x \in U$, or vice versa, depending on whether $(\nabla f)_x = 0$ or $(\nabla f)_x \neq 0$, cf. (5) and (6).) Next, all values of φ lie in the pseudosphere $\Sigma = \{y \in \mathcal{E} : \langle y, y\rangle = 1/K\}$, as one sees, for any $x \in U$, by choosing f as in (7) and evaluating the constant function (6), with $\hat{f} = f$, at x, which gives the required value $\langle f, f\rangle = 1/K$. Furthermore, whenever $x \in U$ and $v \in T_x M$, (7) gives

(8) $\quad d\varphi_x v = \dot{f}$ for the unique $\dot{f} \in \mathcal{E}$ with $\dot{f}(x) = 0$ and $(\nabla \dot{f})_x = v$.

To verify (8), consider an interval $I \subseteq \mathrm{I\!R}$ and a smooth curve $I \ni t \mapsto x = x(t) \in U$. Then $f = \varphi(x(t))$ is a curve of functions $U \to \mathrm{I\!R}$, parametrized by $t \in I$, which allows us to treat it as a function $(t, x) \mapsto f(t, x)$ of $(t, x) \in I \times U$, with $f(t, \,\cdot\,) \in \mathcal{E}$ for each t. We will use the partial derivatives $\dot{f} = \partial f/\partial t$ and $\partial_j f$ (for fixed local coordinates in U), all of which are functions of (t, x). According to (7),

$$f(t, x(t)) \;=\; 1/K\,, \qquad (\partial_j f)(t, x(t)) \;=\; 0\,. \tag{9}$$

Applying d/dt to (7), we obtain, from the chain rule, $\dot{f} + \dot{x}^k \partial_k f = 0$ and $\partial_j \dot{f} + \dot{x}^k \partial_k \partial_j f = 0$ (with each term evaluated at $(t, x(t))$, which is suppressed from our notation). As $f_{,jk} = \partial_k \partial_j f - \Gamma_{kj}^l \partial_l f$, (9) allows us to rewrite these equalities as $\dot{f} = 0$ and $0 = \partial_j \dot{f} + \dot{x}^k f_{,jk} = \partial_j \dot{f} - Kf\dot{x}^l g_{jl} = \partial_j \dot{f} - \dot{x}^k g_{jk}$. With $x(t) = x$ and $\dot{x}(t) = v$ at fixed t, this yields (8).

Finally, by (6) and (8), $\langle \dot{f}, \dot{f}\rangle = g(v, v)$. Thus, φ is isometric and so, locally, is an isometry between open sets in U and open subsets of the pseudosphere Σ, which completes the proof when $K \neq 0$.

Now let $K = 0$. This time $\langle\,,\rangle$ is degenerate, with the nullspace $\mathrm{I\!R} = \mathcal{E}^\perp$ consisting of constant functions. We denote by $\mathcal{A}$ the set of all linear functionals $\mathcal{E} \to \mathrm{I\!R}$, the restriction of which to the line $\mathrm{I\!R} \subseteq \mathcal{E}$ of constant functions equals the identity mapping $\mathbb{R} \to \mathbb{R}$. Thus, $\mathcal{A}$ is an affine subspace of $\mathcal{E}^*$, namely, a coset of the vector subspace $(\mathcal{E}/\mathbb{R})^*$ of $\mathcal{E}^*$ formed by all linear functionals $\mathcal{E} \to \mathbb{R}$ that vanish on $\mathbb{R} \subseteq \mathcal{E}$. Clearly, $\langle\,,\rangle$ descends to a pseudo-Euclidean inner product on $\mathcal{E}/\mathbb{R}$, again denoted by $\langle\,,\rangle$, which induces a further inner product $\langle\,,\rangle$, and thus turns the affine space into $\mathcal{A}$ into a flat pseudo-Riemannian manifold. We now define a mapping $\varphi : U \to \mathcal{A}$ by $\varphi(x) = \delta_x$, with δ_x standing for the ***Dirac delta*** (evaluation) functional, that is, the assignment $\mathcal{E} \ni f \mapsto f(x) \in \mathbb{R}$. Next, if $x \in U$ and $v \in T_xM$, we have $d\varphi_x v = \langle \dot{f}, \cdot \rangle \in (\mathcal{E}/\mathbb{R})^*$ for the unique $\dot{f} \in \mathcal{E}$ with $\dot{f}(x) = 0$ and $[\nabla \dot{f}]_x = v$. In fact, given an interval $I \subseteq \mathbb{R}$ and a smooth curve $I \ni t \mapsto x = x(t) \in U$, the evaluation of $\varphi(x(t))$ on a fixed function $f \in \mathcal{E}$ yields $f(x(t))$, and so, applying d/dt, one gets $df_x \dot{x}$, with $x = x(t)$, which equals $\langle \dot{f}, f \rangle$. Thus, by (6), $\langle \dot{f}, \dot{f} \rangle = g(v, v)$. Consequently, φ is again isometric, as required.

Remark. Here is a common method of constructing injective mappings φ from a manifold (or a more general topological space), denoted here by M, into a vector space: one first selects a "natural" vector space $\mathcal{F}$ of functions $M \to \mathbb{R}$, and then defines $\varphi : M \to \mathcal{F}^*$ by $\varphi(x) = \delta_x$ (the Dirac delta functional), so that $[\varphi(x)](f) = f(x)$ for $x \in M$ and $f \in \mathcal{F}$. This is precisely what we did in the above proof, with $\mathcal{F} = \mathcal{E}$ – not only (quite explicitly) when $K = 0$, but also for $K \neq 0$. In the latter case, nondegeneracy of $\langle\,,\rangle$ leads to the isomorphic identification $\mathcal{F} = \mathcal{F}^*$, via $f \mapsto \langle f, \cdot \rangle$ which, by (6), makes δ_x correspond to $\varphi(x)$ with (7).

Another well-known example of (a modified version of) the above method arises in complex differential geometry: let $\mathcal{F}$ denote the space of all holomorphic sections of a given a holomorphic line bundle L over a compact complex manifold M, assumed to have a "weak-ampleness" property, in the sense that for every $x \in M$ there exists $\psi \in \mathcal{F}$ with $\psi_x \neq 0$. Denoting by $\mathrm{P}(\mathcal{F}^*)$ the projective space of the complex dual $\mathcal{F}^*$, we now define a holomorphic mapping $\varphi : M \to \mathrm{P}(\mathcal{F}^*)$ by $\varphi(x) = \mathbb{C}\delta_x$. Note that, even though $\delta_x : \mathcal{F} \to L_x$ (rather than $\delta_x : \mathcal{F} \to \mathbb{C}$), the complex span $\mathbb{C}\delta_x$ of δ_x equals that of a linear functional $\mathcal{F} \to \mathbb{C}$, obtained by choosing an isomorphic identification $L_x \approx \mathbb{C}$, and independent of the choice of such an identification.