MATH 7711, AUTUMN 2019

Conformal Flatness

[**AC**] stands for *Algebraic Curvature Tensors* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/ac.pdf`
[**TC**] for *Tractor Connections in Tractor Bundles* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/tc.pdf`

A *conformal change* of a pseudo-Riemannian metric g on a manifold M means its replacement by the metric $\tilde{g} = e^{2\sigma}g$, where $\sigma : M \to \mathrm{I\!R}$ is a smooth function. Using the tilde notation $\tilde{}$ for objects corresponding to $\tilde{g}$, we obtain the equality

$$(1) \qquad \tilde{\varGamma}_{ij}^k = \varGamma_{ij}^k + \varPhi_{ij}^k \quad \text{with} \quad \varPhi_{ij}^k = \sigma_{,i}\delta_j^k + \sigma_{,j}\delta_i^k - \sigma^{,k}g_{ij}\,,$$

obvious from the Christoffel-symbol formula $2\varGamma_{ij}^k = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$.

In general, for two linear connections ∇ and $\tilde{\nabla}$ in a vector bundle $\mathrm{pr} : E \to M$ one has $\tilde{\nabla} = \nabla + \varPhi$, where $\varPhi$ is a vector-bundle morphism $TM \to \mathrm{End}\,E$, so that $\tilde{\varGamma}_{ib}^a = \varGamma_{ib}^a + \varPhi_{ib}^a$, and then the curvature formula

$$(2) \qquad \tilde{R}_{ijb}{}^a \,=\, R_{ijb}{}^a + \varPhi_{ib,j}^a - \varPhi_{jb,i}^a + \varPhi_{jc}^a\varPhi_{ib}^c - \varPhi_{ic}^a\varPhi_{jb}^c$$

easily follows since $R_{ijb}{}^a = \partial_j\varGamma_{ib}^a - \partial_i\varGamma_{jb}^a + \varGamma_{jc}^a\varGamma_{ib}^c - \varGamma_{ic}^a\varGamma_{jb}^c$ and, at any given point, one is free to assume that $\varGamma_{kb}^a = 0$, as well as $\varGamma_{ij}^k = 0$ for a torsion-free connection in TM needed to form the covariant derivatives in (2).

In particular, for the Levi-Civita connections of g and $\tilde{g} = e^{2\sigma}g$,

$$(3) \qquad \tilde{R}_{ijk}{}^l \,=\, R_{ijk}{}^l + \varPhi_{ik,j}^l - \varPhi_{jk,i}^l + \varPhi_{jq}^l\varPhi_{ik}^q - \varPhi_{iq}^l\varPhi_{jk}^q\,,$$

with $\varPhi$ as in (1). In this case, defining $\widehat{R}$ by $\widehat{R}_{ijkl} = \tilde{R}_{ijk}{}^p g_{pl}$, we have

$$(4) \qquad \widehat{R} \,=\, R + L + Q\,,$$

where $L_{ijkp} = g_{pl}(\varPhi_{ik,j}^l - \varPhi_{jk,i}^l)$ and $Q_{ijkp} = g_{pl}(\varPhi_{jq}^l\varPhi_{ik}^q - \varPhi_{iq}^l\varPhi_{jk}^q)$. By (1), $L_{ijkp} = -g_{ik}\sigma_{,jp} - g_{jp}\sigma_{,ik} + g_{jk}\sigma_{,ip} + g_{ip}\sigma_{,jk}$, i.e., $L = -2g \wedge \nabla d\sigma$ (notation of [**AC**]). Also, again from (1), $g_{pl}\varPhi_{jq}^l = g_{pq}\sigma_{,j} + g_{jp}\sigma_{,q} - \sigma_{,p}g_{jq}$ and, consequently,

$$\begin{aligned}
g_{pl}\varPhi_{jq}^l\varPhi_{ik}^q \,=\, & g_{pq}\sigma_{,j}(\sigma_{,i}\delta_k^q + \sigma_{,k}\delta_i^q - \sigma^{,q}g_{ik}) \,+\, g_{jp}\sigma_{,q}(\sigma_{,i}\delta_k^q + \sigma_{,k}\delta_i^q - \sigma^{,q}g_{ik}) \\
& - \sigma_{,p}g_{jq}(\sigma_{,i}\delta_k^q + \sigma_{,k}\delta_i^q - \sigma^{,q}g_{ik}) \\
=\, & (g_{kp}\sigma_{,i}\sigma_{,j} + g_{ip}\sigma_{,j}\sigma_{,k} - g_{ik}\sigma_{,j}\sigma_{,p}) + (g_{jp}\sigma_{,i}\sigma_{,k} + g_{jp}\sigma_{,i}\sigma_{,k} - \sigma^{,q}\sigma_{,q}g_{ik}g_{jp}) \\
& - (g_{jk}\sigma_{,i}\sigma_{,p} + g_{ij}\sigma_{,k}\sigma_{,p} - g_{ik}\sigma_{,j}\sigma_{,p}).
\end{aligned}$$

In the last two lines, the third and ninth terms above cancel each other. The first, the eighth, as well as the sum of the second and fourth ones, are all symmetric in i, j. Therefore, as we need to subtract from the above expression its version obtained by switching i with j, only the sum $g_{jp}\sigma_{,i}\sigma_{,k} - \sigma^{,q}\sigma_{,q}g_{ik}g_{jp} - g_{jk}\sigma_{,i}\sigma_{,p}$ of the fifth, sixth, and seventh terms will contribute, producing the difference

$$(g_{jp}\sigma_{,i}\sigma_{,k} - \sigma^{,q}\sigma_{,q}g_{ik}g_{jp} - g_{jk}\sigma_{,i}\sigma_{,p}) \,-\, (g_{ip}\sigma_{,j}\sigma_{,k} - \sigma^{,q}\sigma_{,q}g_{jk}g_{ip} - g_{ik}\sigma_{,j}\sigma_{,p}).$$

Thus, $Q_{ijkp} = g_{ik}\sigma_{,j}\sigma_{,p} + g_{jp}\sigma_{,i}\sigma_{,k} - g_{jk}\sigma_{,i}\sigma_{,p} - g_{ip}\sigma_{,j}\sigma_{,k} - \sigma^{,q}\sigma_{,q}(g_{ik}g_{jp} - g_{jk}g_{ip})$, and so, with $\otimes$ as in [**AC**, formula (1)], $Q = g \wedge [2d\sigma \otimes d\sigma - g(\nabla\sigma,\nabla\sigma)g]$. Now

$$\widehat{R} = R - g \wedge b, \quad \text{where} \quad b = 2(\nabla d\sigma - d\sigma \otimes d\sigma) + g(\nabla\sigma,\nabla\sigma)g, \tag{5}$$

due to (4), since we saw earlier that $L = -2g \wedge \nabla d\sigma$. Consequently, $\widehat{R}$ and R have the same $\mathcal{W}(\mathcal{R})$-component (in dimensions $m \geq 3$), as their difference is $\wedge$-divisible by g (see [**AC**, Remark 1]). This implies *conformal invariance* for the Weyl tensor of type $(1,3)$:

$$\tilde{W}_{ijk}{}^{l} = W_{ijk}{}^{l} \quad \text{whenever} \quad \tilde{g} = e^{2\sigma}g. \tag{6}$$

It is convenient to rewrite (5) in terms of the positive smooth function $\alpha = e^{-\sigma}$. As $\sigma = -\log\alpha$, we have $\sigma_{,i} = -\alpha^{-1}\alpha_{,i}$, that is, $d\sigma = -\alpha^{-1}d\alpha$ or, equivalently, $\nabla\sigma = -\alpha^{-1}\nabla\alpha$, as well as $g(\nabla\sigma,\nabla\sigma) = \alpha^{-2}g(\nabla\alpha,\nabla\alpha)$ and $\sigma_{,ij} = -\alpha^{-1}\alpha_{,ij} + \alpha^{-2}\alpha_{,i}\alpha_{,j}$ or, in coordinate-free form, $\nabla d\sigma = -\alpha^{-1}\nabla d\alpha + \alpha^{-2}d\alpha \otimes d\alpha$. Now (5) reads

$$\widehat{R} = R - g \wedge b, \quad \text{with} \quad b = -2\alpha^{-1}\nabla d\alpha + \alpha^{-2}g(\nabla\alpha,\nabla\alpha)g \tag{7}$$

for a conformal change having the form $\tilde{g} = g/\alpha^2$ and, since one obviously has $\tilde{R}_{ik} = g^{jl}\widehat{R}_{ijkl}$, that is, the g-induced Ricci contraction of $\widehat{R}$ yields the Ricci tensor $\tilde{r}$ of the metric $\tilde{g}$, we obtain

$$\begin{aligned} \tilde{r} &= r + (m-2)\alpha^{-1}\nabla d\alpha + [\alpha^{-1}\Delta\alpha - (m-1)\alpha^{-2}g(\nabla\alpha,\nabla\alpha)]g, \\ \tilde{s} &= \alpha^2 s + 2(m-1)\alpha\Delta\alpha - m(m-1)g(\nabla\alpha,\nabla\alpha) \end{aligned} \tag{8}$$

for the Ricci tensors and scalar curvatures of two conformally related metrics, g and $\tilde{g} = g/\alpha^2$, in dimension m.

The next result is due to Schouten. (See Jan A. Schouten, *Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Maßbestimmung auf eine Mannigfaltigkeit mit euklidischer Maßbestimmung*, Math. Zeitschr. **11**, 1921, 58-88.)

The Conformal-Flatness Theorem. *A pseudo-Riemannian manifold (M,g) of any dimension $m \geq 4$ is conformally flat if and only if its Weyl tensor W is identically zero.*

In dimension three conformal flatness is equivalent to the condition $dh = 0$ for the Schouten tensor h, cf. [**AC**], [**TC**].

Proof: According to Lemma 4 in [**AC**], it suffices to verify that, in all dimensions $m \geq 3$, conformal flatness of a metric g amounts precisely to flatness of its tractor connection $\overline{\nabla}$.

First, let $\overline{\nabla}$ be flat. Given $x \in M$, we may thus choose a local $\overline{\nabla}$-parallel section (v,α,ψ) of the tractor bundle E defined on a neighborhood of x and having $\alpha(x) \neq 0$, as well as $v = \nabla\alpha$, the latter due to the equality $\hat{\alpha} = d_u\alpha - g(u,v)$ in formula (1) of [**TC**]. The middle line of formula (1) in [**TC**] now yields relation (2) in [**TC**] which, combined with Lemmas 1 and 2 of [**TC**], shows that $\tilde{g} = g/\alpha^2$ is an Einstein metric on some, possibly smaller, neighborhood of x. Contracting equality (2) in [**TC**] we also get $\Delta\alpha = m\psi - \alpha s/(m-2)$, where $m = \dim M$. The last line in (8) above, with $v = \nabla\alpha$, thus reads $\tilde{s} = m(m-1)[2\alpha\psi - g(v,v)] - m\alpha^2 s/(m-2)$, for the *constant* scalar curvature $\tilde{s}$ of $\tilde{g}$, so that suitably chosen values of ψ and v at x give $\tilde{s} = 0$. In other words, the Einstein metric $\tilde{g}$ then must be Ricci-flat. However, $W = 0$ by Lemma 4 in [**AC**], and so, according to [**AC**, formula (5)], $\tilde{g}$ is a flat metric.

Conversely, let (M, g) be conformally flat. We then have $W = 0$ in view of conformal invariance of the Weyl tensor (relation (6)) along with the obvious fact that $W = 0$ for any flat metric, cf. [**AC**, the formula preceding (5)]. Formula (1) in [**TC**] along with Lemmas 1 and 2 of [**TC**] state that, locally, we may choose a $\overline{\nabla}$-parallel section $(v, \alpha, \psi) = (\nabla\alpha, \alpha, \psi)$ of the tractor bundle E having $\alpha \neq 0$ everywhere in its domain U. The Ricci identity, applied to the curvature tensor $\overline{R}$ of the tractor connection $\overline{\nabla}$, thus gives $\overline{R}(u, u')(v, \alpha, \psi) = 0$ for all vector fields u, u' on U, and so Lemma 3 in [**TC**], with $W = 0$ and $\alpha \neq 0$, implies that $Dh = 0$ as well as $\overline{R} = 0$ everywhere, completing the proof.