EINSTEIN METRICS IN DIMENSION FOUR

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CONTENTS

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§0. Introduction

A pseudo-Riemannian metric g on a (connected) manifold M is called an *Einstein metric* if its Ricci tensor Ric and scalar curvature s satisfy the relations

$$\mathrm{Ric} = \frac{\mathrm{s}}{n}\, g\,, \qquad d\mathrm{s} = 0\,, \qquad n = \dim M\,. \tag{0.1}$$

One then also refers to the pair (M, g) as an *Einstein manifold*. Note that, since M is assumed connected, condition $d\mathrm{s} = 0$ means here that s is constant.

This article is an exposition of selected results concerning Einstein metrics on four-manifolds. The focus of our discussion on dimension four, with other dimensions mentioned only in passing, follows what has consistently, with relatively few exceptions, seemed to be the main areas of interest for Einstein metrics both in differential geometry and mathematical physics.

We address two main topics concerning four-dimensional Einstein metrics, often treated as completely separate. One is the question of global properties of compact Einstein 4-manifolds which are *Riemannian* in the sense that their metrics are positive definite. The other topic pertains to local properties of Einstein metrics on 4-dimensional manifolds in Riemannian as well as *Lorentzian* and *neutral* cases; the italicized terms refer to the sign patterns $-+++$ and $--++$, of which the former characterizes spacetime metrics in general relativity.

As a result, the material presented here can naturally be divided into *local* and *global* topics. The former are dealt with in most of Part I and Part IV. As for the latter, they are discussed, usually in the context of compact Riemannian manifolds, in Parts II and III.

Here is a brief outline of the contents of this article. Part I covers a variety of elementary topics, ranging from preliminary material on curvature, flatness, submanifolds, and conformal changes of metrics, through local classification theorems for curvature-homogeneous Riemannian Einstein metrics of dimension four (along the special cases of metrics which are locally homogeneous, or locally symmetric), and also including topics such as Einstein hypersurfaces in pseudo-Euclidean spaces, Einstein metrics conformal to Kähler metrics or to Riemannian-product metrics, mobility for Einstein metrics, and potentials for Kähler metrics. Parts II and III deal with topological conditions that guarantee nonexistence (or, in some rare cases, existence) of a Riemannian Einstein metric on a given compact four–manifold. Finally, Part IV is devoted to local properties of indefinite Einstein metrics in dimension four and contains Petrov's classification of curvature types and a local classification of locally symmetric pseudo-Riemannian Einstein 4-manifolds.

For more details, see the detailed summaries at the beginning of each of the four parts, and the table of contents.

Besides the local-global distinction, another division in the material reflects four different levels involved in the presentation. One, which might be termed *basic and self-contained*, appears in Part I, and is both easily accessible to nonexperts and accompanied by full proofs. A 'nonexpert' means here someone whose background includes, but does not reach far beyond, the contents of a typical introductory graduate course in modern differential geometry offered at large U.S. universities. Accordingly, the reader is assumed to have some familiarity with objects such as manifolds, differentiable mappings, tensors, pseudo-Riemannian metrics, and vector bundles (along with differentiable sections of, connections in, and fibrewise algebraic operations on the latter); the notions just listed will not be defined. Other concepts, although expected to be known to the reader, are briefly defined here; these include the Levi-Civita connection, the curvature and Ricci tensors, the Lie bracket of vector fields, etc.

As for Part II, it contains material which is still ***basic***, but this time relies on several non-elementary results that are quoted without proofs (but, once taken for granted, make the rest of the discussion in Part II quite easy). In Part IV, the presentation level is ***intermediate***, which here means that it is elementary and self-contained, but involves lengthy technical details that a nonexpert might not be willing to go into. Finally, Part III contains material on an ***advanced*** level in the form of brief summaries of results relying on repeated uses of techniques the details of which cannot be presented in an article of this size.

As another consequence of its limited size, the present text does not and cannot address all facts concerning four-dimensional Einstein metrics that are deemed important by large groups of experts. The topics left out include:

- techniques of global complex analysis used to prove existence and uniqueness results of the Calabi-Yau-Aubin type;
- moduli spaces of Einstein metrics on compact manifolds;
- the whole subject of cohomogeneity-one Einstein 4-manifolds, with the example of a U(2)-invariant Einstein metric on a compact 4-manifold discovered by Page (1978), and Bérard Bergery's (1981) classification theorem stating that no further examples of this type are possible;
- further topological obstructions to the existence of specific types of Einstein metrics on compact 4-manifolds, such as a theorem of due to Hitchin (1974) for nonpositively or nonnegatively curved Riemannian metrics, or results of Law (1991) for indefinite metrics of the *neutral* sign pattern $--++$;
- a result of DeTurck and Kazdan (1981) which says that every Riemannian Einstein metric is analytic in suitable local coordinates;
- an approach to Einstein metrics using normal coordinates and power-series solutions to (0.1);
- a result of Gasqui (1982) stating that any "Einsteinian" algebraic curvature tensor at a point x is realized as the curvature at x of an Einstein metric g defined in a neighborhood of x, with the prescribed value $g(x)$ at x;

and many more.

This paper benefited from correspondence and conversations with many people. Some of those discussions go back more than a decade. I am particularly obliged to Andrea Sambusetti for numerous comments and helpful criticism of §27. I also greatly appreciate conversations I had in the mid-1980s with Richard Palais and Chuu-Lian Terng, concerning warped-product Einstein metrics, a topic mentioned in §16 and §19 of this text.

PART I: BASICS

Throughout the following twenty-three sections we will usually assume that the metrics in question are both *Riemannian* (positive definite) and *four-dimensional*, unless a generalization to other signatures and/or dimensions is completely straightforward. There are only three exceptions to this rule, namely, all of §8, Remark 13.10 at the very end of §13, and the second half of §15(starting from Lemma 15.7). In those places we discuss some classes of pseudo-Riemannian manifolds that do not contain, and are quite different from, any Riemannian manifolds. Without those three fragments, Part I remains a self-contained whole.

The Einstein condition (0.1) is a fairly complicated system of nonlinear second-order partial differential equations imposed on the local component functions of the metric g. In sections 7 through 19 and 23 we discuss some special cases in which that system of equations becomes more manageable.

The first of such cases is that of locally homogeneous Einstein 4-manifolds. The question of finding a local classification of all such metrics is automatically reduced to an algebraic problem (which is an obvious consequence of homogeneity); however, for Riemannian Einstein metrics in dimension four, it becomes completely straightforward, as explained next.

In general, we have the following chain of inclusions

$$(*)\quad \boxed{\begin{matrix}\text{algebraic}\\ \text{examples}\end{matrix}} \subset \boxed{\begin{matrix}\text{locally}\\ \text{symmetric}\end{matrix}} \subset \boxed{\begin{matrix}\text{locally}\\ \text{homogeneous}\end{matrix}} \subset \boxed{\begin{matrix}\text{curvature}\\ \text{homogeneous}\end{matrix}}$$

between classes of pseudo-Riemannian metrics g of any given dimension and sign pattern. Here "algebraic examples" stands for a specific set of local-isometry types of locally symmetric metrics (see §7) that arise from some direct linear-algebra constructions, while *curvature-homogeneity* means that the metric-curvature pair (g, R) represents the same algebraic type at each point x of the underlying manifold M, i.e., given two points x, y, there exists a linear isomorphism $T_xM \to T_yM$ that takes $g(x)$ onto $g(y)$ and $R(x)$ onto $R(y)$. (See, e.g., Tricerri and Vanhecke, 1989.)

As it turns out, in the category of Riemannian Einstein 4-manifolds, all four inclusions in $(*)$ above are actually equalities:

$$\boxed{\begin{matrix}\text{algebraic}\\ \text{examples}\end{matrix}} = \boxed{\begin{matrix}\text{locally}\\ \text{symmetric}\end{matrix}} = \boxed{\begin{matrix}\text{locally}\\ \text{homogeneous}\end{matrix}} = \boxed{\begin{matrix}\text{curvature}\\ \text{homogeneous}\end{matrix}},$$

with the "algebraic examples" listed in the assertion of Theorem 14.7. More precisely, the first of these equalities is Cartan's classification result for locally symmetric 4-manifolds (Cartan, 1926; see Theorem 14.7 in §14); the second one is a theorem of Jensen (1969), a proof of which is given in §7 (Corollary 7.3); and the third equality is Corollary 7.2 in §7. Note that, for four-dimensional Riemannian Einstein manifolds, curvature-homogeneity is equivalent to the requirement that the curvature operator acting on bivectors have constant eigenvalues; see Remark 6.24 in §6.

It should be noted that, in contrast with the Riemannian case, for *indefinite* Einstein metrics in dimension four all three inclusions in $(*)$ above are strict. See Theorems 41.5 and 41.6 in §41 (compared with Theorem 41.4) and, respectively, Proposition 8.5 in §8 and Corollary 49.2 in §49.

A condition similar to, but weaker than local homogeneity is that of (local) *cohomogeneity one*, where the Riemannian four-manifold in question has a (local) isometry group with 3-dimensional principal orbits. The Einstein condition (0.1) then is obviously equivalent to a system of ordinary differential equations. (See, e.g., Bérard Bergery, 1982; Gibbons and Hawking, 1979.) Lack of space makes it impossible to include a discussion of that situation in the present text.

Another possible source of easy examples of Einstein four-manifolds are those hypersurfaces M in 5-dimensional pseudo-Euclidean vector spaces whose submanifold metrics, i.e., first fundamental forms, happen to be (nondegenerate and) Einstein. In the case where g is Riemannian or Lorentzian, this leads to nothing new; in fact, (M,g) then must be a space of constant curvature (Proposition 15.6). However, among those indefinite metrics g of pseudo-Euclidean hypersurfaces having the *neutral* sign pattern $--++$ there is a surprising wealth of examples that are Einstein (more precisely, Ricci-flat). For details, see Example 15.14 in §15.

A further simple method of constructing Einstein 4-manifolds is through conformal changes of products of surface metrics (§16). The surface metrics involved must be of the special type known as *Calabi's extremal metrics*. In the Lorentz case, the examples obtained in this way include some important spacetime models for general relativity, namely, the Schwarzschild and Kottler metrics (§18, §48). A related class is that of Einstein 4-manifolds (M,g) with g locally conformal to a $(1+3)$-dimensional product metric; see §19.

A fifth case where a simpler-than-general approach to the Einstein condition exists is that of Kähler metrics on complex manifolds. Such metrics can be represented, locally, through a single real-valued C^∞ function ϕ (a *Kähler potential*) and, for Kähler metrics, condition (0.1) becomes a single partial differential equation imposed on ϕ (the *Monge-Ampère equation* (23.29)). See §23.

§1. Remarks on notation

The notations used in this text were chosen in an attempt to have symbols and notational conventions that are, at the same time, simple, internally consistent (within the text) and, finally, in agreement with whatever norms prevail in the existing literature.

The result is, at best, an uneasy compromise. First, in matters as basic as the sign conventions for the curvature tensor R, the second fundamental form b, or the divergence and Laplace operators div and Δ, no general consensus seems to exist. As a result, our usage agrees with some and disagrees with many others among the standard sources. For instance, our sign of R differs from that in Kobayashi and Nomizu (1963), while or Δ and div are the opposites of those in Besse, 1987 (where, by the way, the divergence is denoted δ). Also, our conventions for the universal factors appearing in the component descriptions of the exterior product and exterior derivative differ from Kobayashi and Nomizu (1963).

Simplicity of notation, in a text of this size, is not only a matter of æsthetic appeal, but also involves the requirement that notations be *self-explanatory*, relieving the reader of the need to constantly search the 'Preliminaries' section for clues. This is why "projections" of all kinds are represented by the symbol pr, with superscripts if necessary, while our Ricci tensor, divergence, and volume form are denoted Ric, div, vol. The scalar curvature, however, is s rather than Scal (or something similar), since the latter symbol would be too awkward in expressions like $\mathrm{s}_{,j}$. (The same problem for Ric was resolved by writing its components as R_{jk}, which follows an old tradition.) For similar reasons, ∇ is preferred over grad for the gradient operator. It is also for the sake of simplicity that some less orthodox symbols are used here, such as $t \mapsto e^{tw}$ for the flow of a vector field w. Finally, whenever a fixed pseudo-Riemannian metric is present, we use the associated index-raising and index-lowering operations to identify tensorial objects of various "types" without any special notation; the symbol v used for a tangent vector thus represents the associated cotangent vector $g(v, \cdot)$ as well.

This last identification leads to one particular difficulty which seems to lack a completely satisfactory solution. Namely, when a twice-covariant tensor B is treated, with the aid of a fixed metric g, as an operator acting on tangent vectors v, we always interpret that identification as raising the *second* lower index of B. Thus, Bv stands for $B(v, \cdot)$ (that is, the unique vector w with $g(w,u) = B(v,u)$ for all vectors u; in terms of components, $[Bv]^j = B_k{}^j v^k$ with $B_k{}^j = B_{kl}g^{lj}$). Our choice, which of course makes no difference for *symmetric* two-tensors, is largely motivated by our frequent use of the standard notation $R(v,w)u$ for the curvature tensor of a metric g acting on vectors v,w,u so as to be skew-symmetric in v,w. At the same time, we want to treat $R(v,w)$ as a (skew-adjoint) operator in the tangent space sending u to $R(v,w)u$, while the (skew-symmetric) two-tensor corresponding to that operator should assign the number $R(v,w,u,u') = g(R(v,w)u,u')$ to vectors u,u'. The component version of this convention is $R_{jklp} = R_{jkl}{}^s g_{sp}$, with R_{jklp} skew-symmetric both in j,k and l,p. Note that if we had chosen the first (rather than second) index of B to be the one raised, this curvature convention would require annoying modifications in the form of a minus sign or a changed order of arguments. On the other hand, the drawbacks of our choice include a counterintuitive form of the component description of the composite AB of two twice-covariant tensors A,B, with $(AB)_{jk} = A^s{}_k B_{sj}$. Another inconvenience arises when the covariant derivative ∇w of a tangent vector field w acts as an operator on tangent vector v, with $(\nabla w)v = \nabla_v w$, and we choose to treat w as a cotangent vector field instead. The component notation $\nabla_k w_j$ then fits our convention (as it engages v^k, in $v^k \nabla_k w_j$, via its first lower index k); however, for simplicity, we will in most cases use the symbol $w_{j,k}$ as a stand-in for $\nabla_k w_j$, with the resulting component expression $v^k w_{j,k}$ for $\nabla_v w$ which constitutes an exception to the above rule.

As the examples scattered throughout the last paragraph clearly indicate, out notations include the *summing convention*, so that in a "monomial" type expression, an index repeated twice (once as a subscript and once as a superscript) is to be summed over. Superscripts will therefore be very common, for instance in components v^j of tangent vectors, or in individual coordinate functions x^j forming a given coordinate system. They must not be confused with exponents: x^j is *not* the jth power of x (unless clearly stated to be just that). In those rare instances where both uses of upper indices have to coexist, we use parentheses; an expression such as $(x^2)^2$ stands for the square of the second coordinate function x^2.

Internal consistency of notations leads to yet another problem. The sheer number of topics covered makes it hard to get by with the few commonly used alphabets without introducing symbols that would make a reader cringe. Clashes between notations used in various sections, even "noninteracting" ones, are nevertheless kept to a minimum by being allowed only in those cases where overlapping notations appear completely harmless and cannot be reasonably avoided. A typical example is the symbol A used (in §30 only) to represent a generic U(1)-connection in a complex line bundle; this widely established usage goes back to the traditional symbol A for a four-potential in relativistic electrodynamics (which in turn comes from the symbol $\mathbf{A}$ for a nonrelativistic vector potential). However, due to alphabet scarcity, the symbol A is still used elsewhere in the text as a generic notation for linear operators or functions. Similarly, following the common usage, we denote $\|M\|$ the simplicial volume of a compact orientable manifold M (in §35 only), while in §25, §26, §27 and §34 we let $\|\ \|$ stand for the L^2 norm.

Also, in some places internal consistency and simplicity requirements disagree, making it necessary to settle for a patchwork approach. The most pronounced example is the g-inner product $\langle A, B\rangle$ of twice-covariant tensors A, B in a pseudo-Riemannian manifold (M, g) (treated, with the aid of g, as operators in the tangent space), such that A, B are both symmetric, or both skew-symmetric. We define it by $\langle A, B\rangle = r\,\mathrm{Trace}\,AB$ with $r = 1$ when A, B are symmetric, and $r = -1/2$ when they are skew-symmetric. (Applying the trace to AB^* rather than AB removes the discrepancy in sign, but not the factor of $1/2$.) The reason we adopt this convention is that it leads to simplicity in further formulae; for instance, when g is positive definite and $|\ |$ denotes the norm corresponding to $\langle\,,\rangle$, we have $|v\wedge w| = |v|\,|w|$ for mutually orthogonal vectors v, w, with the "simple" convention that $(v\wedge w)^{jk} = v^j w^k - v^k w^j$;, on the other hand, $|A|^2 = \lambda_1^2 + \ldots + \lambda_n^2$ whenever A is symmetric and $\lambda_1, \ldots, \lambda_n$ are its eigenvalues.

For any given local coordinate system x^j, $j = 1, \ldots, n$, in an n-dimensional manifold, we use the symbols $e_1, \ldots, e_n$ for the corresponding coordinate vector fields (rather than the more commonly accepted $\partial/\partial x^j$ or ∂_j). The reason is that in our notation the directional derivative in the direction of a vector field v sends a function f to something written as $d_v f$, rather than vf. (We never use the latter symbol to avoid confusion with other multiplications involving vector fields, such as the Clifford product.) Having symbols for vector fields generally different from those for the corresponding directional derivatives, it would be awkward to suddenly make an exception in the case of coordinate vector fields.

Finally, we will use the phrase 'locally in (the manifold) M' to mean 'in a suitable neighborhood of any given point of M'.

§2. Preliminaries

This and the next three sections form the reference part of the text, where formulae, definitions and some assertions with proofs can be looked up when needed. Although they can serve as a crash course in basics, they were not designed primarily to fulfill that rôle; consequently, the reader trying to learn material from here may find the presentation too sketchy.

Given a local coordinate system x^j in a manifold M, $j = 1, \ldots, n$ (with $n = \dim M$), we will denote y^j and v^j the components relative to the coordinates x^j of any point y in the coordinate domain U and, respectively, any tangent vector $v \in T_y M$. Thus, $y^j = x^j(y)$ and, given any C^1 curve $t \mapsto x(t) \in U$, we have $[\dot x(t)]^j = \dot x^j(t)$ (where the latter stands for dx^j/dt). The coordinates x^j also give rise to the *coordinate vector fields*, denoted $e_1, \ldots, e_n$, in such a way that, for $k = 1, \ldots, n$, $v = e_k$ has the components (relative to the x^j) given by

$$v^j \,=\, \delta_k^j \quad \text{for} \quad v \,=\, e_k\,. \tag{2.1}$$

For any tangent vector v we thus have

$$v = v^j e_j . \tag{2.2}$$

The e_j are thus dual to the dx^j, i.e.,

$$(dx^j)(e_k) = \delta^j_k . \tag{2.3}$$

Here and in the sequel we adopt, unless stated otherwise, the standard convention of *summing over repeated indices* (a subscript and a superscript), in a monomial-style expression. A tangent vector $v \in T_xM$ gives rise to the corresponding *directional derivative*, associating with every C^1 function f defined in a neighborhood of x the number $d_v f$. When v varies with x (i.e., we are given a tangent vector *field*), so does $d_v f$, i.e., d_v then takes functions to functions. The directional derivative d_v corresponding to the coordinate vector field e_j for a fixed local coordinate system x^j is called the *partial derivative* in the jth coordinate direction and denoted $\partial/\partial x^j$ or simply ∂_j. Thus, by (2.2), $d_v f = v^j \partial_j f$.

The *Lie bracket* of any (local) C^1 vector fields v, w in M is the vector field $[v, w]$ characterized by the local-coordinate formula

$$[v, w]^j = d_v w^j - d_w v^j , \tag{2.4}$$

i.e., $[v, w]^j = v^k \partial_k w^j - w^k \partial_k v^j$. Thus, for a coordinate vector field $v = e_k$ with (2.1), we have

$$[e_k, w]^j = \partial_k w^j . \tag{2.5}$$

In terms of directional derivatives, (2.4) easily gives

$$d_{[v,w]} f = d_v d_w f - d_w d_v f \tag{2.6}$$

for any C^2 function f.

Let (M, g) now be a pseudo-Riemannian manifold. As usual, we denote g_{jk} the component functions of the metric g relative to any local coordinate system x^j in M. Thus,

$$g_{jk} = g(e_j, e_k) . \tag{2.7}$$

The symbols g^{jk} then stand for the component functions of the reciprocal metric g^{-1}, so that, at any point of the coordinate domain, $[g^{jk}] = [g_{jk}]^{-1}$ as matrices, i.e.,

$$g^{jl} g_{lk} = \delta^j_k . \tag{2.8}$$

Using the metric g, we will identify, without further comments, tangent vectors $v \in T_xM$ with cotangent vectors (1-forms) $\xi \in T^*_xM$ at any point $x \in M$. Thus, $\xi = g(v, \cdot)$. In terms of the components v^j, ξ_j of v and ξ relative to any fixed local coordinate system x^j in M, this is known as *lowering* or *raising of the indices*, with $\xi_j = g_{jk} v^k$, $v^j = g^{jk} \xi_k$. From now on, we will simply write $\xi = v$. In particular, the differential df of any real-valued function f corresponds in this way, and will be identified with, its *gradient*, which is the tangent vector field ∇f with the component functions (relative to any local coordinates x^j) denoted $\nabla^j f$ or $f^{,j}$; thus

$$f^{,j} \,=\, \nabla^j f \,=\, g^{jk}\,\partial_k f \,=\, g^{jk} f_{,j}\,, \tag{2.9}$$

Also,

$$\nabla x^j \,=\, g^{jk} e_k\,, \tag{2.10}$$

and

$$[g^{jk}] \,=\, g(\nabla x^j,\, \nabla x^k)\,. \tag{2.11}$$

In fact, by (2.9), for any fixed j the components of ∇x^j are $(\nabla x^j)^k = g^{ks}\,\partial_s x^j = g^{ks}\delta^j_s = g^{jk}$. Evaluating $g(\nabla x^j,\, \nabla x^k)$ via (2.10), we now obtain (2.11).

Similarly, we will identify 2-tensors at any $x \in M$ with linear operators $T_xM \to T_xM$. More precisely, a twice-covariant tensor α will be treated, with the aid of g, as an operator $v \mapsto \alpha v$ acting on tangent vectors $v \in T_xM$, with

$$g(\alpha v,\,\cdot\,) \,=\, \alpha(v,\,\cdot\,)\,, \qquad \text{i.e.,}\quad (\alpha v)_j = \alpha_{kj} v^k\,, \tag{2.12}$$

or, in terms of “index-raising”, $(\alpha v)^j = \alpha_k{}^j v^k$ with $\alpha_k{}^j = \alpha_{kl} g^{lj}$.

Our conventions about the tensor and exterior products of 1-forms ξ, η, the exterior derivative of a 1-form ξ and the g-inner product of 2-forms α, β on the underlying manifold M are such that

$$(\xi \otimes \eta)(u,v) \,=\, \xi(u)\eta(v)\,, \qquad (\xi \otimes \eta)_{jk} \,=\, \xi_j \eta_k \tag{2.13}$$

and

$$\xi \wedge \eta \,=\, \xi \otimes \eta \,-\, \eta \otimes \xi\,, \tag{2.14}$$

i.e.,

$$(\xi \wedge \eta)(u,v) \,=\, \xi(u)\eta(v) \,-\, \xi(v)\eta(u)\,, \qquad (\xi \wedge \eta)_{jk} \,=\, \xi_j\eta_k \,-\, \xi_k\eta_j\,, \tag{2.15}$$

$$(d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v])\,, \quad (d\xi)_{jk} \,=\, \partial_j\xi_k \,-\, \partial_k\xi_j\,, \tag{2.16}$$

$$\langle \alpha, \beta \rangle \,=\, -\frac{1}{2}\,\text{Trace}\,\alpha\beta \,=\, \frac{1}{2}\,\alpha_{jk}\beta^{jk}\,, \tag{2.17}$$

for any tangent vectors (vector fields) u, v and any local coordinates x^j. However, for twice-covariant C^∞ tensor fields A, B which are *symmetric* (and so, treated as operators $T_xM \to T_xM$, are self-adjoint), we define a natural inner product using a different convention, namely

$$\langle A, B \rangle \,=\, \text{Trace}\,AB \,=\, A_{jk}B^{jk}\,, \tag{2.18}$$

cf. (3.1). As usual, by *bivectors* at a point $x \in M$ we mean elements of the *bivector space* $[T_xM]^{\wedge 2}$ (the second exterior power of T_xM; for more on bivectors, see the paragraph preceding Lemma 3.7 in §3). Obvious examples are exterior products $v \wedge w$ of vectors; under the index-raising operation for a fixed metric g, bivectors correspond to exterior 2-forms. Thus, from (2.15) and (2.17), we have, for tangent vectors u, v, w, v', w' and bivectors or 2-forms α,

$$\langle \alpha v, w \rangle \,=\, \alpha(v,w)\,, \tag{2.19}$$

$$\langle \alpha, v \wedge v' \rangle \,=\, \langle \alpha v, v' \rangle \,=\, -\langle v, \alpha v' \rangle \,=\, \alpha(v,v')\,, \tag{2.20}$$

$$\langle v \wedge w,\, v' \wedge w' \rangle \,=\, g(v,v')g(w,w') \,-\, g(v,w')g(v',w)\,, \tag{2.21}$$

and, for α, $u \wedge v$ and $v \wedge w$ treated as operators acting in the tangent space,

$$(u \wedge v)w \,=\, \langle u,w \rangle v \,-\, \langle v,w \rangle u\,, \tag{2.22}$$

(by (2.15) and (2.19)); hence, we have the formula

$$\alpha(v \wedge w) \,=\, v \otimes (\alpha w) \,-\, w \otimes (\alpha v)\,. \tag{2.23}$$

for the composite operator and, consequently, $[u \wedge v, \alpha] = v \wedge (\alpha u) - u \wedge (\alpha v)$ for the commutator. Therefore, for bivectors α, β and tangent vectors u, v, (2.20) along with skew-symmetry of α and β yield

$$\langle [u \wedge v, \alpha], \beta \rangle \,=\, \langle [\alpha, \beta]u, v \rangle \tag{2.24}$$

and

$$\langle v \wedge (\alpha u) - u \wedge (\alpha v), \beta \rangle \,=\, \langle [\alpha, \beta]u, v \rangle\,. \tag{2.25}$$

Similarly, from (2.22) and the relation

$$(v \otimes u)w \,=\, g(v,w)u \tag{2.26}$$

(which is immediate from (2.13) and (2.12)), we have the formula

$$(u \wedge v)(u' \wedge v') \,=\, [g(v,u')v' - g(v,v')u'] \otimes u \,+\, [g(u,v')u' - g(u,u')v'] \otimes v \tag{2.27}$$

for the composite of $u \wedge v$ and $u' \wedge v'$ (treated as operators $T_xM \to T_xM$, with $u, v, u', v' \in T_xM$), and so their commutator is given by

$$\begin{aligned}[u \wedge v,\, u' \wedge v'] \,=\, & g(u,u')\, v \wedge v' \,+\, g(v,v')\, u \wedge u' \\ & -\, g(u,v')\, v \wedge u' \,-\, g(u',v)\, u \wedge v'\,,\end{aligned} \tag{2.28}$$

since $u \wedge v = u \otimes v - v \otimes u$ (see (2.14)).

Let $F : M \to N$ be a C^1 mapping between manifolds. Given a twice-covariant tensor field h on N, by the *pullback* of h under F we mean the twice-covariant tensor field F^*h on M such that, for all $x \in M$ and $v, w \in T_xM$, $[F^*h]_x(v,w) = h_y(Av, Aw)$ with $x = F(y)$ and $A = dF_y$, i.e.,

$$[F^*h]_x(v,w) \,=\, h_{F(x)}(dF_xv, dF_xw)\,. \tag{2.29}$$

In local coordinates y^a for N and x^j for M, this can be expressed as the component formula

$$[F^*h]_{jk} \,=\, h_{\lambda\mu}(F)(\partial_jF^\lambda)\partial_kF^\mu\,. \tag{2.30}$$

If, in addition, g and h are pseudo-Riemannian metrics on M and N, respectively, we will say that a mapping $F : M \to N$ is an *isometry* between the pseudo-Riemannian manifolds (M,g) and (N,h) if it is a C^1 diffeomorphism with $F^*h = g$. One also says that a given pseudo-Riemannian manifold (M,g) is *locally homogeneous* if for any two points $x, y \in M$ there exists an isometry of a neighborhood of x onto a neighborhood of y which sends x to y.

Remark 2.1. Given a pseudo-Riemannian manifold (M,g), let $\sim$ be the binary relation in M with $x \sim y$ if and only if there is an isometry of a neighborhood of x onto a neighborhood of y sending x to y. Clearly, $\sim$ is an equivalence relation, and local homogeneity of (M,g) means that $\sim$ has only one equivalence class.

Lemma 2.2. *Let a C^k vector-bundle morphism $F : \mathcal{E} \to \mathcal{H}$ between C^∞ real or complex vector bundles $\mathcal{E}$ and $\mathcal{H}$ over a manifold M, with $0 \le k \le \infty$, be of* constant rank *in the sense that $\dim[F(\mathcal{E}_x)]$ is the same for all $x \in M$. Then both the image $F(\mathcal{E})$ and the kernel $\operatorname{Ker} F$ of F are C^k vector subbundles of $\mathcal{E}$ and $\mathcal{H}$, respectively.*

Proof. Let q, p and r denote the fibre dimensions of $\mathcal{E}$ and $\mathcal{H}$, respectively, and the rank of F (that is, the fibre dimension of $F(\mathcal{E})$). What needs to be shown is C^k regularity of $F(\mathcal{E})$ and $\operatorname{Ker} F$, i.e., the existence, in a neighborhood of any given point $x \in M$, of local trivializing sections e_a for $F(\mathcal{E})$, $a = 1, \ldots, r$, and e_λ of $\operatorname{Ker} F$, $\lambda = r+1, \ldots, q$, which are of class C^k when regarded as sections of the ambient bundle $\mathcal{H}$ or $\mathcal{E}$. To obtain the e_a, it suffices to fix C^∞ local sections $\psi_1, \ldots, \psi_q$ of $\mathcal{E}$ which trivialize $\mathcal{E}$ in a neighborhood of x and rearrange their order so that the images $e_a = F(\psi_a)$, $a = 1, \ldots, r$, are linearly independent at x. Then, in some neighborhood of x, $F(\psi_\lambda)$, for each $\lambda = r+1, \ldots, q$, must be equal to a combination $h_\lambda^a F(\psi_a)$ of the $e_a = F(\psi_a)$ (summation over $a = 1, \ldots, r$), with some C^k coefficient functions h_λ^a. The required local C^k sections e_λ of $\mathcal{E}$ trivializing $\operatorname{Ker} F$ in a neighborhood of x now can be defined by $e_\lambda = \psi_\lambda - h_\lambda^a\psi_a$. ■

As usual, by the *flow* of a C^1 vector field w on a manifold M is the mapping

$$(t,x) \,\mapsto\, e^{tw}x \in M\,, \tag{2.31}$$

defined on an open subset of $\mathbf{R} \times M$, and characterized by the requirement that, for each $x \in M$, $t \mapsto x(t) = e^{tw}x$ be the unique C^1 solution, defined on the largest possible interval in $\mathbf{R}$ containing 0, to the initial value problem $\dot{x}(t) = w(x(t))$, $x(0) = x$. (More generally, C^1 curves $t \mapsto x(t) \in M$ with $\dot{x}(t) = w(x(t))$ for all

t are called *integral curves* of w.) Note that the flow of w has the *homomorphic property*

$$e^{tw}e^{sw}x = e^{(t+s)w}x \tag{2.32}$$

valid for those t, s, x for which it makes sense. In fact, $t \mapsto y(t) = e^{(t+s)w}x$ is an integral curve of w with $y(0) = e^{sw}x$, and so (2.32) follows from the uniqueness-of-solutions theorem for ordinary differential equations.

Remark 2.3. In §17 we will need the fact that every C^1 vector field w on a manifold M, which vanishes outside a compact set Ω, is *complete* in the sense that its flow (2.31) is defined on all of $\mathbf{R} \times M$. In fact, all integral curves $t \mapsto e^{tw}x$ can be defined on a common interval $[-a, a]$ for some $a > 0$; the existence of such a for all $x \in \Omega$ is immediate from compactness of Ω, while for $x \notin \Omega$ we may set $e^{tw}x = x$ for all t. Piecing together integral curves defined on intervals of length $2a$ we thus obtain integral curves defined on all of $\mathbf{R}$. Completeness of w now gives rise to the assignment

$$\mathbf{R} \ni t \mapsto e^{tw} \in \mathrm{Diff}\,(M) \tag{2.33}$$

valued in the group of all C^∞ diffeomorphisms of M which, in view of (2.32), is a group homomorphism.

Remark 2.4. As suggected by the notation, expression $e^{tw}x$ in (2.31) depends only on x and tw (rather than t and w individually). In facts, for any nonzero real number c, the assignment $t \mapsto y(t) = e^{(t/c)(cw)}x$ is easily seen to be an integral curve of w with $y(0) = x$, and so $y(t) = e^{tw}x$ in view of the uniqueness-of-solutions theorem. Given a C^1 vector field w on a manifold M, we may therefore use the symbol $e^w : U \to M$, which stands for the mapping e^{tw} with $t = 1$; its domain U is a (possibly empty) open subset of M formed by all x for which $(1, x)$ is in the domain of the flow (2.31).

Lemma 2.5. *Let w be a C^∞ vector field on a manifold M of dimension $n \geq 1$ and let $z \in M$. Then, for any codimension-one submanifold $N \subset M$ equipped with the subset topology, containing z, and transverse to w at z in the sense that $w(z) \notin T_zN$, and for any real number q, there exists a coordinate system x^j, $j = 1, \ldots, n$ with a domain U containing z such that*

(i) *w coincides on U with the coordinate field e_1 in the direction of x^1,*
(ii) *$N \cap U$ is precisely the subset of U given by the equation $x^1 = q$, and*
(iii) *The image of U under the coordinate mapping $(x^1, \ldots, x^n)$ is a convex set in $\mathbf{R}^n$.*

Furthermore, the functions $x^2, \ldots, x^n$ may be chosen so as to coincide, on a neighborhood of z in N, with any prescribed coordinate system for N at z.

Proof. In view of the inverse mapping theorem, there exist neighborhoods I of c in $\mathbf{R}$ and N' of z in N such that the formula $F(t, y) = e^{(t-q)w}y$ (notation of (2.31)) defines a C^∞ diffeomorphism $F : I \times N' \to U$ onto a neighborhood U of z in M with $N' = N \cap U$, while N' itself is the domain of a coordinate system y^a, $a = 2, \ldots, n$, which identifies N' with a convex set in $\mathbf{R}^{n-1}$. (The y^a thus may be completely arbitrary, as long as we then replace N' by a suitable smaller neighborhood of z in N.) The required coordinates x^j now may be defined so as to assign to any $x \in U$ the n-tuple $(x^1, x^1, \ldots, x^{n-1})$ with $x^1 = t$ and $x^a = y^a$, $a = 2, \ldots, n$, where $F^{-1}(x) = (t, y)$ and y^a stand for the y^a-components of y. Assertions (ii),(ii) now are obvious, and (i) follows from the fact that the assignment $t \mapsto x(t)$ with $x^1(t) = t$ and with any $x^a = x^a(t)$ that are constant (i.e., independent of t), is an integral curve of w. This completes the proof. ■

Lemma 2.6. *Given a pseudo-Riemannian manifold (M, g) of dimension $n \geq 1$, let t be a C^∞ function on M with $\langle \nabla t, \nabla t \rangle \neq 0$ everywhere in M, where ∇t is the g-gradient of t and $\langle\,,\rangle$ stands for g. Then, for any point $z \in M$, there exists a coordinate system x^j, $j = 1, \ldots, n$, defined on a neighborhood U of z, such that the components g_{jk} of g relative to the x^j satisfy*

$$g_{11} = \langle \nabla t, \nabla t \rangle, \qquad g_{1a} = 0 \quad \text{for} \quad a = 2, \ldots, n, \tag{2.34}$$

everywhere in U and, letting w stand for the coordinate vector field e_1 in the direction of x^1, we have, on U,

$$t = x^1, \qquad \nabla t = \frac{w}{\langle w, w \rangle}, \tag{2.35}$$

and

$$w = \frac{\nabla t}{\langle \nabla t, \nabla t \rangle}. \tag{2.36}$$

Furthermore, denoting N the codimension-one submanifold defined by $N = t^{-1}(q)$ with $q = t(z)$, we may choose the functions $x^2, \ldots, x^n$ so as to coincide, near z in N, with any given local coordinate system for N at z.

Proof. Defining the vector field w by (2.36), we clearly obtain the second equality in (2.35). Choosing the coordinates x^j, $j = 1, \ldots, n$, as in Lemma 2.5 for our w and z and for the codimension-one submanifold $N = t^{-1}(q)$, $q = t(z)$, we now have $w = e_1$. Hence, by (2.36), $g_{11} = \langle w, w \rangle \langle \nabla t, \nabla t \rangle$. On the other hand, by Lemma 2.5(ii), the intersection of N with the coordinate domain U is given by $x^1 = q$. The coordinate vector fields e_a, $a = 2, \ldots, n$, thus are tangent to $N \cap U$ along $N \cap U$ and, since t is constant on N, (2.36) implies that w is normal to $N \cap U$ along $N \cap U$, that is, $g_{1a} = 0$. Finally, $\partial_1 t = d_w t = \langle w, \nabla t \rangle = 1$ and so, by (2.35), $\partial_1 t = 1$. Hence, for $a = 2, \ldots, n$, $\partial_1 \partial_a t = \partial_a \partial_1 t = 0$, i.e., $\partial_a t$ is constant in the direction of x^1. Since t is constant on N, we have $\partial_a t = 0$ wherever $x^1 = q$ (that is, along $N \cap U$), so that $\partial_a t = 0$ everywhere in U. As $t = x^1 = q$ on $N \cap U$, these relations show that $t = x^1$ on U. This completes the proof. ■

§3. Some linear algebra

Most arguments appearing in this text are algebraic in nature. Some facts from linear algebra are, however, invoked much more often than others, and it is convenient to have those facts gathered in one place for easy reference. This section serves such a purpose.

Lemma 3.1. *Any real- or complex-valued function f of three variables u, v, w which is skew-symmetric in u, v and symmetric in v, w, must be identically equal to zero.*

In fact, writing, for simplicity, $f(u,v,w) = uvw$, we then have $uvw = -\,vuw = -\,vwu = wvu = wuv = -\,uwv = -\,uvw$. ■

For a quick reference, let us note here that

$$\text{Trace}\,AB \;=\; \text{Trace}\,BA\,. \tag{3.1}$$

whenever $A, B : V \to V$ are linear operators in a finite-dimensional real or complex vector space V.

Remark 3.2. A linear operator $F : V \to V$ in a real or complex vector space V is called an *involution* if $F^2 = \text{Id}$. This is the case if and only if V admits a direct-sum decomposition

$$V \;=\; V_+ \oplus V_- \tag{3.2}$$

such that $F = \pm\,\text{Id}$ on $V_\pm$. In other words, for either sign $\pm$,

$$V_\pm \;=\; \text{Ker}\,(F \mp \text{Id})\,, \tag{3.3}$$

so that $V_\pm$ = is the (± 1)-eigenspace of F unless $V_\pm = \{0\}$. The direct-sum projections $\text{pr}^\pm : V \to V_\pm$ then are given by $2\,\text{pr}^\pm = F \pm \text{Id}$.

Lemma 3.3. *Let $F : V \to V$ be a complex-linear operator in a complex vector space V of dimension m with $1 \le m < \infty$.*

(i) *There exists a basis $e_1, \ldots, e_m$ of V in which the matrix $[F_j^k]$ of F, characterized by*

$$Fe_j \;=\; F_j^k e_k \qquad \text{(summed over} \quad k = 1, \ldots, m)\,,$$

is upper triangular *in the sense that $F_j^k = 0$ whenever $k > j$.*

(ii) *The complex and real traces of F, obtained by treating it as a complex-linear or, respectively, real-linear operator, are related by*

$$\text{Trace}_{\mathbf{R}} F \;=\; 2\,\text{Re}\,[\text{Trace}_{\mathbf{C}} F]\,. \tag{3.4}$$

Proof. (i): Induction on m. Assuming the assertion to be true in dimension $m-1$, pick an eigenvector $e_1 \in V$ of F and select $e_2, \ldots, e_m$ so that their cosets in the quotient space $V' = V/\mathbf{C}e_1$ make the operator $F' : V' \to V'$ that F descends to appear upper triangular. (ii): Evaluate both traces using any fixed (complex) basis $e_1, \ldots, e_m$ of V and, respectively, the real basis

$$e_1\,,\; ie_1\,,\; \ldots\,,\; e_m\,,\; ie_m\,. \tag{3.5}$$

This completes the proof. ■

Given a finite-dimensional vector space V over the field $\mathbf{K}$ of real or complex numbers, let $\mathcal{B}(V)$ stand for the set of all bases of V, and let $\text{GL}(V)$ be the group of all $\mathbf{K}$-linear isomorphisms $V \to V$. When $V = \mathbf{K}^n$, the group $\text{GL}(V)$ is also denoted $\text{GL}(n, \mathbf{K})$, and consists of all invertible $n \times n$ matrices with entries in $\mathbf{K}$. Either of the groups $\text{GL}(V)$ and $\text{GL}(n, \mathbf{K})$, $n = \dim V$, then has a natural simply transitive actions on the set $\mathcal{B}(V)$: The former via the transformations $(e_1,\dots,e_n)\mapsto(Fe_1,\dots,Fe_n)$, with $F\in\mathrm{GL}(V)$ and $(e_1,\dots,e_n)\in\mathcal{B}(V)$, the latter by matrix multiplication from the right, applied to bases now treated as row matrices $[e_1,\dots,e_n]$. Fixing any basis, we thus obtain a diffeomorphic identification $\mathcal{B}(V)\approx\mathrm{GL}(V)$. Note that both sets involved amay be treated as manifolds, since they are open subsets of vector spaces; specifically,

$$\mathrm{GL}(V)\subset\mathfrak{gl}(V)\,,\qquad \mathcal{B}(V)\subset V^n\,,\tag{3.6}$$

where $\mathfrak{gl}(V)=\mathrm{Hom}\,(V,V)$ is the space of all $\mathbf{K}$-linear operators $V\to V$, and V^n is the nth Cartesian power of $\mathcal{V}$, $n=\dim V$. In the case where $\mathbf{K}=\mathbf{R}$, the group $\mathrm{GL}(V)$ contains the open subgroup $\mathrm{GL}^+(V)$ of all $\mathbf{R}$-linear isomorphisms $F:V\to V$ which are *orientation-preserving* in the sense that $\det F>0$. When $V=\mathbf{R}^n$, the corresponding matrix group is denoted $\mathrm{GL}^+(n,\mathbf{R})$.

Lemma 3.4. *For any complex vector space V with $\dim V<\infty$, the sets $\mathrm{GL}(V)$ and $\mathcal{B}(V)$ are connected.*

Proof. Let us set $m=\dim V$ and fix $A\in\mathrm{GL}(V)$. According to Lemma 3.3(i), we can find a basis $e_1,\dots,e_m$ of V which makes A upper triangular, that is, $Ae_j=\sum_k a_{jk}e_k$ for $j=1,\dots,m$, with $a_{jk}=0$ whenever $k>j$. As $\det A=a_{11}\dots a_{mm}\neq 0$, we then also have $a_{jj}=e^{z(j)}$, $j=1,\dots,m$, with some $z(j)\in\mathbf{C}$. A continuous curve $[0,1]\ni t\mapsto A_t\in\mathrm{GL}(V)$ joining Id to A now can be defined by $A_te_j=\sum_k a_{jk}(t)e_k$, $j=1,\dots,m$, with $a_{jk}(t)=ta_{jk}$ when $j\neq k$ and $a_{jj}(t)=e^{tz(j)}$. Thus, $\mathrm{GL}(V)$ is connected, and our assertion follows since $\mathcal{B}(V)\approx\mathrm{GL}(V)$. ■

Lemma 3.5. *Let there be given a real vector space V with $1\le n=\dim V<\infty$, and a positive-definite inner product $\langle\,,\rangle$ in V.*

- (i) *The group $\mathrm{GL}^+(V)$ is connected, while $\mathrm{GL}(V)$ has two connected components.*
- (ii) *The set $\mathcal{B}(V)$ of all bases of V, and the set of all $\langle\,,\rangle$-orthonormal bases of V, each have two connected components.*

Proof. Since the standard orthonormalization process can be treated as a continuous deformation, all we need to show is that two orthonormal bases with a positive transition determinant can be joined by a continuous curve of orthonormal bases. Such a curve may consist of segments each of which brings successive vectors of one basis in agreement with the other basis, by employing a continuous rotation in a plane containing the two vectors, complemented by the identity transformation in the orthogonal complement of the plane. ■

For V as in Lemma 3.5, the two connected components of $\mathcal{B}(V)$ are called *orientations* of V. When one orientation is chosen, V is said to be *oriented.* Bases belonging to that fixed orientation then are referred to as *positive-oriented.*

Remark 3.6. A real vector space V with $\dim V=n$, $1\le n<\infty$, becomes naturally oriented if one chooses any fixed connected set of bases of V. (In fact, they all represent the same orientation of V.) As an example, the underlying real space of an m-dimensional complex vector space V $(1\le m<\infty)$ has a *natural orientation* determined in this way by the set of all real bases (3.5) obtained using complex bases $e_1, \ldots, e_m$ of V. In fact, according to Lemma 3.4, such real bases form a connected set.

By a *p-vector* in a finite-dimensional real or complex vector space V we will mean, as usual, any element of the pth exterior power $V^{\wedge p}$. When $p = 2$, the term *bivector* is also used. One of many equivalent definitions of $V^{\wedge p}$ is

$$V^{\wedge p} = L_{\mathrm{skew}}(V^*, \ldots, V^*; \mathbf{K}) \tag{3.7}$$

(with V^* in parentheses repeated p times), that is, the space of all skew-symmetric p-linear mappings $V^* \times \ldots \times V^* \to \mathbf{K}$, where $\mathbf{K}$ is the scalar field ($\mathbf{R}$ or $\mathbf{C}$), and V^* stands for the dual space of V. There is a natural p-linear skew-symmetric mapping $V \times \ldots \times V \to V^{\wedge p}$ called *exterior multiplication* and denoted

$$(v_1, \ldots, v_p) \mapsto v_1 \wedge \ldots \wedge v_p\,, \tag{3.8}$$

which, for $V^{\wedge p}$ as in (3.7), is defined by $(v_1 \wedge \ldots \wedge v_p)(\xi^1, \ldots, \xi^p) = \det \mathfrak{M}$ for $\xi^1, \ldots, \xi^p \in V^*$, $\mathfrak{M}$ being the $p \times p$ matrix $[\xi^j(v_k)]$. For any basis $e_1, \ldots, e_n$ of V, $n = \dim V$, pne easily verifies that the set of exterior products

$$\{e_{j_1} \wedge \ldots \wedge e_{j_p} : 1 \le j_1 < \ldots < j_p \le n\} \tag{3.9}$$

forms a basis of $V^{\wedge p}$. Therefore, the exterior product $v_1 \wedge \ldots \wedge v_p \in V^{\wedge p}$ is nonzero if and only if the vectors $v_1, \ldots, v_p \in V$ are linearly independent. (To see this, complete $v_1, \ldots, v_p$ to a basis of V.) A nonzero p-vector α which is *decomposable* (i.e., $\alpha = v_1 \wedge \ldots \wedge v_p$ for some $v_1, \ldots, v_p \in V$) uniquely determines the p-dimensional subspace of V spanned by $v_1, \ldots, v_p$; for instance,

$$\mathrm{Span}\,\{v_1, \ldots, v_p\} = \{v \in V : v \wedge (v_1 \wedge \ldots \wedge v_p) = 0\}\,. \tag{3.10}$$

as one sees, again, by completing $v_1, \ldots, v_p$ to a basis $v_1, \ldots, v_n$ of V and using the corresponding basis of type (3.9) for $V^{\wedge p}$. The exterior multiplication of vectors has an extension to a bilinear pairing of p-vectors β and p'-vectors β' for any p, p', sending them to a $(p+p')$-vector $\beta \wedge \beta'$, and uniquely determined by the requirement of associativity.

Lemma 3.7. *Given a bivector $\alpha \in V^{\wedge 2}$ in a finite-dimensional real or complex vector space V,*

(a) *A nonzero vector $v \in V$ satisfies $v \wedge \alpha = 0$ if and only if $\alpha = v \wedge w$ for some vector $v \in V$.*
(b) *Condition $\alpha \wedge \alpha = 0$ holds if and only if α is decomposable, that is, $\alpha = v \wedge w$ for some vectors $v, w \in V$.*

Proof. (a) is obvious from (3.9) for a basis $e_1, \ldots, e_n$ of V, $n = \dim V$, with $e_1 = v$. The 'if' assertion in (b) is also immediate. Finally, it is easy to verify that, for a vector $v \in V$, conditions $v \wedge \alpha = 0$ and $\alpha \wedge \alpha = 0$ are, respectively, equivalent to $v^k \alpha^{lm} + v^l \alpha^{mk} + v^m \alpha^{kl} = 0$ and $\alpha^{jk} \alpha^{lm} + \alpha^{jl} \alpha^{mk} + \alpha^{jm} \alpha^{kl} = 0$ for all indices $j, k, l, m = 1, \ldots, n$, where $e_1, \ldots, e_n$ is any fixed basis of V and $v = v^j e_j$, $\alpha = \alpha^{jk} e_j \wedge e_k$ are the expansions of v and α, with the coefficients of the latter made unique by the skew-symmetry requirement $\alpha^{jk} = -\alpha^{kj}$. To prove the 'only if' part of (b), let us suppose that $\alpha \wedge \alpha = 0$. Clearly, we may assume that $\alpha \neq 0$, and so $\alpha^{jk} \neq 0$ for some fixed j and some k. Defining the vector v by $v^k = \alpha^{jk}$ for all k (with this fixed j), we thus have $v \neq 0$ and $v^k\alpha^{lm} + v^l\alpha^{mk} + v^m\alpha^{kl} = 0$, i.e, $v \wedge \alpha = 0$. Our assertion now follows from (a), which completes the proof. ■

Let ω be a p-linear skew-symmetric mapping $V \times \ldots \times V \to W$ for some real or complex vector spaces V and W. Then

$$\omega(u_1, \ldots, u_p) = \det \mathfrak{B} \cdot \omega(v_1, \ldots, v_p) \tag{3.11}$$

whenever $v_1, \ldots, v_p \in V$ and $u_1, \ldots, u_p \in V$ are vectors such that each v_j, $j = 1, \ldots, p$, is a combination of the v_j, with the coefficient matrix $\mathfrak{B} = [B_j^k]$, so that $u_j = B_j^k v_k$, $j, k \in \{1, \ldots, p\}$. This is clear since, denoting $\varepsilon_{j_1 \ldots j_p}$ is the *Ricci symbol* (equal to the signum of the permutation $(j_1, \ldots, j_p)$, if the $j_1, \ldots, j_p$ are all distinct, and to 0, if they are not), we have

$$\begin{aligned} \omega(B_1^{j_1} v_{j_1}, \ldots, B_p^{j_p} v_{j_p}) \,&=\, B_1^{j_1} \ldots B_p^{j_p}\, \omega(v_{j_1}, \ldots, v_{j_p}) \\ &=\, \varepsilon_{j_1 \ldots j_p} B_1^{j_1} \ldots B_p^{j_p}\, \omega(v_1, \ldots, v_p)\,. \end{aligned} \tag{3.12}$$

On he other hand, $\varepsilon_{j_1 \ldots j_p} B_1^{j_1} \ldots B_p^{j_p} = \det\,[B_j^k]$.

Remark 3.8. An n-dimensional real or complex vector space V admits an n-linear skew-symmetric function $\omega : V \times \ldots \times V \to \mathbf{K}$, where $\mathbf{K}$ is the scalar field ($\mathbf{R}$ or $\mathbf{C}$), and such a function is unique up to a scalar factor. In fact, by (3.11), such ω is uniquely determined by the value $\omega(e_1, \ldots, e_n)$ for any fixed basis $e_1, \ldots, e_n$ of V.

Given a finite-dimensional real or complex vector space V, let us fix a basis e_j of V, $j = 1, \ldots, n$ $(n = \dim V)$ and an n-linear skew-symmetric mapping $\omega : V \times \ldots \times V \to V'$, valued in a vector space V'. For any linear operator $F : V \to V$ we then have

$$\begin{aligned} (\det F) \cdot \omega(e_1, \ldots, e_n) \,&=\, \omega(Fe_1, \ldots, Fe_n)\,, \\ (\mathrm{Trace}\, F) \cdot \omega(e_1, \ldots, e_n) \,&=\, \sum_{j=1}^{n} \omega(e_1, \ldots, e_{j-1}, Fe_j, e_{j+1}, \ldots, e_n)\,, \end{aligned} \tag{3.13}$$

as one sees using (3.11) for the matrix $\mathfrak{B} = [F_j^k]$ with $Fe_j = F_j^k e_k$. Therefore, for any C^1 curve $t \mapsto F = F(t) \in \mathrm{GL}\, V \subset \mathrm{Hom}\,(V, V)$ of isomorphisms $V \to V$, we have

$$(\det F)^{\boldsymbol{\cdot}} = \det F \cdot \mathrm{Trace}\,(F^{-1}\dot{F}) \tag{3.14}$$

with $(\;)^{\boldsymbol{\cdot}} = d/dt$, that is, for all t, $d\,[\det F(t)]/dt = [\det F(t)] \cdot \mathrm{Trace}\,[F^{-1}(t) \circ \dot{F}(t)]$. To see this, fix $\omega \neq 0$ as above with $V' = \mathbf{R}$ (or $V' = \mathbf{C}$) and apply d/dt to (3.13): $(\det F)^{\boldsymbol{\cdot}} \cdot \omega(e_1, \ldots, e_n) = [\omega(Fe_1, \ldots, Fe_n)]^{\boldsymbol{\cdot}} = \sum_\alpha \omega(Fe_1, \ldots, \dot{F}e_\alpha, \ldots, Fe_n) = \sum_\alpha \omega(Fe_1, \ldots, FF^{-1}\dot{F}e_\alpha, \ldots, Fe_n) = (\det F) \cdot \sum_\alpha \omega(e_1, \ldots, F^{-1}\dot{F}e_\alpha, \ldots, e_n) = \det F \cdot (\mathrm{Trace}\, F^{-1}\dot{F})\, \omega(e_1, \ldots, e_n)$. (Note that the first relation in (3.13) remains valid whether or not the $e_1, \ldots, e_n$ are linearly independent; if they are not, both sides must equal zero.)

Remark 3.9. We will repeatedly use the obvious fact that a complex vector space V is nothing else than its underlying real vector space along with the operator $J : V \to V$ of complex multiplication by i. The operator J is often called a *complex structure* in the real space V, and is subject to the single condition

$$J^2 = -\mathrm{Id}\,. \tag{3.15}$$

Remark 3.10. Let V now be a real or complex vector space, and let $\mathbf{K}$ stand for the scalar field ($\mathbf{R}$ or $\mathbf{C}$). Suppose that $\langle\,,\rangle$ is a function $V \times V \to \mathbf{K}$ such that one of the following three cases occurs:

(a) $\mathbf{K} = \mathbf{R}$, $\langle\,,\rangle$ is real-bilinear and symmetric;
(b) $\mathbf{K} = \mathbf{C}$, $\langle\,,\rangle$ is sesquilinear and Hermitian;
(c) $\mathbf{K} = \mathbf{C}$, $\langle\,,\rangle$ is complex-bilinear and symmetric.

Two vectors $v, w \in V$ then are called *orthogonal* if $\langle v, w\rangle = 0$. An *orthonormal system* in V is an ordered q-tuple $v_1, \ldots, v_q$ of vectors in V, for any integer $q \ge 0$, such that $\langle v_a, v_b\rangle = 0$ for all $a, b = 1, \ldots, q$ with $a \ne b$, while, for $a = 1, \ldots, q$, $\langle e_a, e_a\rangle = \varepsilon_a$ with some numbers ε_a such that $\varepsilon_a \in \{1, -1\}$ in cases (a), (b), and $\varepsilon_a = 1$ in case (c). An *orthonormal basis* of V is any orthonormal system in V which also happens to be a basis of V. We also define the *orthogonal complement* of any set $\mathcal{K} \subset V$ to be the vector subspace

$$\mathcal{K}^\perp = \{v \in V : \langle v, x\rangle = 0 \text{ for all } x \in \mathcal{K}\}\,. \tag{3.16}$$

When $\mathcal{K} = \{v\}$, we use he simplified symbol $v^\perp$ rather than $\{v\}^\perp$.

The remainder of this section is devoted to vector spaces endowed with three possible kinds of *inner products*. Specifically, let V and $\langle\,,\rangle$ be as in Remark 3.10 and, in addition, let $\dim V < \infty$. We then will call $\langle\,,\rangle$ an *inner product* in V if, besides having property (a), (b) or (c) of Remark 3.10, it is also *nondegenerate* in the sense that

$$V^\perp = \{0\}\,, \tag{3.17}$$

i.e., if no nonzero vector is *orthogonal* to all of V, that is, for each $v \in V$ with $v \ne 0$ there exists $w \in V$ with $\langle v, w\rangle \ne 0$. Thus, another requirement obviously equivalent to (3.17) is that the operator

$$V \ni v \mapsto \langle\,\cdot\,, v\rangle \in V^* \tag{3.18}$$

be an (anti)linear isomorphism. A further condition equivalent to nondegeneracy of $\langle v, w\rangle$ obviously is

$$\det[\langle e_j, e_k\rangle] \ne 0 \tag{3.19}$$

for some, or any, basis $e_1, \ldots, e_n$ of V, with $n = \dim V$. A vector space carrying a fixed inner product will be referred to as an *inner-product space*, and its inner product (unless specified otherwise) will be represented by the generic symbol $\langle\,,\rangle$.

Remark 3.11. A form $\langle\,,\rangle$ in a finite-dimensional complex vector space V, having property (b) or (c) of Remark 3.10, is nondegenerate if and only if so is its real part, treated as a bilinear form in the underlying real space. In fact, a vector $v \in V$ with $\mathrm{Re}\,\langle v, x\rangle = 0$ for all $x \in V$ must have $\langle v, x\rangle = 0$ for all x, as one sees considering x and ix.

Given an inner product $\langle\,,\rangle$ in V, we have

$$\dim V \,=\, \dim \mathcal{T} \,+\, \dim \mathcal{T}^\perp \tag{3.20}$$

for any vector subspace $\mathcal{T} \subset V$. In fact, $\mathcal{T}^\perp$ is the kernel of the composite of the (anti)isomorphism (3.18) with the (obviously surjective) restriction operator $V^* \to \mathcal{T}^*$. Consequently, for dimensional reasons,

$$\mathcal{T}^{\perp\perp} \,=\, \mathcal{T}\,, \tag{3.21}$$

since the inclusion $\mathcal{T} \subset \mathcal{T}^{\perp\perp}$ is obvious. A vector subspace $\mathcal{T} \subset V$ is called *nondegenerate* if $\langle\,,\rangle$ restricted to $\mathcal{T}$ is nondegenerate, that is, constitutes an inner product in $\mathcal{T}$. Clearly, a subspace $\mathcal{T}$ of V is nondegenerate if and only if

$$\mathcal{T} \cap \mathcal{T}^\perp \,=\, \{0\}\,. \tag{3.22}$$

Thus, by (3.21),

$$\mathcal{T} \quad \text{is nondegenerate if and only if} \quad \mathcal{T}^\perp \quad \text{is.} \tag{3.23}$$

Another condition equivalent to nondegeneracy of $\mathcal{T}$ is

$$\mathrm{Span}\,(\mathcal{T} \cup \mathcal{T}^\perp) \,=\, V\,. \tag{3.24}$$

In fact, this is immediate, for dimensional reasons, from (3.11) and (3.22). Hence, for any nondegenerate subspace $\mathcal{T}$ of V,

$$V \,=\, \mathcal{T} \oplus \mathcal{T}^\perp\,. \tag{3.25}$$

A vector subspace $\mathcal{W}$ of an inner-product space V is called *null* if

$$\mathcal{W} \subset \mathcal{W}^\perp\,, \tag{3.26}$$

that is, if $\langle\,,\rangle$ restricted to $\mathcal{W}$ is identically zero. An equivalent condition is the existence of a basis e_a of $\mathcal{W}$, $a = 1,\dots,q$, $q = \dim \mathcal{W}$ which is *null and orthogonal*, that is, $\langle e_a, e_b\rangle = 0$ for all $a, b = 1,\dots,q$.

Remark 3.12. Suppose that we are given real/complex vector spaces V, V' and a mapping $B : V \times V \to V'$ which is real-bilinear and symmetric (V, V' real) or complex-bilinear and symmetric (V, V' complex) or, finally, sesquilinear and Hermitian (V complex, $V' = \mathbf{C}$). It is well-known (and easy to verify) that B then is uniquely determined by its *quadratic function* $V \ni x \mapsto B(x,x) \in V'$. Thus, a subspace $\mathcal{W}$ of an inner-product space V is null if and only if all its elements $w \in \mathcal{W}$ are *null vectors* in the sense that $\langle w, w\rangle = 0$. Another consequence is that a mapping $B : V \times V \to V'$ which is bilinear (or, sesquilinear with $V' = \mathbf{C}$) is skew-symmetric (or, skew-Hermitian) if and only if $B(x,x) = 0$ (or, respectively, $B(x,x) \in i\mathbf{R}$) for all $x \in V$.

In the real-bilinear and complex-sesquilinear cases, the inner product $\langle\,,\rangle$ will be referred to *pseudo-Euclidean* or, respectively, *pseudo-Hermitian.* If, in addition, $\langle\,,\rangle$ happens to be positive definite, we will call it *Euclidean* or, respectively, (positive-definite) *Hermitian.*

A form $\langle\,,\rangle$ satisfying (a), (b) or (c) of Remark 3.10 in a vector space V with $\dim V < \infty$ is nondegenerate, i.e., constitutes an inner product in V, if and only if it admits an orthonormal basis. The 'if' part is clear from (3.19). As for the 'only if' assertion, it follows if one assumes that $V \neq \{0\}$ and proceeds to select such a basis by first choosing e_1 with $\langle e_j, e_j\rangle = \pm 1$ (which is possible according to Remark 3.12), and then using induction on $n = \dim V$ along with (3.23) and the decomposition (3.25) for $\mathcal{T} = \mathbf{K}e_1$. Similarly one sees that every orthonormal system in an inner-product space can be extended to an orthonormal basis.

Remark 3.13. Let our inner-product space V now be pseudo-Euclidean or pseudo-Hermitian. For a fixed orthonormal basis $e_1, \ldots, e_n$ of V, $n = \dim V$, let q^- and q^+ be the number of minuses and , respectively, of pluses among the signs of $\langle e_j, e_j\rangle$, $j = 1, \ldots, n$. Then q^- (or, respectively, q^+) is the maximum dimension of a vector subspace of V on which the inner product is negative semidefinite (or, respectively, positive semidefinite). In fact, any subspace $\mathcal{T}$ with $\pm\langle x, x\rangle \leq 0$ for all $x \in \mathcal{T}$ satisfies $\mathcal{T} \cap \mathcal{T}^{\pm} = \{0\}$, and so $\dim \mathcal{T} \leq \dim V - q^{\pm} = q^{\mp}$, as required. $\dim \mathcal{T} + q^+ \leq \dim V = q^- + q^+$. Thus, q^- and q^+ are algebraic invariants of the inner-product space V. The pair (q^-, q^+) will be referred to as the *sign pattern* of the inner product of V and often written in the form $- \ldots - + \ldots +$ (with q^- minuses, q^+ pluses).

For any pseudo-Euclidean or pseudo-Hermitian inner-product space V with the sign pattern (q^-, q^+),

$$\min(q^-, q^+) \text{ is the maximum dimension of a null subspace of } V. \tag{3.27}$$

In fact, $\dim \mathcal{W} \leq q^{\pm}$ for any null subspace $\mathcal{W}$ (Remark 3.13). On the other hand, given an orthonormal basis $v_1, \ldots, v_{q^-}, w_1, \ldots, w_{q^+}$ of V with $\langle v_a, v_a\rangle = -1$ for $a = 1, \ldots, q^-$ and $\langle w_\lambda, w_\lambda\rangle = 1$ for $\lambda = 1, \ldots, q^+$, the vectors $v_j + w_j$ with $1 \leq j \leq \min(q^-, q^+)$ obviously span a null subspace of dimension $\min(q^-, q^+)$.

Lemma 3.14. *Let $u_1, \ldots, u_r$ be mutually orthogonal, linearly independent null vectors in an n-dimensional inner-product space V, $0 \leq r \leq n/2$. Then V admits a basis of the form*

$$u_1, \ldots, u_r, v_1, \ldots, v_r, w_1, \ldots, w_{n-2r}\,,$$

where $n = \dim V$, such that $\langle u_a, v_a\rangle = \langle v_a, u_a\rangle = 1$ for all $a \in \{1, \ldots, r\}$, while $\langle w_\lambda, w_\lambda\rangle \in \{1, -1\}$ for $\lambda = 1, \ldots, n - 2r$, and all other inner products involving vectors of the basis are zero.

Proof. Induction on $r \in \{0, 1, \ldots, n/2\}$. If $r = 0$, this is just the existence of an othonormal basis. Now suppose that $r \geq 1$ and our assertion holds if r is replaced by $r-1$ Completing $u_1, \ldots, u_r$ to a basis of V and then choosing $v \in V$ such that $\langle\,\cdot\,, v\rangle$ is the first element of the dual basis in V^* (which is possible since (3.18) is bijective), we obtain $\langle u_1, v\rangle = 1$ and $\langle u_a, v\rangle = 0$ for $a = 2, \ldots, r$. Thus, $v_1 = v - \langle v, v\rangle\, u_1/2$ satisfies the relations just listed for v and, in addition, $\langle v_1, v_1\rangle = 0$. The subspace $\mathcal{T} = \operatorname{Span}\{u_1, v_1\}$ is nondegenerate (by (3.19)) and our conclusion follows from the inductive assumption applied to the vectors $u_2, \ldots, u_r \in \mathcal{T}^{\perp}$ (cf. (3.25) and (3.23). ■

Lemma 3.15. *Let vectors u, w in an inner-product space V with $n = \dim V \geq 4$ satisfy the conditions $u \neq 0$, $\langle u, u\rangle = 0$ and $\langle w, w\rangle \in \{1, -1\}$. Then*

(a) *There exists $w' \in V$, orthogonal to u and w, with $\langle w', w'\rangle \in \{1, -1\}$.*
(b) *For any w' with the properties listed in* (i)*, there exists $v \in V$, orthogonal to w and w', with $\langle u, v\rangle = 1$ and $\langle v, v\rangle = 0$.*

Proof. The orthogonal complement $P^\perp$ of the plane $P = \mathrm{Span}\,\{u, w\}$ is of dimension $n - 2 \geq n/2$ (see (3.20)) and so it cannot be a null subspace of V. In fact, we have relation (3.27) with $\min(q^-, q^+) \leq n/2$ which, if $P^\perp$ were null, would give $n - 2 = n/2$, that is, $n = 4$. Then (3.26) with $\mathcal{W} = P^\perp$ would imply that, for dimensional reasons, $P^\perp = P^{\perp\perp} = P$ (cf. (3.21)), i.e., P would be null, contradicting the assumption that $\langle w, w\rangle \neq 0$.

According to Remark 3.12, this proves (a). On the other hand, the orthogonal complement $Q^\perp$ of the plane $Q = \mathrm{Span}\,\{w, w'\}$ is nondegenerate by (3.23) and contains $u \neq 0$. Thus, some vector $v \in Q^\perp$ is not orthogonal to u, and may be normalized so that $\langle u, v\rangle = 1$. Assertion (b) now follows if we replace v with $v - \langle v, v\rangle u/2$. This completes the proof. ■

In any inner-product space V one identifies linear operators $F : V \to V$ with bilinear (or, sesqilinear) forms B on V, in such a way that

$$B(v, w) \,=\, \langle Fv, w\rangle \tag{3.28}$$

for all $v, w \in V$. Note that the assignment $F \mapsto B$ with (3.28) is injective due to nondegeneracy of $\langle\,,\rangle$, and so, for dimensional reasons, it is a linear isomorphism. Nondegeneracy of $\langle\,,\rangle$ also guarantees that every linear operator $F : V \to V$ has a unique *adjoint* $F^* : V \to V$ with

$$\langle Fv, w\rangle \,=\, \langle v, F^*w\rangle \tag{3.29}$$

for all $v, w \in V$. Under the identification (3.28), the adjoint $F \mapsto F^*$ corresponds to the operation $B \mapsto B^*$ such that $B^*(v, w)$ equals $B(w, v)$ (B bilinear) or $\overline{B(w, v)}$ (B sesqilinear). One calls $F : V \to V$ *self-adjoint* or *skew-adjoint* if $F^* = F$ (or, respectively, $F^* = -F$). This is clearly the case if and only if the corresponding B is symmetric [Hermitian] or, respectively, skew-symmetric [skew-Hermitian].

The skew-adjoint operators $F : V \to V$ are, explicitly, characterized by

$$\langle Fv, w\rangle \,=\, -\langle v, Fw\rangle \qquad \text{for all} \quad v \in V\,. \tag{3.30}$$

Given a finite-dimensional real or complex vector space V, we denote $\mathfrak{gl}(V) = \mathrm{Hom}\,(V, V)$ the space of all linear operators $V \to V$. The commutator multiplication $[\,,\,]$, with $[A, B] = AB - BA$, obviously turns $\mathfrak{gl}(V)$ into a real/complex Lie algebra. If V now is a real space and $\langle\,,\rangle$ is pseudo-Euclidean, one uses the symbol $\mathfrak{so}(V)$ for the subspace of $\mathfrak{gl}(V)$ formed by all skew-adjoint linear operators $F : V \to V$, and one easily sees that $\mathfrak{so}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Due to the identification (3.28) with skew-symmetric forms, we have

$$\dim\,\mathfrak{so}(V) \,=\, \frac{n(n-1)}{2}\,, \qquad n \,=\, \dim V\,. \tag{3.31}$$

Given a bilinear mapping $B : V \times V \to V'$ of a pseudo-Euclidean vector space V into *any* vector space V' (finite-dimensional or not), we define its $\langle\,,\rangle$-*trace* to be the element $\operatorname{Trace} B$ of V' given by

$$\operatorname{Trace} B \,=\, \sum_{j=1}^{n} \varepsilon_j B(e_j, e_j)\,, \tag{3.32}$$

where $n = \dim V$ and e_j, $j = 1, \dots, n$, is any orthonormal basis of V, with $\langle e_j, e_j\rangle = \varepsilon_j \in \{1, -1\}$. Note that, if $V' = \mathbf{R}$, we have $\operatorname{Trace} B = \operatorname{Trace} F$, F being the operator $V \to V$ corresponding to B via the identification (3.28). In general, the reason why (3.32) does not depend on the orthonormal basis used is that, for two such bases $e_1, \dots, e_n$ and $u_1, \dots, u_n$ which, in addition, are *sign-coordinated* (that is, $\langle u_j, u_j\rangle = \langle e_j, e_j\rangle$ for all j), the transition matrix $\mathfrak{A} = [a_{jk}]$ with $u_j = \sum_k a_{jk} e_k$ satisfies

$$\mathfrak{C}\mathfrak{C}^* \,=\, \mathbf{1}\,, \tag{3.33}$$

where $\mathfrak{C}$ is the matrix with the entries $c_{jk} = \varepsilon_j a_{jk}$ (no summation), and $\mathfrak{C}^*$ is the transpose of $\mathfrak{C}$, while $\mathbf{1}$ stands for the identity matrix. (In fact, this is nothing else than the condition $\langle u_j, u_k\rangle = \langle e_j, e_k\rangle$ for all j, k.) Relation (3.33) clearly implies $\mathfrak{C}^*\mathfrak{C} = \mathbf{1}$, which in turn easily shows (3.32).

Remark 3.16. Another consequence of (3.33) is that transition matrix $\mathfrak{A}$ between any two orthonormal bases of a pseudo-Euclidean space has $\det \mathfrak{A} = \pm 1$. In fact, (3.33) yields $\det \mathfrak{C} = \pm 1$ for a sign-coordinated change of basis; however, rearranging the order of a basis, as well as passing from $\mathfrak{A}$ to $\mathfrak{C}$, both involve multiplications by matrices of deteminant ± 1. Any oriented n-dimensional pseudo-Euclidean space V thus has a naturally distinguished *volume element*, which is the n-vector $\mathrm{vol} \in V^{\wedge n}$, given by

$$\mathrm{vol} \,=\, e_1 \wedge \ldots \wedge e_n \tag{3.34}$$

for any positive-oriented orthonormal basis $e_1, \dots, e_n$ of V. In fact, (3.11) now implies that (3.34) does not depend on the choice of such a basis.

Remark 3.17. Here are some more well-known facts, listed for easy reference: For a self-adjoint or skew-adjoint operator $F : V \to V$ in an inner-product space V,

(i) The eigenspaces of F are mutually orthogonal;
(ii) The orthogonal complement of any F-invariant subspace of V is also F-invariant.

Remark 3.18. Let $\langle v, w\rangle$ be a real-bilinear inner product in a complex vector space V treated here as a real space with a fixed complex-structure operator J satisfying (3.15). Then, for $\langle v, w\rangle$ to be the real part of a complex-bilinear (or, sesquilinear) inner product $\langle\,,\rangle_{\mathbf{c}}$ in the complex vector space V, it is obviously necessary and sufficient that the operator J be anti-isometric (or, respectively, isometric) relative to $\langle v, w\rangle$ in the sense that $\langle Jv, Jw\rangle = -\langle v, w\rangle$ (or, respectively, $\langle Jv, Jw\rangle = \langle v, w\rangle$) for all $v, w \in V$. In view of (3.15), the requirement that J be anti-isometric (or, respectively, isometric) is equivalent to its self-adjointness (or, respectively, skew-adjointness) relative to $\langle v, w\rangle$. Moreover, if such $\langle\,,\rangle_{\mathbf{c}}$ exists, it is uniquely determined by $\langle\,,\rangle$ via the formula

$$\langle v, w\rangle_{\mathbf{c}} \,=\, \langle v, w\rangle \,-\, i\,\langle Jv, w\rangle \tag{3.35}$$

for $v, w \in V$.

§4. Basic facts about curvature

Let (M,g) be a pseudo-Riemannian manifold. We will denote Γ_{jk}^l the *Christoffel symbols* of (M,g) relative to any fixed local coordinate system x^j in M. Thus,

$$\Gamma_{jk}^l = \frac{1}{2}\, g^{ls}(\partial_j g_{ks} + \partial_k g_{js} - \partial_s g_{jk})\,. \tag{4.1}$$

(See Remark 4.1 below.) More generally, given *any* connection ∇ in the tangent bundle TM of a manifold M, one introduces the component functions Γ_{jk}^l of ∇ relative to any local coordinate system x^j in M by

$$\nabla_{e_j} e_k \;=\; \Gamma_{jk}^l e_l\,. \tag{4.2}$$

A connection ∇ in TM is called *torsionfree* if

$$\Gamma_{jk}^l \;=\; \Gamma_{kj}^l\,. \tag{4.3}$$

A coordinate-independent characterization of torsionfree connections is

$$\nabla_v w \;-\; \nabla_w v \;=\; [v,w]\,, \tag{4.4}$$

for any (local) C^1 vector fields v,w on M, where $[\,,\,]$ denotes the Lie bracket (see (2.4)). Another important class of connections ∇ in TM are those ∇ which are *compatible* with a given pseudo-Riemannian metric g on M in the sense that $\nabla g = 0$ or, equivalently, the *Leibniz rule*

$$d_u[g(v,w)] \;=\; g(\nabla_u v,\, w) \;+\; g(v,\, \nabla_u w) \tag{4.5}$$

holds for arbitrary C^1 vector fields u,v,w defined on any open set in M.

Remark 4.1. The Levi-Civita connection ∇ of a given pseudo-Riemannian metric g on M is the unique connection in TM which is both compatible with g (i.e., makes g parallel) and torsionfree. In fact, for any ∇ we may set

$$\Gamma_{jkl} \;=\; \Gamma_{jk}^s g_{sl} \tag{4.6}$$

in fixed local coordinates x^j, with Γ_{jk}^s as in (4.2). Note that, by (4.2) and (4.6),

$$\Gamma_{jkl} \;=\; g(\nabla_{e_j} e_k, e_l)\,. \tag{4.7}$$

Obviously, the requirement that ∇ be compatible with g (or, torsionfree) is equivalent to

$$\partial_j g_{kl} \;=\; \Gamma_{jkl} \;+\; \Gamma_{jlk} \tag{4.8}$$

(or, respectively, $\Gamma_{jkl} = \Gamma_{kjl}$). Hence both conditions hold simultaneously if and only if

$$2\,\Gamma_{jkl} \;=\; \partial_j g_{kl} \;+\; \partial_k g_{jl} \;-\; \partial_l g_{jk}\,, \tag{4.9}$$

that is, if ∇ is the connection characterized by (4.1).

Next we have two formulae which, although important, will not be needed until §23. First, for any pseudo-Riemannian metric g (or, respectively, a C^1 curve $t \mapsto g(t)$ of metrics), and any coordinate system x^j, we have

$$g^{jk}\,\frac{d}{dt}\,g_{jk} \;=\; \frac{d}{dt}\,\log|\det[g_{jk}]|\,, \qquad g^{jk}\partial_l g_{jk} \;=\; \partial_l \log|\det[g_{jk}]|\,, \tag{4.10}$$

with g^{jk} given by (2.8). Furthermore, the Christoffel symbols of g satisfy the conditon

$$2\Gamma^s_{sl} \;=\; \partial_l \log|\det[g_{jk}]|\,. \tag{4.11}$$

In fact, (4.10) is nothing else than (3.14) with F standing for the matrix $[g_{jk}]$ of the component functions of g, and the second relation in (4.10) is obtained by choosing t to be the coordinate x^l. Now (4.11) is immediate from (4.10) and (4.1).

Let ∇ again be any connection in TM. For local C^1 vector fields w in M and vectors v tangent to M, we have

$$[\nabla_v w]^j \;=\; w^j{}_{,k}v^k \qquad \text{with} \qquad w^j{}_{,k} = \partial_k w^j \,+\, \Gamma^j_{kl}w^l\,, \tag{4.12}$$

in any local coordinates x^j. Similarly, for a C^1 vector field $t \mapsto w(t) \in T_{x(t)}M$ along a C^1 curve $t \mapsto x(t) \in M$, the covariant derivative $\nabla_{\dot{x}}w$ is characterized by the local-coordinate formula

$$[\nabla_{\dot{x}}w]^j \;=\; \dot{w}^j \,+\, \Gamma^j_{kl}(x)\dot{x}^k w^l\,, \tag{4.13}$$

where $(\;)^{\dot{}} = d/dt$ and (x) stands for $x(t)$. If the curve $t \mapsto x(t) \in M$ is of class C^2, by its *velocity* we mean the C^1 vector field $\dot{x}$ along the curve, given by $t \mapsto \dot{x}(t) \in T_{x(t)}M$. Those C^2 curves in M for which

$$\nabla_{\dot{x}}\dot{x} \;=\; 0 \tag{4.14}$$

identically are called *geodesics* of the connection ∇ in TM. In terms of a local coordinate system x^j, they are characterized by the system of equations

$$\ddot{x}^j \,+\, \Gamma^j_{kl}(x)\dot{x}^k\dot{x}^l \;=\; 0\,, \tag{4.15}$$

that is, (4.13) with $w^j(t) = \dot{x}^j(t)$. Every geodesic is automatically of class C^∞ and, for any fixed $a \in \mathbf{R}$, $y \in M$ and $v \in T_yM$, there exists an open interval $I \subset \mathbf{R}$ containing a and a unique geodesic $I \ni t \mapsto x(t) \in M$ with $x(a) = y$ and $\dot{x}(a) = v$. This allows us to introduce the *exponential mapping*

$$\exp_x : U_x \;\to\; M \tag{4.16}$$

of the given connection ∇ at any point $x \in M$, defined as follows. Its domain U_x is a subset of T_xM consisting of those $v \in T_xM$ for which there exists a geodesic $t \mapsto x(t)$, defined on the whole interval $[0,1]$, and such that $x(0) = x$, $\dot{x}(0) = v$. For such v and $x(t)$, we set $\exp_x v = x(1)$. (One traditionally writes $\exp_x v$, without parentheses, rather than $\exp_x(v)$.) It is obvious from the dependence-on-parameters theorem for ordinary differential equations that the set U_x is open in

T_xM (and contains 0), and the mapping $\exp_x$ is of class C^∞. Furthermore, the geodesic $x(t)$ with $x(0) = x$ and $\dot{x}(0) = v$ is given by $x(t) = \exp_x tv$, as one sees fixing $t \in [0,1]$ and noting that $[0,1] \ni t' \mapsto x(tt')$ then is a geodesic with the value and velocity at $t' = 0$ equal to x and, respectively, tv. In particular, $d[\exp_x tv]/dt$ at $t = 0$ equals v while, obviously, $\exp_x 0 = x$; in other words, the differential of the mapping (4.16) at the point $0 \in U_x$ is given by

$$d(\exp_x)_0 \,=\, \mathrm{Id} : T_xM \to T_xM\,. \tag{4.17}$$

According to the inverse mapping theorem, there exist a neighborhood U of y in M and a neighborhood U' of 0 in T_xM such that $U' \subset U_x$ and $\exp_x : U' \to U$ is a C^∞-diffeomorphism. Its inverse diffeomorphism may be thought of as a coordinate system $x^1, \dots, x^n$ with the domain U (after one has identified T_xM with $\mathbf{R}^n$, $n = \dim M$, using any fixed linear isomorphism). A coordinate system obtained as a local inverse of $\exp_x$ is called a *normal*, or *geodesic*, coordinate system at x, for the given connection ∇ in TM. Note that if the connection ∇ is torsionfree, its component functions $\varGamma_{jk}^l$ satisfy

$$\varGamma_{jk}^l(x) \,=\, 0 \quad \text{in normal coordinates at} \quad x\,. \tag{4.18}$$

To see this, note that under the identification $U' \approx U$ provided by $\exp_x$, geodescis emanating from $0 \approx x$ appear as the radial line segments $t \mapsto tv$, and so we have $\ddot{x}^j = 0$. For such a geodesic, the system (4.15) gives, at $t = 0$, $\varGamma_{kl}^j(x)v^kv^l = 0$ for all v, which in view of the symmetry (4.3) implies (4.18) (cf. Remark 3.12).

A fixed connection ∇ in TM gives rise to the corresponding ("dual") connection in the cotangent bundle T^*M (also denoted ∇), which acts on local C^1 cotangent vector fields ξ so that, for vectors v tangent to M,

$$[\nabla_v\xi]_j \,=\, \xi_{j,k}v^k \qquad \text{with} \qquad \xi_{j,k} \,=\, \partial_k\xi_j \,-\, \varGamma_{kj}^l\xi_l\,, \tag{4.19}$$

with the same $\varGamma_{kj}^l$ as in (4.12). If ∇ is the Levi-Civita connection of a pseudo-Riemannian manifold (M,g), and v is a local C^1 vector field in M treated, with the aid of g, as a dual vector field with the components $v_j = g_{jk}v^k$, (4.19) combined with (4.6) yields

$$v_{j,k} \,=\, \partial_kv_j \,-\, \varGamma_{kjl}v^l\,. \tag{4.20}$$

For any torsionfree connection ∇ and a 1-form ξ of class C^1, we have

$$(d\xi)(u,v) \,=\, [\nabla_u\xi](v) \,-\, [\nabla_v\xi](u)\,, \qquad (d\xi)_{jk} \,=\, \xi_{k,j} \,-\, \xi_{j,k}\,. \tag{4.21}$$

Hence, if ∇ is torsionfree,

$$\nabla\xi \,=\, 0 \qquad \text{implies} \qquad d\xi \,=\, 0\,, \tag{4.22}$$

for any 1-form ξ of class C^1. The same is true for 2-forms α as well; the components of $d\alpha$ for a 2-form α are $(d\alpha)_{jkl} = \partial_j\alpha_{kl} + \partial_k\alpha_{lj} + \partial_l\alpha_{jk}$, and one easily sees using the formula $\alpha_{jk,l} = \partial_l\alpha_{jk} - \varGamma_{lj}^s\alpha_{sk} - \varGamma_{lk}^s\alpha_{js}$ (analogous to (4.19)) that, if ∇ is torsionfree, i.e., satisfies (4.3), we also have $(d\alpha)_{jkl} = \alpha_{jk,l} + \alpha_{kl,j} + \alpha_{lj,k}$.

Our sign convention for the curvature tensor R of any connection ∇ in TM, including the case where ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g on M, is such that

$$R(v,w)u \,=\, \nabla_w\nabla_v u \,-\, \nabla_v\nabla_w u \,+\, \nabla_{[v,w]}u \tag{4.23}$$

for C^2 vector fields u, v, w. In any local coordinates x^j, the curvature component functions $R_{jkl}{}^m$, characterized by the formula

$$R(v,w)u \,=\, v^j w^k u^l R_{jkl}{}^m e_m\,, \tag{4.24}$$

thus satisfy the relation

$$R_{jkl}{}^m \,=\, \partial_k\Gamma_{jl}^m \,-\, \partial_j\Gamma_{kl}^m \,+\, \Gamma_{ks}^m\Gamma_{jl}^s \,-\, \Gamma_{js}^m\Gamma_{kl}^s\,. \tag{4.25}$$

As an immediate consequence of (4.25), we have the *Ricci-Weitzenböck* identity for for C^2 vector fields w in pseudo-Riemannian manifolds (M,g):

$$w^l{}_{,jk} - w^l{}_{,kj} = R_{jks}{}^l w^s\,. \tag{4.26}$$

Similarly, given a twice-covariant C^2 tensor field F, viewed as a bundle morphism $TM\to TM$, and C^2 vector fields v, w, we have

$$\nabla_w\nabla_v F \,-\, \nabla_v\nabla_w F \,+\, \nabla_{[v,w]}F \,=\, [F, R(v,w)]\,, \tag{4.27}$$

$[\,,\,]$ being the ordinary commutator of bundle morphisms. Here $R(v,w)$ is the bundle morphism $TM\to TM$ given by (4.23), i.e, with the local-coordinate components

$$[R(v,w)]_l^m \,=\, v^j w^k R_{jkl}{}^m\,. \tag{4.28}$$

Thus, the local coordinate version of (4.27) is

$$F_{lm,jk} \,-\, F_{lm,kj} \,=\, R_{jkl}{}^p F_{pm} \,+\, R_{jkm}{}^p F_{lp}\,, \qquad F_{jk} \,=\, F_j^l g_{lk}\,. \tag{4.29}$$

Let R_{jklm} denote, as usual, the g-modified components of the curvature tensor of (M,g), with

$$R_{jklm} \,=\, R_{jkl}{}^p g_{pm}\,. \tag{4.30}$$

From (4.25), (4.8) and (4.6), we thus have

$$R_{jklm} \,=\, \partial_k\Gamma_{jlm} \,-\, \partial_j\Gamma_{klm} \,+\, g^{pq}\,[\Gamma_{klp}\Gamma_{jmq} \,-\, \Gamma_{jlp}\Gamma_{kmq}]\,. \tag{4.31}$$

The well-known algebraic symmetries of R_{jklm} are

$$R_{jklm} \,=\, -\,R_{kjlm} \,=\, -\,R_{jkml} \,=\, R_{lmjk} \tag{4.32}$$

and the *first Bianchi identity*

$$R_{jklm} \,+\, R_{jlmk} \,+\, R_{jmkl} \,=\, 0\,. \tag{4.33}$$

(Relation $R_{jklm} = R_{lmjk}$ happens to be an algebraic consequence of the remaining symmetries; see Remark 38.1 in §38.)

The Ricci tensor Ric of any connection ∇ in TM is a twice-covariant tensor field on M which assigns to each point $x \in M$ the bilinear mapping $\mathrm{Ric}(x) : T_xM \times T_xM \to \mathbf{R}$ given by

$$\mathrm{Ric}(v,w) \,=\, \mathrm{Trace}\,[u \mapsto R(v,u)w]\,. \tag{4.34}$$

(We write $\mathrm{Ric}(v,w)$, with no reference to x, in order to simplify the notation.) In local coordinates x^j, Ric has the component functions

$$R_{jk} \,=\, \mathrm{Ric}\,(e_j, e_k)\,, \tag{4.35}$$

where e_j are the coordinate vector fields with (2.1), (2.2). Thus, $\mathrm{Ric}(v,w) = v^j w^k R_{jk}$ for $v,w \in T_xM$. From (4.34) and (4.24) we obtain

$$R_{jk} \,=\, R_{jlk}{}^l\,, \tag{4.36}$$

In the case where ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g on M, (4.36) becomes

$$R_{jk} \,=\, g^{lm}R_{jlkm} \,=\, g^{lm}R_{ljmk} \,=\, -\,g^{lm}R_{jlmk}\,, \tag{4.37}$$

cf. (4.32). Note that, by (4.32) and symmetry of g^{jk}, we also have $g^{lm}R_{jlkm} = g^{lm}R_{kmjl} = g^{lm}R_{kljm}$, that is, the Ricci tensor any pseudo-Riemannian manifold (M,g) is symmetric:

$$R_{jk} \,=\, R_{kj}\,. \tag{4.38}$$

Contracting (4.26) in $k = l$ and using (4.37), we now obtain the *contracted Ricci-Weitzenböck formula*

$$R_{jk}w^k \,=\, w^k{}_{,jk} \,-\, w^k{}_{,kj}\,, \tag{4.39}$$

valid for local C^2 vector fields w in any pseudo-Riemannian manifold (M,g).

With the aid of g, we may treat the Ricci tensor Ric of (M,g) as a self-adjoint bundle morphism $TM \to TM$ (see (3.28)). Its components then can also be written as $R_j^k = g^{kl}R_{jl}$. The *scalar curvature* of (M,g) is defined to be the function $\mathrm{s} : M \to \mathbf{R}$ equal, at every point x of M, to the trace of this morphism in T_xM (that is, the g-trace of Ric at x, cf. (3.32)). Thus,

$$\mathrm{s} \,=\, \mathrm{Trace\ Ric} \,=\, R_j^j \,=\, g^{jk}R_{jk}\,. \tag{4.40}$$

Remark 4.2. Given a pseudo-Riemannian metric g on a manifold M and a real constant $a \neq 0$, the product $\tilde g = ag$ is another metric on M, whose Levi-Civita connection $\tilde\nabla$, Ricci tensor $\tilde{\mathrm{Ric}}$ and scalar curvature $\tilde{\mathrm{s}}$ are given by $\tilde\nabla = \nabla$, $\tilde{\mathrm{Ric}} = \mathrm{Ric}$ and $\tilde{\mathrm{s}} = \mathrm{s}/a$, where ∇, Ric and s denote the analogous objects corresponding to g. In fact, ∇ coincides with $\tilde\nabla$, as one sees using Remark 4.1 (since ∇ is torsionfree and makes $\tilde g$ parallel), or directly from (4.1) with $\tilde g^{jk} =$

$a^{-1}g^{jk}$. The other two relations are clear from (4.40) and the fact that Ric is determined by ∇ alone (see (4.34), (4.23)).

The *divergence* operator div acting on differential forms of any degree on a pseudo-Riemannian manifold (M,g) is given by

$$\operatorname{div}\alpha \,=\, \sum_{j=1}^{n}\varepsilon_j(\nabla_{e_j}\alpha)(e_j,\,\dots\,)\,, \qquad n \,=\, \dim M\,, \tag{4.41}$$

e_j being any local orthonormal vector fields with $g(e_j,e_j)=\varepsilon_j=\pm 1$, $j=1,\dots,n$. (See (3.32).) The same definition of div applies, more generally, to differential forms on (M,g) that are valued in a vector bundle over M carrying a fixed connection. In particular, treating a C^1 vector field w on M as a 1-form (with the aid of g), we obtain from (4.12)

$$\operatorname{div} w \,=\, \operatorname{Trace}\nabla w \,=\, w^j{}_{,j}\,. \tag{4.42}$$

The *Laplacian* or *Laplace operator* Δ acting on C^2 functions f in a given pseudo-Riemannian manifold (M,g) is defined by

$$\Delta \,=\, \operatorname{div}\circ\nabla\,, \tag{4.43}$$

that is,

$$\Delta f \,=\, f_{,j}{}^{j} \,=\, g^{jk}f_{,jk}\,. \tag{4.44}$$

As another example, the contracted Ricci-Weitzenböck formula (4.39) can be rewritten as

$$\operatorname{Ric}(w,\,\cdot\,) \,=\, \operatorname{div}(\nabla w) \,-\, d\,(\operatorname{div} w)\,, \tag{4.45}$$

where ∇w is treated as a 1-form valued in tangent vectors. Also, as a consequence of (2.12) and (4.41), for a 2-form α and a 1-form ξ, we have

$$\operatorname{div}(\alpha\xi) \,=\, -\,\langle\operatorname{div}\alpha,\xi\rangle \,-\, \langle\alpha,d\xi\rangle\,. \tag{4.46}$$

In §17 we will need the fact that

$$\operatorname{div}\operatorname{div}\alpha \,=\, 0\,, \qquad \text{i.e.,} \qquad \alpha^{jk}{}_{,jk} \,=\, 0 \tag{4.47}$$

for any C^2 bivector field α on a pseudo-Riemannian manifold (M,g). To see this, note that, due to skew-symmetry of α, $2\alpha^{jk}{}_{,jk}=\alpha^{jk}{}_{,jk}-\alpha^{jk}{}_{,kj}$. The last expression is zero for any twice-contravariant tensor field α (skew-symmetric or not), since, by (4.29) and (4.37), it equals $R_{jk}{}^{j}{}_{s}\alpha^{sk}+R_{jk}{}^{k}{}_{s}\alpha^{js}=(R_{jk}-R_{kj})\alpha^{jk}$, and so it vanishes in view of (4.38).

Beside connections in the tangent bundle TM, we will also have to discuss the more general case of connections in arbitrary vector bundles.

Let $\mathcal{E}$ be a real or complex vector bundle over a manifold M, and let ∇ be a connection in $\mathcal{E}$. Any local trivialization e_a of $\mathcal{E}$ associates with every (local) section ϕ its component functions ϕ^a, characterized by $\phi=\phi^a e_a$. Similarly, ∇

is represented by its component functions Γ_{ja}^b relative to the local trivialization e_a of $\mathcal{E}$ and any given local coordinate system x^j in M, both with the same domain. The Γ_{ja}^b are given by

$$\nabla_{e_j} e_a \,=\, \Gamma_{ja}^b e_b\,. \tag{4.48}$$

Thus, for any tangent vector (field) v and any local C^1 section ϕ of $\mathcal{E}$,

$$\nabla_v \phi \,=\, v^j \left(\partial_j \phi^a \,+\, \Gamma_{jb}^a \phi^b\right) e_a\,. \tag{4.49}$$

As in (4.13), we can use a connection ∇ in $\mathcal{E}$ to define a covariant-derivative operation that can be applied to "sections" of $\mathcal{E}$ that are defined only along a curve in M. More precisely, for a C^1 section $t \mapsto \psi(t) \in \mathcal{E}_{x(t)}M$ of $\mathcal{E}$ along a C^1 curve $t \mapsto x(t) \in M$, we have the component formula

$$[\nabla_{\dot{x}}\psi]^a \,=\, \dot{\psi}^a \,+\, \Gamma_{jb}^a \dot{x}^j \psi^a\,, \tag{4.50}$$

with $(\ )^\cdot = d/dt$. As before, the covariant derivative $\nabla_{\dot{x}}\psi$ thus obtained is a new section of $\mathcal{E}$ along the same curve, and it does not depend on the local trivialization and local coordinates used in (4.50). A section $t \mapsto \psi(t)$ along the given curve is called *parallel*, or ∇*-parallel*, if $\nabla_{\dot{x}}\psi = 0$ identically. The ∇*-parallel transport* in $\mathcal{E}$ along such a fixed curve associates with any two parameter values t, t', the *parallel transport along the curve from* t *to* t', which the linear isomorphism $\mathcal{E}_x \to \mathcal{E}_y$, with $x = x(t)$, $y = x(t')$, sending any ϕ to $\psi(t')$, where ψ is the unique parallel section along the curve with $\psi(t) = \phi$.

Given any connection ∇ in the tangent bundle TM of a manifold M and a C^1 vector field $t \mapsto w(t) \in T_{x(t)}M$ along any C^1 curve $t \mapsto x(t) \in M$ which is a geodesic for ∇, one says that w is a *Jacobi field* if it satisfies the *Jacobi equation*

$$\nabla_{\dot{x}}\nabla_{\dot{x}} w \,=\, R(w, \dot{x})\dot{x}\,, \tag{4.51}$$

where R is the curvature tensor of ∇ and the operation $\nabla_{\dot{x}}$ is defined by (4.13). For more on Jacobi fields, see §17 and §28, especially Remark 28.5.

Our conventions for the curvature tensor R^∇ of a connection ∇ in a vector bundle $\mathcal{E}$ over M, and its component functions $R_{jka}{}^b$, are analogous to the special case in (4.23) and (4.25) (where $\mathcal{E} = TM$, the e_a are the e_j, and ∇ is the Levi-Civita connection of (M, g)): for C^2 (local) sections v, w of TM and ϕ of $\mathcal{E}$,

$$R^\nabla(v, w)\phi \,=\, \nabla_w \nabla_v \phi \,-\, \nabla_v \nabla_w \phi \,+\, \nabla_{[v,w]}\phi\,, \tag{4.52}$$

and so $R^\nabla(v, w)\phi = v^j w^k \phi^a R_{jka}{}^b e_b$ with

$$R_{jka}{}^b \,=\, \partial_k \Gamma_{ja}^b \,-\, \partial_j \Gamma_{ka}^b \,+\, \Gamma_{kc}^b \Gamma_{ja}^c \,-\, \Gamma_{jc}^b \Gamma_{ka}^c\,. \tag{4.53}$$

Most vector bundles we are going to encounter will carry *fibre metrics*. Such a metric $\langle\,,\rangle$ in a vector bundle $\mathcal{E}$ over a manifold M assigns to each point $x \in M$ a nondegenerate inner product in the fibre $\mathcal{E}_x$ whose dependence on x is C^∞-differentiable and which is, for real bundles, real-bilinear and symmetric and, for complex bundles, usually sesquilinear and Hermitian (except for some cases where it is complex-bilinear and symmetric). In the first two cases, the metric may (but does not have to be) positive definite; we speak, in general, of *pseudo-Riemannian* or *pseudo-Hermitian* fibre metrics, dropping the prefix 'pseudo' when the metric is positive definite. Complex-bilinear symmetric fibre metrics in complex bundles will be of no importance for us except in Part IV of this text.

A connection ∇ in a real (complex) vector bundle $\mathcal{E}$ is called *compatible* with a given fibre metric $\langle\,,\rangle$ in $\mathcal{E}$ if $\langle\,,\rangle$ is ∇-parallel or, equivalently, the *Leibniz rule*

$$d_u[\langle\phi,\psi\rangle] \,=\, \langle\nabla_u\phi,\,\psi\rangle \,+\, \langle\phi,\,\nabla_u\psi\rangle \tag{4.54}$$

holds for arbitrary C^1 sections ϕ,ψ of $\mathcal{E}$ and vector fields u, all defined on any open set in M

Example 4.3. The restriction of a given connection ∇ in $\mathcal{E}$ to a parallel subbundle $\mathcal{P}$ (Remark 4.7(ii)) has "the same" curvature as ∇, as one sees applying (4.52) to sections ϕ of $\mathcal{P}$. More precisely, for every $x\in M$ and any vectors $v,w\in T_xM$, the curvature operator $R^\nabla(v,w):\mathcal{E}_x\to\mathcal{E}_x$ given by $\phi\mapsto R^\nabla(v,w)\phi$, with (4.52), then obviously leaves the subspace $\mathcal{P}_x\subset\mathcal{E}_x$ invariant, and the analogous the curvature operator for the restricted connection is the restriction of R^∇ to $\mathcal{E}_x$.

Remark 4.4. Given a connection ∇ in a vector bundle $\mathcal{E}$ over a manifold M, a point $x\in M$ and an element ψ of the fibre $\mathcal{E}_x$ of $\mathcal{E}$ over x, we can always find a C^∞ local section ϕ of $\mathcal{E}$ defined on a neighborhood of x which realizes the prescribed value ψ at x, that is, $\phi(x)=\psi$, and is *parallel at the point* x in the sense that $[\nabla\phi](x)=0$. (In fact, this can be done by properly choosing the $\phi^a(x)$ and $(\partial_j\phi^a)(x)$ in (4.49).) Using sections parallel at the given point leads to enormous simplifications in calculations, based on (4.52), of the curvature tensors of various connections we are going to construct; specifically, we may always omit the terms containing the first covariant derivatives of the sections involved. At the same time, the vector fields v,w in (4.52) may be chosen so as to have $\partial_jv^k(x)=\,\partial_jv^k(x)=0$ in some fixed coordinates, and so $[v,w](x)=0$ by (2.4). Consequently, the term $\nabla_{[v,w]}\phi$ in (4.52) can always be assumed to vanish at the point in question. The result of the computation will not be affected due to the tensorial nature of the curvature; in other words, the terms we omit would have added up to zero anyway.

Example 4.5. As an immediate application of the simplifications offered by Remark 4.4, let us note that for a connection ∇ in $\mathcal{E}$, compatible with a fibre metric $\langle\,,\rangle$, the curvature operators $R^\nabla(v,w):\mathcal{E}_x\to\mathcal{E}_x$ (Example 4.3) are all skew-adjoint relative to $\langle\,,\rangle$. In fact, differentiating (4.52) by parts against ϕ and using the simplifications described in Remark 4.4, we get $2\langle R^\nabla(v,w)\phi,\,\phi\rangle=2d_w\langle\nabla_v\phi,\,\phi\rangle-2d_v\langle\nabla_w\phi,\,\phi\rangle$. For (real or complex) bilinear fibre metrics, this equals $d_wd_v\langle\phi,\phi\rangle-d_vd_w\langle\phi,\phi\rangle=0$ (by (2.6), since $[v,w]$ is assumed to vanish at the point in question). This establishes the required skew-adjointness property, cf. the final clause in Remark 3.12. For pseudo-Hermitian fibre metrics $\langle\,,\rangle$, applying the previous conclusion to the real metric $\mathrm{Re}\,\langle\,,\rangle$, we see that $\langle R^\nabla(v,w)\phi,\,\phi\rangle$ is always imaginary, as required (again, by Remark 3.12).

Remark 4.6. Given a vector bundle $\mathcal{E}$ with a fixed connection ∇ over a manifold M, we will sometimes obtain a local section ϕ of $\mathcal{E}$ defined on a neighborhood

U of a given point x by choosing its value $\phi(x)$ at x and then *spreading it via radial parallel transports*, which means selecting U which has a fixed diffeomorphic identification with a convex open set in R^n, $n = \dim M$, and then defining $\phi(y)$ at any $y \in U$ to be the result of the ∇-parallel transport of $\phi(x)$ from x to y along the straight-line segment connecting x and y. In view of the dependence-on-parameters theorem for ordinary differential equations, the resulting section ϕ is of class C^∞.

Note that any natural relation between a local section ϕ on U obtained as above and any parallel object, if satisfied at x, must automatically be satisfied everywhere in U. If the value $\phi(y)$ of such a section, at each point y, determined uniquely, up to finitely many choices, by the parallel object in question, then speading $\phi(x)$ as above produces a section which is *parallel* (in virtue of being invariant, due to its "almost uniqueness", under all parallel transports).

Remark 4.7. Given a connection ∇ in a vector bundle $\mathcal{E}$ over a manifold M (cf. §11), by a *parallel subbundle* of $\mathcal{E}$ we will mean any assignment $x \mapsto \mathcal{P}_x$ which associates with every $x \in M$ a vector subspace $\mathcal{P}_x$ of the fibre $\mathcal{E}_x$ of $\mathcal{E}$, in a manner invariant under ∇-parallel transports along all piecewise C^1 curves in M. (In other words, the parallel transport along any such curve connecting x to y in M sends $\mathcal{P}_x$ onto $\mathcal{P}_y$.) Note that the $\mathcal{P}_x$ then are all of some fixed dimension q, independent of x, and form the fibres of a C^∞ vector subbundle $\mathcal{P}$ of M. (In fact, let $e_\lambda(x)$, $\lambda = 1, \ldots, q$, be a basis of $\mathcal{P}_x$; spreading each e_λ radially via parallel transports, as described in Remark 4.6, produces a local trivialization of $\mathcal{P}$ by C^∞ local sections of $\mathcal{E}$.) Furthermore,

(i) For a subbundle $\mathcal{P}$ of fibre dimension q in a vector bundle $\mathcal{E}$ with a fixed connection ∇, the following four conditions are equivalent:
   a) $\mathcal{P}$ is parallel;
   b) For any local trivialization e_a of $\mathcal{E}$ whose initial q sections e_λ, $\lambda = 1, \ldots, q$, lie in $\mathcal{P}$, the component functions of ∇ defined by (4.48) satisfy $\Gamma_{j\lambda}^c = 0$ whenever $\lambda \le q$ and $c > q$;
   c) $\mathcal{P}$ is closed under taking ∇-covariant derivatives of its local C^1 sections in all directions;
   d) $\mathcal{P}$ is closed under taking ∇-covariant derivatives of its C^1 sections along all C^1 curves in M.

   To see this, it suffices to consider the system

$$\dot{\psi}^a = -\Gamma_{jb}^a \dot{x}^j \psi^a \,, \tag{4.55}$$

   of equations which, according to (4.50), characterizes those C^1 sections $t \mapsto \psi(t)$ along a given curve which are parallel. Condition b) above obviously holds if and only if every solution $\psi(t)$ to (4.55) whose value at some t satisfies $\psi^c(t) = 0$ for all $c > q$, satisfies the same condition for all t. Thus, b) is equivalent to a). Equivalences of b) with c), and of b) with d), now are immediate from (4.48) and (4.50). Finally, let us note that

(ii) The connection ∇ in $\mathcal{E}$ gives rise to a connection in every parallel subbundle $\mathcal{P}$, called its *natural restriction to the parallel subbundle* $\mathcal{P}$, and obtained just by applying ∇ to sections of $\mathcal{P}$ (see (i)c)).

The remainder of this section is devoted to a proof of de Rham's local decomposition theorem for pseudo-Riemannian metrics. We will use this theorem only once, in Remark 16.10 (§16), which is of relatively minor importance; also, Lemma 4.9 below which we use to prove de Rham's Theorem 4.10, will have just one more application (in the proof of Proposition 46.10, §46). The reason for this restraint is to enable the reader (who choses to do so) to go through the most important topics covered in Part I of this text with a minimum amount of prerequisite material. As an example, even though Theorem 14.5 (§14) is usually derived from de Rham's decomposition theorem, its proof we give in §14 is different, cf. Remark 14.6. It is because of these considerations that some facts used in our proof of Lemma 4.8 (see below) have not been included in this section and, instead, will appear for the first time in §28.

Lemma 4.8. *Suppose that* ∇ *is a torsionfree connection in the tangent bundle* TM *of a manifold* M*, while* $\mathcal{P}$ *is a parallel subbundle of* TM*, as defined in* Remark 4.7*. Given a point* $x \in M$*, let* U' *be a convex neighborhood of* 0 *in* T_xM *such that* $\exp_x$ *sends* U' *diffeomorphically onto an open set in* M*. The submanifold* $N = \exp_x(U' \cap \mathcal{P}_x)$ *then is an* integral manifold of the subbundle $\mathcal{P}$ *of* TM *in the sense that* $T_yN = \mathcal{P}_y$ *for each* $y \in N$*.*

Proof. In view of Proposition 28.9, it suffices to show that any Jacobi field $t \mapsto w(t)$ along a geodesic $t \mapsto x(t) = \exp_x tv$ such that v, $w(0)$ and $[\nabla_{\dot{x}} w](0)$ all lie in $\mathcal{P}_x$, must satisfy $w(t) \in \mathcal{P}_{x(t)}$ for all t. However, the Jacobi equation (4.51), with the fixed geodesic, may be treated as a condition imposed on sections $t \mapsto w(t) \in \mathcal{P}_{x(t)}$ of the bundle $\mathcal{P}$ along the geodesic. In fact, as $\mathcal{P}$ is parallel and $\dot{x}$ is parallel along the geodesic, for any such $w(t)$ the derivatives $[\nabla_{\dot{x}} w](t)$, $[\nabla_{\dot{x}} \nabla_{\dot{x}} w](t)$ and $\dot{x}(t)$ must lie in $\mathcal{P}_{x(t)}$ for every t. Consequently, $R(u, u')\dot{x}(t) \in \mathcal{P}_{x(t)}$ for any $u, u' \in T_{x(t)}M$, since $R(u, u')$ leaves the subspace $\mathcal{P}_x \subset T_{x(t)}M$ invariant (Example 4.3). Our assertion now is easily obtained by solving (4.51) for sections of $\mathcal{P}$ along the geodesic, and using the uniqueness-of-solutions theorem for ordinary differential equations. This completes the proof. ■

The following lemma is a special case of Frobenius's integrability theorem for involutive distributions; see, e.g., Kobayashi and Nomizu (1963).

Lemma 4.9. *Let* ∇ *be a torsionfree connection in the tangent bundle* TM *of a manifold* M*, and let* $\mathcal{P}$ *be a parallel subbundle of* TM*, as defined in* Remark 4.7*. Given a point* $y \in M$ *and a cotangent vector* $\xi \in T_y^*M$*, that is, a linear function* $\xi : T_yM \to \mathbf{R}$*, satisfying the condition* $\mathcal{P}_y \subset \operatorname{Ker}\xi$*, there exists a* C^∞ *function* f *on a neighborhood* U *of* y *in* M *such that* $df(y) = \xi$ *and* f *is* constant in the direction of $\mathcal{P}$*, everywhere in* U*, i.e.,* $d_vf = 0$ *for all* $x \in U$ *and all* $v \in \mathcal{P}_x$*.*

Proof. Let us assume that $\xi \neq 0$. (Otherwise, we may set $f = 0$.) Using a local coordinate system, we can find a submanifold N of M containing y along with a C^∞ function $\varphi : N \to \mathbf{R}$ such that $T_zM = \mathcal{P}_z \oplus T_zN$ for evry $z \in N$ and $d\varphi_y$ is the restriction of ξ to T_N. Replacing N by a smaller neighborhood of y in N, we can also assume that the bundle $\mathcal{P}$ restricted to N is trivial, with some trivializing C^∞ sections e_a. Let V be the vector space spanned by these sections e_a that is, formed by their constant-coefficient combinations). Making N smaller again, we can now find a neighborhood U_0 of 0 in V such that the C^∞ mapping F given by the formula $F(z, v) = \exp_z v(z)$, with $\exp_z$ as in (4.16) is well-defined on $N \times U_0$. Its differential $dF_{(y,0)}$ at $(y, 0)$ sends $(u, 0)$ to u and $(0, w)$ to $w(y)$, for any $u \in_y N$ and $w \in V$ (since $d(\exp_y)_0 = \mathrm{Id}$ by (4.17)). Thus, $dF_{(y,0)}$ is an isomorphism onto T_yM and, in view of the inverse mapping theorem, we can find a neighborhood U of y in M on which there exists an inverse for F, sending each $x \in U$ to a pair $(z,v) = (h(x), A(x))$. The C^∞ function $f : U \to \mathbf{R}$ given by $f(x) = \varphi(h(x))$ appears, under the identification $N \times U_0 \approx U$ provided by F, to be $(z,v) \mapsto \varphi(z)$, and so it satisfies $df(y) = \xi$ and is constant on the submanifolds $F(\{z\} \times U_0)$ which, according to Lemma 4.9, form a decomposition of U into integral manifolds of $\mathcal{P}$. This completes the proof. ■

The following result is known as (the local version of) de Rham's decomposition theorem.

Theorem 4.10 (de Rham, 1952)**.** *Let the tangent bundle TM of a pseudo-Riemannian manifold (M, g) admit a direct-sum decomposition*

$$TM \,=\, \mathcal{P} \oplus \mathcal{Q}\,, \tag{4.56}$$

into subbundles $\mathcal{P}$ and $\mathcal{Q}$, which are both parallel in the sense of Remark 4.7*, and g-orthogonal to each other. Then g is locally isometric to a product metric for which the subbundles $\mathcal{P}$ and $\mathcal{Q}$ represent the product-factor directions. In other words, let $n = \dim M$ and let p stand for the fibre dimension of $\mathcal{P}$; a neighborhood of any given point $y \in M$ then admits a coordinate system x^j, $j = 1, \ldots, n$, with*

$$g_{a\lambda} \,=\, g_{\lambda a} \,=\, 0 \tag{4.57}$$

and

$$\partial_\lambda g_{jk} \,=\, \partial_a g_{\lambda\mu} \,=\, 0 \tag{4.58}$$

for all a, b, λ, μ with

$$a, b \,\in\, \{1, 2, \ldots, p\}\,, \qquad \lambda, \mu \,\in\, \{p+1, \ldots, n\}\,, \tag{4.59}$$

as well as

$$\mathcal{P} \,=\, \mathrm{Span}\,\{e_1, \ldots, e_p\}\,, \qquad \mathcal{Q} \,=\, \mathrm{Span}\,\{e_{p+1}, \ldots, e_n\}\,, \tag{4.60}$$

where the symbols e_j, $g_{jk} = g(e_j, e_k)$ and $\partial_j = \partial/\partial x^j$ for $j, k = 1, \ldots, n$ denote, respectively, the coordinate vector fields, the component functions of g and the partial derivatives.

Proof. Let us fix any basis $e_1(y), \ldots, e_n(y)$ of T_yM such that (4.80) holds at the point y. Using Lemma 4.9 we can now find C^∞ functions $x^1, \ldots, x^n$ defined on a neighborhood U' of y and such that their differentials $dx^1, \ldots, dx^n$ form, at y, a basis of T^*_yM dual to $e_1(y), \ldots, e_n(y)$ while, at every point of U', such that $x^1, \ldots, x^p$ are constant in the direction of $\mathcal{P}$ and $x^{p+1}, \ldots, x^n$ are constant in the direction of $\mathcal{Q}$. According to the inverse mapping theorem, restricted to some smaller neighborhood U of y, the x^j form a local coordinate system in M. Since the corresponding coordinate vector fields e_j satisfy (2.3), we now have (4.80) everywhere in U and so, due to mutual orthogonality of $\mathcal{P}$ and $\mathcal{Q}$, also (4.57), identically in U. By (4.80) and (4.2), the Christoffel symbols of g now satisfy $\Gamma^\lambda_{ab} = \Gamma^a_{\lambda\mu} = 0$ for all indices with (4.59) and so, in view of (4.6) with (4.57), we have $\Gamma_{ab\lambda} = \Gamma_{\lambda\mu a} = 0$ (indices as before). Using (4.57) and formula (4.9), we now obtain (4.58), which completes the proof. ■

§5. Einstein manifolds

The curvature tensor R of every pseudo-Riemannian manifold satisfies the *second Bianchi identity*

$$dR = 0\,, \qquad \text{i.e.,} \qquad R_{jkl}{}^{p}{}_{,\,m} + R_{kml}{}^{p}{}_{,\,j} + R_{mjl}{}^{p}{}_{,\,k} = 0 \tag{5.1}$$

which, via two successive contractions, implies the relation

$$2\,\mathrm{div}\,\mathrm{Ric} = d\mathrm{s}\,, \qquad \text{that is,} \qquad 2\,R^{k}_{j,\,k} = \mathrm{s}_{\,,j} \tag{5.2}$$

known as the *Bianchi identity for the Ricci tensor.*

According to (0.1), a pseudo-Riemannian manifold (M, g) is Einstein if and only if

$$\mathrm{Ric} = \kappa g \tag{5.3}$$

for some constant $\kappa \in \mathbf{R}$. In fact, the g-contraction of both sides of (5.3) then yields

$$\kappa = \frac{\mathrm{s}}{n}\,, \qquad n = \dim M\,. \tag{5.4}$$

s being the scalar curvature of (M, g). Thus, Einstein manifolds (M, g) are characterized by constancy of s along with the condition $\mathrm{E} = 0$, where

$$\mathrm{E} = \mathrm{Ric} - \frac{\mathrm{s}}{n}\,g\,, \qquad n = \dim M \tag{5.5}$$

denotes the *traceless part of the Ricci tensor*, sometimes called the *Einstein tensor* of (M, g). However, in dimensions other than $n = 2$, the constancy requirement is redundant. Namely, we have

Theorem 5.1 (Schur, 1869)**.** *Let a pseudo-Riemannian manifold* (M, g) *satisfy* (5.3) *for some function* $\kappa : M \to \mathbf{R}$*. Then* κ *is constant unless* $\dim M = 2$*. In other words, in dimensions* $n \neq 2$*, Einstein manifolds are characterized by condition* $\mathrm{E} = 0$ *alone, with* E *given by* (5.5)*.*

Proof. Relation (5.4) reads $nR^{k}_{j} = \mathrm{s}\,\delta^{k}_{j}$, whence $nR^{k}_{j,\,l} = \mathrm{s}_{\,,l}\delta^{k}_{j}$, and so $nR^{k}_{j,\,k} = \mathrm{s}_{\,,j}$. By (5.2), $(n-2)\,\mathrm{s}_{\,,j} = 0$ with $n = \dim M$, and so s and κ are constant, as $n > 2$ and our manifolds are connected by definition. ■

To further put this discussion into perspective, we may note that Einstein manifolds are characterized by vanishing of one irreducible curvature component; by the *irreducible components of the curvature* of any pseudo-Riemannian manifold (M, g) we mean its scalar curvature s with (4.40), traceless Ricci tensor E (see (5.5)) and the *Weyl conformal curvature tensor* W. The latter is defined for $n = \dim M \geq 3$, by the relation

$$W = R - \frac{2}{n-2}\,g \circledast \mathrm{Ric} + \frac{\mathrm{s}}{(n-1)(n-2)}\,g \circledast g\,. \tag{5.6}$$

Here $\circledast$ is a natural bilinear pairing of symmetric twice-covariant tensors B, C, valued in covariant 4-tensors (see Besse, 1987), given by

$$\begin{aligned} 2\,(B \circledast C)(v,w,v',w') \,&=\, B(v,v')C(w,w') \,+\, B(w,w')C(v,v') \\ &\quad -\, B(w,v')C(v,w') \,-\, B(v,w')C(w,v')\,. \end{aligned} \tag{5.7}$$

Thus, the local-coordinate version of (5.6) is

$$\begin{aligned} W_{jklm} \,&=\, R_{jklm} \,-\, \frac{1}{n-2}\,(g_{jl}R_{km} \,+\, g_{km}R_{jl} \,-\, g_{kl}R_{jm} \,-\, g_{jm}R_{kl}) \\ &\quad +\, \frac{\mathrm{s}}{(n-1)(n-2)}\,(g_{jl}g_{km} \,-\, g_{kl}g_{jm})\,. \end{aligned} \tag{5.8}$$

Since, by (5.5), $\mathrm{Ric} \,=\, \mathrm{E} \,+\, \mathrm{s}g/n$, (5.6) can be rewritten as

$$R \,=\, W \,+\, \frac{2}{n-2}\,g \circledast \mathrm{E} \,+\, \frac{\mathrm{s}}{n(n-1)}\,g \circledast g\,. \tag{5.9}$$

For Einstein manifolds (M,g), characterized by $\mathrm{E} \,=0$, this becomes

$$R \,=\, W \,+\, \frac{\mathrm{s}}{n(n-1)}\,g \circledast g\,, \qquad n = \dim M\,. \tag{5.10}$$

It is convenient to single out those four-times covariant tensors A at a point x of a pseudo-Riemannian manifold (M,g) which satisfy the conditions

$$A_{jklm} \,=\, -\,A_{kjlm} \,=\, -\,A_{jkml} \,=\, A_{lmjk}\,. \tag{5.11}$$

An obvious example of such a tensor is $A = \alpha\otimes\alpha$ for any bivector α at x treated, with the aid of g, as an exterior 2-form. The components of A then are

$$(\alpha\otimes\alpha)_{jklm} \,=\, \alpha_{jk}\alpha_{lm}\,. \tag{5.12}$$

Any tensor A with (5.11) at a point x of (M,g) can be regarded as a self-adjoint linear operator sending the space $[T_xM]^{\wedge 2}$ of bivectors β at x into itself, with

$$[A\beta]_{jk} \,=\, \frac{1}{2}\,A_{jklm}\beta^{lm}\,, \tag{5.13}$$

where the metric is again used to identify bivectors and 2-forms. For instance, using (5.12) and (2.17), we see that

$$(\alpha\otimes\alpha)\beta \,=\, \langle\alpha,\beta\rangle\alpha \tag{5.14}$$

for bivectors α,β. Also, given a symmetric 2-tensor B and a bivector α, by combining (5.7) with (5.13) (for $A = g \circledast B$) and (2.12), we obtain

$$2\,(g \circledast B)\alpha \,=\, \{B,\,\alpha\}\,. \tag{5.15}$$

Here $\{B,\,\alpha\} \,=\, B\alpha \,+\, \alpha B$ is the *anticommutator* of B and α, that is, the sum of the composites $B\alpha$ and αB obtained by treating both B and α as operators in the tangent space. (Note that, for a symmetric 2-tensor B and a bivector α,

$B\alpha + \alpha B$ is again a bivector.) Since g is the identity operator when acting on tangent vectors, (5.15) for $B = g$ shows that, as an operator acting on bivectors,

$$g \circledast g \,=\, \mathrm{Id}\,. \tag{5.16}$$

We can now introduce the *Weitzenböck formula for the Weyl tensor acting on bivectors*, which reads

$$\begin{aligned} W\alpha \,=\, & \frac{1}{2}\,[\mathrm{div}\,(\nabla\alpha \,-\, d\alpha) \,-\, d\,(\mathrm{div}\,\alpha)] \\ & +\, \frac{n-4}{2(n-2)}\,\{\mathrm{Ric},\,\alpha\} \,+\, \frac{\mathrm{s}}{(n-1)(n-2)}\,\alpha\,, \end{aligned} \tag{5.17}$$

and holds for any bivector field α of class C^2 on any n-dimensional pseudo-Riemannian manifold (M, g), where W, Ric and s denote, as usual, the Weyl tensor, Ricci tensor and scalar curvature, while $\{\mathrm{Ric},\,\alpha\} \,=\, \mathrm{Ric}\circ\alpha \,+\, \alpha\circ\mathrm{Ric}$ is the anticommutator of Ric and α (cf. (5.15)). Formula (5.17) is of crucial importance for many arguments presented in this paper, the first of which appears in §9.

To establish (5.17), let us first note that its local-coordinate form is

$$\begin{aligned} W_{jklm}\alpha^{lm} \,=\, & -\alpha_{lj,k}{}^{l} \,-\, \alpha_{kl,j}{}^{l} \,-\, \alpha_{lk,}{}^{l}{}_{j} \,+\, \alpha_{lj,}{}^{l}{}_{k} \\ & +\, \frac{n-4}{n-2}\left(R^l_k\alpha_{jl} \,+\, R^l_j\alpha_{lk}\right) \,+\, \frac{2\mathrm{s}}{(n-1)(n-2)}\,\alpha_{jk}\,, \end{aligned} \tag{5.18}$$

as one easily verifies using (5.13) (for $A = W$, $\beta = \alpha$), (4.41), (4.21) and the paragraph following formula (4.22) in §4, as well as (2.12). Setting $P_{jk} = \alpha_{jl,}{}^{l}{}_{k} - \alpha_{jl,k}{}^{l}$ and contracting (4.29) (with $F = \alpha$), we find that $P_{jk} \,=\, R^l{}_{kj}{}^s\alpha_{sl} + R^l{}_{kl}{}^s\alpha_{js} = R_{ljks}\alpha^{ls} + R^s_k\alpha_{js}$ (cf. also (4.32) and (4.37)). Furthermore, $(R_{lkjs} - R_{ljks})\alpha^{ls} = R_{jkls}\alpha^{ls}$, in view of (4.32) and the relation $(R_{ljks} + R_{lksj} + R_{lsjk})\alpha^{ls} = 0$, immediate from the Bianchi identity (4.33). In other words, if we also set $\beta = \mathrm{div}\,(\nabla\alpha - d\alpha) - d\,(\mathrm{div}\,\alpha)$, we have $\beta_{jk} = P_{kj} - P_{jk}$ and so, from these equalities and (5.13), $\beta = 2R\alpha - \{\mathrm{Ric},\,\alpha\}$. Since, by (5.6), (5.15) and (5.16), $2(n-1)(n-2)W\alpha = 2(n-1)(n-2)R\alpha - 2(n-1)\{\mathrm{Ric},\,\alpha\} + 2\,\mathrm{s}\alpha$, replacing $2R\alpha$ with $\beta + \{\mathrm{Ric},\,\alpha\}$ we now obtain $2(n-1)(n-2)W\alpha = (n-1)(n-2)\beta + (n-1)(n-4)\{\mathrm{Ric},\,\alpha\} + 2\,\mathrm{s}\alpha$, which proves (5.17).

Formula (5.17) becomes particularly simple (and useful) for *parallel* bivector fields α on *four-dimensional* pseudo-Riemannian manifold (M, g). Namely,

$$W\alpha \,=\, \frac{\mathrm{s}}{6}\,\alpha \qquad \text{if} \qquad \nabla\alpha \,=\, 0 \quad \text{and} \quad \dim M = 4\,. \tag{5.19}$$

Moreover, equality (5.17) has the following interesting consequence for parallel bivector fields α on pseudo-Riemannian *Einstein* manifolds (M, g) of any dimension $n > 2$:

$$W\alpha \,=\, \frac{(n-2)\,\mathrm{s}}{n(n-1)}\,\alpha \qquad \text{if} \quad \nabla\alpha \,=\, 0\,, \quad \mathrm{Ric} \,=\, \frac{\mathrm{s}}{n}\,g\,, \quad n = \dim M \geq 3\,. \tag{5.20}$$

Again, given a tensor A with (5.11) at a point x of (M, g) and tangent vectors $u, v, u', v' \in T_xM$ we have, by (2.15) and (2.17),

$$\langle A(u,v)u', v'\rangle \,=\, \langle A(u\wedge v), u'\wedge v'\rangle\,. \tag{5.21}$$

We will then also use the notation

$$A(u \wedge v) \,=\, A(u,v)\,. \tag{5.22}$$

By an *algebraic curvature tensor* at a point x of a pseudo-Riemannian manifold (M,g) we mean any tensor A with (5.11) whose components also satisfy $A_{jklm} + A_{jlmk} + A_{jmkl} = 0$, that is, share all the algebraic symmetries of R_{jklm} listed in (4.32) and (4.33). Besides $A = R(x)$, obvious examples of algebraic curvature tensors include the products $B \circledast C$ of any two symmetric twice-covariant tensors B, C at x given by (5.7). Thus, by (5.6), the Weyl tensor W is an algebraic curvature tensor field as well, that is, it satisfies the *first Bianchi identity*

$$W_{jklm} \,+\, W_{kljm} \,+\, W_{ljkm} \,=\, 0 \tag{5.23}$$

and the *skew-symmetry relations*

$$W_{jklm} \,=\, -\,W_{kjlm} \,=\, -\,W_{jkml} \,=\, W_{lmjk}\,, \tag{5.24}$$

with $W_{jklm} = W_{jkl}{}^p g_{pm}$. In addition, W satisfies the relation

$$W_{jkl}{}^k \,=\, 0\,, \tag{5.25}$$

i.e., the "Ricci contraction" (in fact, any contraction) of W is zero. (This is immediate from (5.8).) Note that notation (5.22) then agrees with the usage of $R(u,v)$ and $W(u,v)$ in (4.23), (4.28) (and in formula (5.34) below).

Any tensor field A with (5.8) on a pseudo-Riemannian manifold (M,g) (such as $A = R$ or $A = W$) can be treated as a differential 2-form valued in 2-forms. The divergence $\operatorname{div} A$ of A then can be defined as in (4.41). Thus, $\operatorname{div} A$ is a differential 1-form valued in 2-forms, with the local component functions

$$[\operatorname{div} A]_{klm} \,=\, A^j{}_{klm,j}\,. \tag{5.26}$$

The divergences of R or W and satisfy the well-known identities

$$\operatorname{div} R \,=\, d\,\mathrm{Ric}\,, \qquad d\mathrm{s} \,=\, -\,2\,\mathrm{ctr}\,[d\,\mathrm{Ric}] \tag{5.27}$$

(where ctr is a specific contraction), and, when $n = \dim M \geq 3$,

$$2(n-1)(n-2)\operatorname{div} W \,=\, (n-3)\,d\,H\,, \quad \text{with} \quad H \,=\, 2(n-1)\,\mathrm{Ric} \,-\, \mathrm{s}\,g\,, \tag{5.28}$$

where the exterior derivative is applied to symmetric twice-covariant C^∞ tensor fields (such as Ric), viewed as differential 1-forms valued in 1-forms. Both identities, the local-coordinate versions of which are $R_{jklm,}{}^j \,=\, R_{km,l} - R_{kl,m}$, $\mathrm{s}_{,k} \,=\, 2\,R^j{}_{k,j}$, and

$$2(n-1)(n-2)\,W_{jklm,}{}^j \,=\, (n-3)\,[H_{km,l} \,-\, H_{kl,m}]\,, \tag{5.29}$$

with $H_{jk} = 2(n-1)\,R_{jk} - \mathrm{s}\,g_{jk}$, can be easily obtained by contracting the second Bianchi identity in dimension n and using (5.8). We consequently have

Lemma 5.2. *Every Einstein manifold (M,g) satisfies the conditions*

$$\operatorname{div} R = \operatorname{div} W = 0\,, \tag{5.30}$$

where R and W are treated as 2-forms valued in 2-forms. ■

Any differential 1-form valued in 2-forms, such as $\operatorname{div} W$, acts on bivectors α, assigning to each of them the 1-form $[\operatorname{div} W]pt\alpha$ with (cf. (5.26))

$$([\operatorname{div} W]\alpha)_k = \frac{1}{2}[\operatorname{div} W]_{klm}\alpha^{lm} = \frac{1}{2}\,W^j{}_{klm,j}\alpha^{lm}\,. \tag{5.31}$$

Lemma 5.3. *Given bivectors $\alpha,\beta\in[T_xM]^{\wedge 2}$ at a point x in a pseudo-Riemannian four-manifold (M,g) and a four-times covariant tensor A at x, let us define an exterior 2-form γ at x by $\gamma(u,v)=\langle[A(u,v),\alpha],\,\beta\rangle$, with $A(u,v)$ as in* (5.22)*, where $[\,,\,]$ is the commutator of bivectors treated, with the aid of g, as skew-adjoint operators $T_xM\to T_xM$. Treating γ as a bivector, we then have $\gamma=A[\alpha,\beta]$.*

Proof. For any bivectors $\alpha,\beta\in[T_xM]^{\wedge 2}$, $\langle[\beta,\alpha],\,\beta\rangle=0$ in view of (2.17) and (3.1). Therefore (see the final clause in Remark 3.12), for any three bivectors α,β,ζ, we have $\langle[\zeta,\alpha],\,\beta\rangle=-\langle[\beta,\alpha],\,\zeta\rangle=\langle[\alpha,\beta],\,\zeta\rangle$. Applying this to $\zeta=A(u,v)=A(u\wedge v)$ with fixed vectors u,v, and using self-adjointness of A, we obtain $\gamma(u,v)=\langle[A(u\wedge v),\alpha],\,\beta\rangle=\langle[\alpha,\beta],\,A(u\wedge v)\rangle=\langle A[\alpha,\beta],\,u\wedge v\rangle$. By (2.20), this gives $\gamma(u,v)=(A[\alpha,\beta])(u,v)$, where the bivector $A[\alpha,\beta]$ now is treated as a 2-form. This completes the proof. ■

The g-inner product of algebraic curvature tensors is given by

$$g(A,\tilde A) = \frac{1}{4}\,A^{jklm}\tilde A_{jklm} = \operatorname{Trace} A\tilde A\,. \tag{5.32}$$

If g is positive definite, we define the norm $|A|$ of such a tensor A by $|A|^2=g(A,A)$.

In view of (5.16), relation (5.10), characterizing Einstein metrics in dimension n, becomes the condition

$$R = W + \frac{\mathrm{s}}{n(n-1)}\,,\qquad n=\dim M\,, \tag{5.33}$$

imposed on bundle morphisms $[TM]^{\wedge 2}\to[TM]^{\wedge 2}$, where the scalar on the right-hand side stands for the corresponding multiple of Id. In view of (5.22), this can be rewritten as

$$R(u,v) = W(u,v) + \frac{\mathrm{s}}{n(n-1)}\,u\wedge v\,,\qquad n=\dim M\,, \tag{5.34}$$

for any tangent vectors u,v, with $R(u,v)$ as in (4.23) or (4.28).

We will also need the fact that, for bivectors α, β, tangent vectors u, v and symmetric 2-tensors A, B,

$$2\,\langle[(A\circledast B)(u,v),\alpha],\beta\rangle = \langle(A[\alpha,\beta]B + B[\alpha,\beta]A)u,v\rangle\,. \tag{5.35}$$

(notation as in (5.7) or (4.28)). In fact, (5.7) yields

$$2\,(A \circledast B)(u,v) \,=\, (Au)\wedge(Bv) \,+\, (Bu)\wedge(Av)\,. \tag{5.36}$$

On the other hand, relation

$$(B\circledast B)_{jklm} \,=\, B_{jl}B_{km} \,-\, B_{kl}B_{jm} \tag{5.37}$$

for any symmetric twice-covariant tensor B, immediate from (5.7), shows that

$$2\,(v\otimes v)\circledast(w\otimes w) \,=\, (v\wedge w)\otimes(v\wedge w)\,, \tag{5.38}$$

for tangent vectors v, w. In fact, $(v\otimes v)_{jk} = v_j v_k$ (see (2.13)), while the components of the right-hand side can be evaluated from (5.12) and (2.15). Consequently,

$$A\circledast A \,=\, \pm\,\alpha\otimes\alpha \qquad \text{if} \quad A \,=\, v\otimes v \,\pm\, w\otimes w \quad \text{and} \quad \alpha \,=\, v\wedge w\,, \tag{5.39}$$

and

$$A\circledast A \,=\, -\,\alpha\otimes\alpha \qquad \text{if} \quad A \,=\, v\otimes w \,+\, w\otimes w \quad \text{and} \quad \alpha \,=\, v\wedge w\,. \tag{5.40}$$

§6. Special properties of dimension four

This section lists some facts showing the extent to which four-dimensional Riemannian geometry differs from what one has in other dimensions. The "ultimate reason", if any, may well be reducibility of the Lie algebra $\mathfrak{so}(4)$ (Remark 6.7), unique in this respect among all $\mathfrak{so}(n)$.

The order in which topics are covered here is dictated by how soon they will be used. The initial part (up to and including Lemma 6.18) is needed immediately, that is, in §7. The reader can in this way skip the remainder of this section until it is called for by further applications (as indicated below).

Let (M,g) be an oriented Riemannian 4-manifold. There exists a unique bundle morphism $*:[TM]^{\wedge 2}\to[TM]^{\wedge 2}$, called the *Hodge star* (acting on bivectors), such that

$$*(e_1\wedge e_2) \,=\, e_3\wedge e_4 \tag{6.1}$$

for any $x\in M$ and any positive-oriented orthonormal basis $e_1,\dots,e_4$ of T_xM. (For details, see formulae (37.9) and (37.13) in §37.) It is now clear from (6.1) that the operator $*$ is an involution, i.e.,

$$*^2 \,=\, \mathrm{Id}\,, \tag{6.2}$$

and it easily follows from (6.1) that

$$*:[T_xM]^{\wedge 2}\to[T_xM]^{\wedge 2} \quad \text{is self-adjoint} \tag{6.3}$$

relative to the inner product $\langle\,,\rangle$ of bivectors, given by (2.17). (See also §37, formulae (37.21) and (37.10).)

According to Remark 3.2, equality (6.2) gives rise to a direct-sum decomposition

$$[TM]^{\wedge 2} \,=\, \Lambda^+M\oplus\Lambda^-M\,, \qquad \Lambda^\pm M \,=\, [\Lambda^\mp M]^\perp\,, \tag{6.4}$$

of $[TM]^{\wedge 2}$ into the subbundles $\Lambda^\pm M$; specifically, the fibre $\Lambda^\pm_x M$ of $\Lambda^- M$ over any $x\in M$ is the (± 1)-eigenspace of $*$ at x. Elements of $\Lambda^+_x M$ and $\Lambda^-_x M$ are called *self-dual* and, respectively, *anti-self-dual* bivectors at x. Note that mutual orthogonality of $\Lambda^+_x M$ and $\Lambda^-_x M$, i.e., the second relation in (6.4), now is obvious from (6.3) (see Remark 3.17(i)). The $\Lambda^\pm_x M$-components $\alpha^\pm$ of any bivector $\alpha\in[T_xM]^{\wedge 2}$, $x\in M$, are obviously (cf. Remark 3.2) given by

$$2\,\alpha^\pm \,=\, \alpha \,\pm\, *\,\alpha\,. \tag{6.5}$$

Lemma 6.1. *Suppose that we are given a point* x *of an oriented Riemannian* 4*-manifold* (M,g) *and a bivector* $\alpha \in \Lambda^+_x M$*. Defining the real number* $r \ge 0$ *by* $2r^2 = \langle \alpha, \alpha \rangle$*, we then have*

$$\alpha \ = r\,[e_1 \wedge e_2 \ + \ e_3 \wedge e_4] \tag{6.6}$$

for some positive-oriented orthonormal basis $e_1, \ldots, e_4$ *of* $T_x M$*, and*

$$2\alpha^2 \ = \ -\langle \alpha, \alpha \rangle\,, \tag{6.7}$$

where α *is treated, with the aid of* g*, as a skew-adjoint operator* $T_x M \to T_x M$*. Any two bivectors* $\alpha, \beta \in \Lambda^+_x M$ *satisfy*

$$\alpha\beta \ + \ \beta\alpha = -\langle \alpha, \beta \rangle\,. \tag{6.8}$$

In both (6.7) *and* (6.8)*, the scalar on the right-hand side stands for the corresponding multiple of* Id*.*

Proof. Due to skew-adjointness of $\alpha : T_x M \to T_x M$, there exists a 2-dimensional α-invariant vector subspace $P \subset T_x M$. In fact, a nonreal complex root of the characteristic polynomial of α immediately leads to such a subspace; on the other hand, if one of these roots is real, then α has an eigenvector $v \in T_x M$ and, as $v^\perp$ is a 3-dimensional α-invariant subspace of $T_x M$ (see Remark 3.17(ii)), α also has an eigenvector $w \in v^\perp$, and we may set $P = \operatorname{Span}\{v, w\}$.

Obviously, $P^\perp$ then is α-invariant as well. Choosing any positive-oriented orthonormal basis $e_1, \ldots, e_4$ of $T_x M$ with $e_1, e_2 \in P$ and $e_3, e_4 \in P^\perp$, we have (by skew-adjointness)

$$\alpha e_1 = r e_2\,, \quad \alpha e_2 = -\,r e_1\,, \quad \alpha e_3 = q e_4\,, \quad \alpha e_4 = -\,q e_3 \tag{6.9}$$

for some $r, q \in \mathbf{R}$. Hence, by (2.22), $\alpha \ = \ r\, e_1 \wedge e_2 + q\, e_3 \wedge e_4$. However, as $*\alpha = \alpha$, formula (6.1) now gives $q = r$. Thus, (6.6) follows and, changing the signs of both e_2, e_4 if necessary, we may assume that $r \ge 0$. Now (6.7) is immediate from (6.9) with $q = r$. Since both sides of (6.8) are bilinear and symmetric in α and β, equality (6.8) follows from (6.7) (see Remark 3.12). This completes the proof. ∎

The spaces $\Lambda^\pm_x M$ are all 3-dimensional. More precisely, we have

Lemma 6.2. *Let* (M,g) *be an oriented Riemannian* 4*-manifold and let* $x \in M$*. For any positive-oriented orthonormal basis* $e_1, \ldots, e_4$ *of* $T_x M$*, formula*

$$\frac{\pm\, e_1 \wedge e_2 \ + \ e_3 \wedge e_4}{\sqrt{2}}\,, \quad \frac{\pm\, e_1 \wedge e_3 \ + \ e_4 \wedge e_2}{\sqrt{2}}\,, \quad \frac{\pm\, e_1 \wedge e_4 \ + \ e_2 \wedge e_3}{\sqrt{2}}\,, \tag{6.10}$$

defines a basis of $\Lambda^\pm_x M$*, which is orthonormal relative to the inner product of bivectors characterized by* (2.17)*. The bundles* $\Lambda^+ M$ *and* $\Lambda^- M$ *carry natural orientations such that the bases* (6.10) *all are positive-oriented.*

Proof. Our assertion is immediate from (6.1), (2.21) and the fact that positive-oriented orthonormal bases of $T_x M$ form a connected set (cf. Lemma 3.5(ii)). ∎

Corollary 6.3. *Let (M,g) be an oriented Riemannian 4-manifold, and let the bivector space $[T_xM]^{\wedge 2}$ at any point $x \in M$ be identified, as in (2.12), with the Lie algebra $\mathfrak{so}(T_xM)$ of all skew-adjoint operators $T_xM \to T_xM$. Then Λ_x^+M and Λ_x^-M commute in the Lie algebra $[T_xM]^{\wedge 2} = \mathfrak{so}(T_xM)$, in the sense that $[\alpha^+, \alpha^-] = 0$ whenever $\alpha^\pm \in \Lambda_x^\pm M$.*

Proof. Let us write a fixed bivector $\alpha \in \Lambda_x^+M$ in the form (6.6) for some positive-oriented orthonormal basis $e_1, \dots, e_4$ of T_xM (see Lemma 6.1). The numerators of the expressions (6.10) with the sign $\pm$ equal to 'minus' then form a basis of Λ_x^-M (Lemma 6.2). On the other hand, for mutually orthogonal vectors $e_1, \dots, e_4$, (2.28) implies that $e_1 \wedge e_2 + e_3 \wedge e_4$ commutes with $-e_1 \wedge e_2 + e_3 \wedge e_4$, $-e_1 \wedge e_3 + e_4 \wedge e_2$ and $-e_1 \wedge e_4 + e_2 \wedge e_3$, which completes the proof. ■

Corollary 6.4. *Suppose that x is a point of an oriented Riemannian 4-manifold (M,g), and bivectors at x are treated, according to (2.12), as skew-adjoint operators $T_xM \to T_xM$. For any fixed nonzero self-dual bivector $\alpha \in \Lambda_x^+M$ and a bivector $\beta \in [T_xM]^{\wedge 2}$, the following two conditions are equivalent:*

(a) *β anticommutes with α; in other words, $\alpha\beta + \beta\alpha = 0$;*
(b) *β is self-dual and orthogonal to α, i.e., $\beta \in \Lambda_x^+M$ and $\langle \alpha, \beta \rangle = 0$.*

Proof. (b) implies (a) as a consequence of (6.8). Conversely, let β satisfy (a). Since $\alpha \neq 0$, we may normalize α so that $\langle \alpha, \alpha \rangle = 2$. By (6.7) we now have $\alpha^2 = -\mathrm{Id}$ and, in particular, α is an isomorphism. We can now write $\beta = c\alpha + \beta' + \beta^-$ with $c \in \mathbf{R}$, $\beta' \in \Lambda_x^+M$, $\langle \alpha, \beta' \rangle = 0$ and $\beta^- \in \Lambda_x^-M$. Denoting $\{\,,\,\}$ the anticommutator, with $\{\alpha, \beta\} = \alpha\beta + \beta\alpha$, we now have $\{\alpha, \beta'\} = 0$ (see (6.8)), and so, since $\beta^-\alpha = \alpha\beta^-$ by Corollary 6.3, we have $0 = \{\alpha, \beta\} = -2c + 2\alpha\beta^-$, where c stands for c times the identity. Taking the trace, we obtain, by (2.17), $c = 0$ (as $\alpha \in \Lambda_x^+M$ is orthogonal to $\beta^- \in \Lambda_x^-M$, cf. (6.4)). Thus, $\alpha\beta^- = 0$ and, since α is an isomorphism, we obtain $\beta^- = 0$. In other words, $\beta = \beta'$, i.e., (b) follows from (a). This completes the proof. ■

Corollary 6.5. *Suppose that (M,g) is a four-dimensional oriented Riemannian manifold, $x \in M$, and $\alpha_1, \alpha_2, \alpha_3$ is a positive-oriented orthogonal basis of Λ_x^+M or Λ_x^-M consisting of vectors of length $\sqrt{2}$, i.e., such that*

$$\langle \alpha_j, \alpha_k \rangle = 2\,\delta_{jk}\,, \qquad \text{for} \quad j,k \in \{1,2,3\}\,. \tag{6.11}$$

Treated as skew-adjoint operators $T_xM \to T_xM$, the α_j then satisfy the quaternion-units relations

$$\alpha_j^2 = -\mathrm{Id}\,, \qquad \alpha_j\alpha_k = \alpha_l = -\alpha_k\alpha_j \qquad \text{if} \quad \varepsilon_{jkl} = 1\,, \tag{6.12}$$

where, for any indices $j,k,l \in \{1,2,3\}$, ε_{jkl} is the Ricci symbol, *equal to the signum of the permutation (j,k,l) of $\{1,2,3\}$, if j,k and l are all distinct, and to 0 otherwise.*

Proof. Reversing the orientation, if necessary, we may assume that $\alpha_1, \alpha_2, \alpha_3 \in \Lambda_x^+M$. Relations $\alpha_j^2 = -\mathrm{Id}$ for $j = 1,2,3$ are obvious from (6.7). In view of (6.8), α_1 and α_2 anticommute, and so their composite $\beta = \alpha_1\alpha_2$ is again skew-adjoint. Moreover, β anticommutes with both α_1 and α_2 (again by (6.8)), and so Corollary 6.4 implies that $\beta \in \Lambda_x^+M$ and $\langle \alpha_1, \beta \rangle = \langle \alpha_2, \beta \rangle = 0$. Also, $\beta^2 = -\mathrm{Id}$ (since the same is true for α_1 and α_2), and so (2.17) yields $\langle\beta,\beta\rangle = 2$. In this way, $\beta = \pm\,\alpha_3$. The sign $\pm$ appearing here is the same for all bases $\alpha_1,\alpha_2,\alpha_3$ with the stated properties. Namely, it equals $\mathrm{sgn}\langle\alpha_1,\alpha_2\alpha_3\rangle$ which, in turn, does not depend on the basis used since positive-oriented orthonormal bases of T_xM form a connected set (cf. Lemma 3.5(ii)). To see that this sign is actually 'plus', it now suffices to test it, using (2.27), on just one basis, for instance, one formed by the numerators of the expressions (6.10) (with either fixed sign $\pm$ preceding e_1), and any given positive-oriented orthonormal basis $e_1,\dots,e_4$ of T_xM. This completes the proof. ■

Corollary 6.6. *At any point x of an oriented Riemannian 4-manifold (M,g), the spaces Λ^+_xM and Λ^-_xM are Lie subalgebras of the Lie algebra $[T_xM]^{\wedge 2} = \mathfrak{so}(T_xM)$, both isomorphic to the Lie algebra formed by $\mathbf{R}^3$ with the vector product. More precisely, for $\alpha_1,\alpha_2,\alpha_3$ as in* Corollary 6.5*, we have*

$$[\alpha_j,\,\alpha_k] \,=\, 2\,\alpha_l \qquad \text{if} \quad \varepsilon_{jkl} = 1\,. \tag{6.13}$$

This is obvious from Corollary 6.5. ■

Remark 6.7. The assertions of Corollaries 6.3 and 6.6 describe a well-known Lie-algebra isomorphism $\mathfrak{so}(4) \approx \mathfrak{so}(3)\oplus\mathfrak{so}(3)$.

It will be useful to have the following analogue of Corollary 6.4.

Corollary 6.8. *Let there be given an oriented Riemannian 4-manifold (M,g), a point $x\in M$, and a fixed nonzero self-dual bivector $\alpha\in\Lambda^+_xM$. For any bivector $\beta\in[T_xM]^{\wedge 2}$, the following two conditions are equivalent*:

- (i) *β is anti-self-dual, that is, $\beta\in\Lambda^-_xM$.*
- (ii) *β commutes with α and is orthogonal to α; in other words, $\alpha\beta = \beta\alpha$ and $\langle\alpha,\beta\rangle = 0$.*

Proof. (i) implies (ii) as a consequence of (6.4) and Corollary 6.3. Conversely, let β satisfy (ii). As $\alpha\neq 0$, we may normalize α so that $\langle\alpha,\alpha\rangle = 2$ and, by (6.7), $\alpha^2 = -\,\mathrm{Id}$. Thus, α is an isomorphism. Writing $\beta = c\alpha+\beta'+\beta^-$ with $c\in\mathbf{R}$, $\beta'\in\Lambda^+_xM$, $\langle\alpha,\beta'\rangle = 0$ and $\beta^-\in\Lambda^-_xM$, we have $c=0$ since $\langle\alpha,\beta\rangle = 0$, and the commutator relation with $[\alpha,\beta]=0$ now becomes $0=[\alpha,\beta']+[\alpha,\beta^-]$. Since $[\alpha,\beta^-]=0$ (Corollary 6.3), we thus have $\alpha\beta'=\beta'\alpha$ while, by (6.8) with $\beta=\beta'$, $\alpha\beta' = -\,\beta'\alpha$. Therefore, $\alpha\beta'=0$ and, since α is an isomorphism, we obtain $\beta'=0$. Consequently, $\beta=\beta^-$, and so (i) follows from (ii). This completes the proof. ■

Lemma 6.9. *Given an oriented Riemannian 4-manifold (M,g), let W and $*$ be the Hodge star and the Weyl tensor of (M,g), both treated as bundle morphisms $[TM]^{\wedge 2}\to[TM]^{\wedge 2}$. Then*

- (i) *W and $*$ commute.*
- (ii) *We have $\mathrm{Trace}\,W = \mathrm{Trace}\,[W*] = \mathrm{Trace}\,[*W] = 0$ everywhere in M.*

Proof. (i) follows from (6.1) and (5.25); for details, see the paragraph following formula (38.7) in §38. As for (ii), note that, according to (5.13), $2\,\mathrm{Trace}\,W$ equals $W_{jk}{}^{jk}$, which is zero by (5.25). On the other hand, we obviously have $\mathrm{Trace}\,[W*] = \sum_{j<k}\langle W(*(e_j\wedge e_k)),e_j\wedge e_k\rangle$, where $e_1,\dots,e_4$ is any positive-oriented orthonormal

basis of T_xM (and so, by (2.21), the $e_j \wedge e_k$ with $j < k$ form an orthonormal basis of $[T_xM]^{\wedge 2}$). Hence, in view of (6.1), $\mathrm{Trace}\,[W\,*]$ equals $W_{3412} + W_{4213} + W_{2314} + W_{1423} + W_{3124} + W_{1234}$ with $W_{jklm} = g(W(e_j, e_k)e_l, e_m) = \langle W(e_j \wedge e_k), e_l \wedge e_m\rangle$ (cf. (5.21)), which is zero in view of (5.23) and (5.24). ■

In view of Lemma 6.9(i), for any oriented Riemannian 4-manifold (M, g), the direct-summand subbundles $\Lambda^{\pm}M$ of $[TM]^{\wedge 2}$ are invariant under the bundle morphism $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$, i.e.,

$$W(\Lambda^{\pm}M) \subset \Lambda^{\pm}M\,. \tag{6.14}$$

One denotes $W^{\pm}$ the restriction of W to the $\Lambda^{\pm}M$. In this way, $W^{\pm}$ becomes a bundle morphism $\Lambda^{\pm}M \to \Lambda^{\pm}M$.

It follows from (6.5) that, for any tangent vectors u, v, the operator $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$ satisfies

$$[W^{\pm}(u,v), \alpha^{\pm}] \,=\, [W(u,v), \alpha]^{\pm}\,, \qquad [W^{\pm}(u,v), \alpha^{\mp}] \,=\, 0\,. \tag{6.15}$$

(Note, again, that the commutator of bivectors at any point $x \in M$ makes sense since they can be viewed as skew-adjoint operators $T_xM \to T_xM$.) In fact, (6.15) is clear from Corollary 6.3, as $W(u,v) = W(u \wedge v)$ (and similarly for W^+, W^-), while $W^{\pm}$ is a bundle morphism with

$$W^{\pm} : [TM]^{\wedge 2} \to \Lambda^{\pm}M\,, \qquad W^{\pm} \,=\, 0 \quad \text{on} \quad \Lambda^{\mp}M\,. \tag{6.16}$$

In other words, the same symbols $W^{\pm}$ as for $W^{\pm} : \Lambda^{\pm}M \to \Lambda^{\pm}M$ are also used for their obvious extensions of to bundle morphisms $[TM]^{\wedge 2} \to [TM]^{\wedge 2}$ characterized by (6.16). We then clearly have

$$2W^{\pm} \,=\, W \,\pm\, *W\,, \tag{6.17}$$

and so, by Lemma 6.9(ii),

$$\mathrm{Trace}\, W^{\pm} \,=\, 0\,. \tag{6.18}$$

In other words, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of W^+ (or, W^-) at any given point satisfy

$$\lambda_1 \,+\, \lambda_2 \,+\, \lambda_3 \,=\, 0\,. \tag{6.19}$$

Remark 6.10. Since the Hodge star $*$ is parallel (due to its naturality), either of the subbundles $\Lambda^{\pm}M$ of $[TM]^{\wedge 2}$ is parallel, as defined in Remark 4.7. This leads to connections in the bundles $\Lambda^{+}M$ and $\Lambda^{-}M$ obtained as the natural restrictions to these parallel subbundles of the connection in $[TM]^{\wedge 2}$ induced by the Levi-Civita connection ∇ of TM. They both amount to nothing else than the ordinary covariant derivative operation applied to (anti)self-dual bivector fields. This is why they will also be denoted ∇ and referred to as the *Levi-Civita connections* in $\Lambda^{+}M$ and $\Lambda^{-}M$.

Lemma 6.11. *Given a point x of any oriented Riemannian 4-manifold (M,g) and a traceless, twice-covariant, symmetric tensor B at x, the anticommutator operator*

$$[T_xM]^{\wedge 2} \ni \alpha \mapsto \{B,\,\alpha\} \in [T_xM]^{\wedge 2}\,,$$

with $\{B,\,\alpha\} = B\alpha + \alpha B$, as in (5.15)*, interchanges Λ_x^+M and Λ_x^-M, that is, sends $\Lambda_x^\pm M$ into $\Lambda_x^\mp M$.*

Proof. Fix $\alpha \in \Lambda_x^\pm M$. For B as above, $\beta = B\alpha + \alpha B$ is obviously skew-adjoint, i.e., a bivector and, by (6.7), β commutes with α. Similarly, in view of (2.17), (3.1) and the assumption that $\operatorname{Trace} B = 0$, we have $\langle\alpha,\beta\rangle = 0$. From Corollary 6.8 aplied to this or the opposite orientation, we now conclude that $\beta \in \Lambda_x^\mp M$ unless $\alpha = 0$. Since $\alpha = 0$ implies $\beta = 0$, this completes the proof. ■

Remark 6.12. Lemma 6.11 has the following important consequence which, however, will be needed only for Lemma 6.25 near the end of this section. Namely, we have the relations

$$[R\alpha]^\pm \,=\, (W \,+\, \mathrm{s}/12)\,\alpha\,, \qquad \text{for} \quad \alpha \in \Lambda_x^\pm M\,, \tag{6.20}$$

$$[R\alpha]^\mp \,=\, (g \circledast \mathrm{E})\alpha \,=\, \mathrm{E}\alpha \,+\, \alpha\mathrm{E} \qquad \text{for} \quad \alpha \in \Lambda_x^\pm M\,, \tag{6.21}$$

valid for the curvature tensor R acting on (anti)self-dual bivectors α at any point x of any oriented Riemannian 4-manifold (M,g), where E is the traceless Ricci tensor of (M,g) (notation of (5.5), (5.13), (6.5)). In fact, given $\alpha \in \Lambda_x^\pm M$, we have $R\alpha = (W + \mathrm{s}/12)\alpha + (g \circledast \mathrm{E})\alpha$ from (5.9) with $n = 4$. However, $W\alpha = W^\pm\alpha$, while (5.15) gives $2(g \circledast \mathrm{E})\alpha = \mathrm{E}\alpha \,+\, \alpha\mathrm{E}$. We thus obtain the following characterization, due to Singer and Thorpe (1969), of the irreducible components of the curvature (§5) at a point x in any oriented Riemannian 4-manifold: For any $\alpha \in \Lambda_x^\pm M$,

$$12\,R\alpha \,=\, \mathrm{s}\alpha \,+\, 12\,W^\pm\alpha \,+\, 6\,(\mathrm{E}\alpha \,+\, \alpha\mathrm{E})\,. \tag{6.22}$$

The first two terms on the right-hand side are in $\Lambda_x^\pm M$ (see (6.14)), while the last term is in $\Lambda_x^\mp M$ (Lemma 6.11). This proves (6.20) and (6.21), and shows that each of s, $W^\pm$, and E accounts for one of four "parts" of R acting on bivectors a part which is a multiple of Id, a part which is zero on $\Lambda_x^\mp M$ and leaves $\Lambda_x^\pm M$ invariant, forming a traceless operator in it, for either sign $\pm$ (cf. Lemma 6.9(ii)); and a part that interchanges Λ_x^+M and Λ_x^-M.

Remark 6.13. Much of what was established here in the Riemannian case remains valid also for pseudo-Riemannian 4-manifolds with indefinite metrics of the *neutral* sign pattern $- - + +$. Namely, we then still have relations (6.2), (6.3) (see (37.21), (37.10) in §37). Consequently, we also then have (6.4) and (6.5). Relations (6.7) and (6.8) still hold for any two bivectors $\alpha,\beta \in \Lambda_x^+M$ (Proposition 37.5). The spaces Λ_x^+M and Λ_x^-M still are mutually commuting Lie subalgebras of $[T_xM]^{\wedge 2} = \mathfrak{so}(T_xM)$ (Proposition 37.2), and Corollaries 6.4 and 6.8 become valid, with exactly the same proofs, if we replace the word 'nonzero' by 'non-null'. Similarly, Lemma 6.9 and formulae (6.14) – (6.16) remain valid, without any change. As for Lemma 6.11, it still holds; the proof works for α with $\langle\alpha,\alpha\rangle \neq 0$, and so the assertion still follows since $B\alpha + \alpha B$ is linear in α. Hence, we also have (6.20) and (6.21).

The bundle morphism $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$ may of course be regarded as a $[TM]^{\wedge 2}$-valued 2-form on M. Then $W^{\pm}$ are the $\Lambda^{\pm}M$-components of W, both when the decomposition is applied valuewise (as it would be for a $[TM]^{\wedge 2}$-valued form of any degree), or argumentwise (as it would be done for real-valued 2-forms and, more generally, for 2-forms valued in any vector bundle over M).

Similarly, $\operatorname{div} W^{\pm}$ then are nothing else than the $\Lambda^{\pm}M$-components of $\operatorname{div} W$, treated as a $[TM]^{\wedge 2}$-valued 1-form on M. (This is clear from (4.41) and Remark 6.10.) Thus, Lemma 5.2 yields

Lemma 6.14. *For every oriented Riemannian Einstein 4-manifold we have*

$$\operatorname{div} W^{+} \,=\, \operatorname{div} W^{-} \,=\, 0\,. \tag{6.23}$$

■

Lemma 6.15. *Let $F : \mathcal{E} \to \mathcal{E}$ be a C^k morphism, $1 \le k \le \infty$, in a C^∞ real/complex vector bundle $\mathcal{E}$ of fibre dimension 2 or 3 over a manifold M. Suppose that the restriction $F(x) : \mathcal{E}_x \to \mathcal{E}_x$ of F to the fibre $\mathcal{E}_x$ is diagonalizable for each $x \in M$, and let $\#\operatorname{spec} F : M \to \{1,2,3\}$ be the function which assigns to each $x \in M$ the number of distinct eigenvalues of the operator $F(x) : \mathcal{E}_x \to \mathcal{E}_x$. Finally, let M' denote the subset of M consisting of all points x such that $\#\operatorname{spec} F$ is constant on some neighborhood of x. Then*

- (i) *M' is open and dense in M.*
- (ii) *Every point $x \in M'$ has a neighborhood U' on which there exist C^k-differentiable* eigenvalue functions *λ_a of F, $a = 1, \ldots, q$, where $q = 2$ or $q = 3$, such that at each $x \in U'$ the unordered system $\{\lambda_1, \ldots, \lambda_q\}$ represents the eigenvalues of the operator $F(x)$ along with their correct multiplicities. If the eigenvalues of F are all real at every point of M, then such functions λ_a may be defined globally on each connected component U' of M'.*
- (iii) *Given a point $x \in M'$ with $\#\operatorname{spec} F = m$ at x, there exists C^k-differentiable* eigenvector sections *e_a of F, defined on a neighborhood U of x contained in M', such that with $Fe_a = \lambda_a e_a$ and, at every $y \in U$, the $e_a(y)$ form a basis of $\mathcal{E}_y$.*
- (iv) *If, in addition, $\mathcal{E}$ carries a positive-definite Riemannian/Hermitian fibre metric that makes all $F(x)$ self-adjoint, then the eigenvector sections e_a in* (iii) *may be chosen orthonormal at each point.*

Proof. Suppose that μ is a simple eigenvalue of $F(z) : \mathcal{E}_z \to \mathcal{E}_z$ at some given point $z \in M$. Then there exist neighborhoods U of z in M and $\varOmega$ of μ in the scalar field $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$) with a C^k function $\lambda : U \to \varOmega$ such that, for each $x \in U$, $\lambda(x)$ is both a simple eigenvalue of the operator $F(x) : \mathcal{E}_x \to \mathcal{E}_x$, as well as its only eigenvalue ν with the property that $\nu \in \varOmega$. In fact, this is an obvious consequence of the implicit function theorem applied to the function $\varPhi : M \times \mathbf{K} \to \mathbf{K}$ given by

$$\varPhi(t,\lambda) \,=\, \frac{d}{d\nu}\bigg|_{\nu=\lambda} \det\,[F(x) - \nu]\,.$$

(In fact, since μ is a simple eigenvalue of $F(z)$, we have $\varPhi = 0$ at $(x,\lambda) = (z,\mu)$, and $\partial\varPhi/\partial\lambda \ne 0$ in a neighborhood of (z,μ).) Here and in the sequel we treat scalar-valued functions f as bundle morphisms $\mathcal{E} \to \mathcal{E}$, namely, multiplication operators; thus, ν above also stands for $\nu \cdot \mathrm{Id}$.

Openness of M' in M is obvious. To prove its denseness, i.e., assertion (i), it suffices to show that M' intersects every nonempty open subset U of M. In fact, given such U, let $z \in U$ be a point at which $\#\operatorname{spec} F$ attains its maximum value p in U. If $p = 3$, $F(z)$ has three distinct (simple) eigenvalues which, as we have seen, give rise to C^k differentiable eigenvalue functions of F, defined and pairwise distinct at all points of some neighborhood U' of z. If $p = 2$, $F(x)$ has one simple and one double eigenvalue, for every point x of some neighborhood U' of z with $U' \subset U$. In fact, this is true at $x = z$, and so the simple eigenvalue at z can be propagated, as before, to form an eigenvalue function $\lambda_1 : U' \to \mathbf{K}$ of class C^k, while $\#\operatorname{spec} F = 2$ everywhere in U', since λ is valued in *simple* eigenvalues. The double eigenvalues then can similarly be organized into a C^k eigenvalue function $\lambda_2 : U' \to \mathbf{K}$, given by $2\lambda_2 = \operatorname{Trace} F - \lambda_1$. Finally, if $p = 1$, we have $\#\operatorname{spec} F = 1$ and $3F = \operatorname{Trace} F$ everywhere in $U' = U$, which gives us the triple-eigenvalue function $\lambda : U' \to \mathbf{K}$, of class C^k, with $3\lambda = \operatorname{Trace} F$. Thus, M' and U have a nonempty intersection (which contains U'), and (i) follows. At the same time, we have also established (ii). Note that, when the λ_a are all real-valued, we can make them defined globally on each connected component of M' by ordering them so that $\lambda_1 \le \ldots \le \lambda_q$.

Finally, assertion (iii) is nothing else than Lemma 2.2 for the kernel of each of the morphisms $F - \lambda_a$, with λ_a as in (ii), while orthonormaliity in (iv) can be achieved by orthonormalizing the original e_a. This completes the proof. ■

We now proceed to develop some local notations and calculations that will be needed later in §7, §10, §20 and §22. Let (M, g) be an arbitrary oriented Riemannian 4-manifold, and let M' denote the set of all points $x \in M$ such that the number of distinct eigenvalues of W^+, regarded as a vector bundle morphism $\Lambda^+ M \to \Lambda^+ M$, is constant in a neighborhood of x. According to Lemma 6.15, M' is an open dense subset of M and, in a connected neighborhood U of any given point of M', W^+ has C^∞ *eigenvector sections* α_j, $j = 1, 2, 3$, of $\Lambda^+ M$ defined on U and mutually orthogonal at each point, which correspond to C^∞ *eigenvalue functions* λ_j, i.e.,

$$W^+ \alpha_j \,=\, \lambda_j \alpha_j \,, \qquad \langle \alpha_j, \alpha_k \rangle \,=\, 2\,\delta_{jk} \,, \qquad j, k \in \{1, 2, 3\} \,, \tag{6.24}$$

the second relation being obtained by normalization, as in (6.11). We also have

$$2\,W^+ \,=\, \sum_{j=1}^{3} \lambda_j \, \alpha_j \otimes \alpha_j \,. \tag{6.25}$$

In fact, by (6.24) along with (5.13), (2.17) and (5.12), both sides produce the same result when applied to the basis α_j of $\Lambda^+_x M$ at any $x \in U$. From now on we will also assume (6.12), which, by Corollary 6.5, can always be achieved by changing the signs of some α_j, if necessary.

In view of Remark 6.10 and (6.11), there exist 1-forms ξ_1, ξ_2, ξ_3 such that

$$\nabla_v \alpha_j \,=\, \xi_l(v)\, \alpha_k \,-\, \xi_k(v)\, \alpha_l \,, \qquad \text{if} \quad \varepsilon_{jkl} = 1 \,, \tag{6.26}$$

for any tangent vector v.

An oriented Riemannian 4-manifold (M, g) is called *self-dual* if $W^- = 0$ identically on M, and *anti-self-dual* if $W^+ = 0$ everywhere.

The following two lemmas will be proved simultaneously.

Lemma 6.16. *Let an oriented Riemannian 4-manifold (M,g) satisfy the conditions $W^+ = 0$ and $\mathrm{s} = 0$ at every point of M, so that (M,g) is anti-self-dual and its scalar curvature is identically zero. Then the Levi-Civita connection ∇ in Λ^+M is flat.*

Lemma 6.17. *Given an oriented Riemannian 4-manifold (M,g), let the self-dual bivector fields α_j, i.e., sections of Λ^+M, as well as the functions λ_j, and 1-forms ξ_j, $j = 1,2,3$, all defined on an open connected subset U of M, be chosen so as to be of class C^∞ and satisfy* (6.24), (6.12) *and* (6.26). *Denoting* E *the traceless Ricci tensor, we have,*

$$(6.27) \qquad d\xi_j + \xi_k \wedge \xi_l \,=\, (\lambda_j + \mathrm{s}/12)\,\alpha_j \,+\, (\mathrm{E}\alpha_j \,+\, \alpha_j\mathrm{E})/2 \qquad \text{if} \quad \varepsilon_{jkl} = 1\,.$$

Proof of Lemmas 6.16 and 6.17. Let β_j, $j = 1,2,3$, be the 2-forms given by $\beta_j(u,v) = \langle[R(u,v),\alpha_k],\alpha_l\rangle$ if $\varepsilon_{jkl} = 1$, for tangent vectors u,v. According to Lemma 5.3 and (6.13), we have $\beta_j = R[\alpha_k,\,\alpha_l] = 2R\alpha_j$, which in view of (6.22) equals twice the right-hand side of (6.27). However, applying (4.27) to $F = \alpha_k$ and using (6.26), (2.15), (2.16) and (6.11), we in turn obtain, from our original definition of β_j, $\beta_j(u,v) = 2\,[d\xi_j + \xi_k \wedge \xi_l](u,v)$, $\varepsilon_{jkl} = 1$, which yields (6.27) and hence proves Lemma 6.17.

If, moreover, s and W^+ are identically zero, on any open set U as above we have relation (6.27) with $\mathrm{s} = 0$ and $\lambda_j = 0$, $j = 1,2,3$ (cf. (6.24)), and so the last equality shows that β_1, β_2 and β_3 are local sections of Λ^-M. The definition of β_j now gives $[R(u,v),\alpha] = 0$ for any local bivector field α in M which is a section of Λ^+M. In view of (4.27), this completes the proof of Lemma 6.16. ■

Let us now introduce the vector fields u_j, $j = 1,2,3$, defined, locally in M_+, by the formula

$$(6.28) \qquad u_j \,=\, (\lambda_k - \lambda_l)\,\alpha_j\xi_j\,, \qquad \text{if} \quad \varepsilon_{jkl} = 1\,,$$

with $\alpha_j\xi_j$ as in (2.12).

Lemma 6.18. *Given an oriented Riemannian 4-manifold (M,g), let the open connected subset U of M, bivector fields α_j, 1-forms ξ_j, vector fields u_j and functions λ_j on U, $j = 1,2,3$, be chosen so as to satisfy* (6.24) (6.12), (6.26) *and* (6.28). *We have*

(a) $\nabla W^+ = 0$ *identically in U if and only if*

$$(6.29) \qquad u_j \,=\, d\lambda_j \,=\, 0 \quad \text{for} \quad j = 1,2,3\,.$$

(b) $\operatorname{div} W^+ = 0$ *identically in U if and only if*

$$(6.30) \qquad d\lambda_j = u_l - u_k \qquad \text{for all} \quad j,k,l \quad \text{with} \quad \varepsilon_{jkl} = 1\,.$$

Proof. We have $[\nabla_v W^+]\,\alpha_j = \langle v, d\lambda_j\rangle\,\alpha_j + (\lambda_j - \lambda_k)\langle v,\xi_l\rangle\,\alpha_k + (\lambda_l - \lambda_j)\langle v,\xi_k\rangle\,\alpha_l$ for any tangent vector v, whenever $\varepsilon_{jkl} = 1$, as one easily verifies combining (6.24) and (6.26). By (6.12) and (6.28), this can be rewritten as

$$(6.31) \qquad \left[\nabla_v W^+\right]\alpha_j \,=\, \langle v, d\lambda_j\rangle\,\alpha_j \,-\, \langle v,\,\alpha_l u_l\rangle\,\alpha_k \,-\, \langle v,\,\alpha_k u_k\rangle\,\alpha_l\,, \qquad \varepsilon_{jkl} = 1\,.$$

Assertion (a) now is immediate. (Here and in the sequel we use the fact that the α_j are isomorphisms in each tangent space T_xM, due to (6.12).) Furthermore, we have $[\operatorname{div} W^+]\,\alpha_j = \alpha_j(d\lambda_j - u_l + u_k)$, $\varepsilon_{jkl} = 1$ (notation of (5.31) with $A = W^+$ and (2.12) with $\xi = d\lambda_j - u_l + u_k$), in view of (4.41), (6.31) and (6.12). This implies (b), and completes the proof. ■

We have now covered all material in this section that will be used immediately in §7. The following paragraph will not be needed until §9 and §10.

Remark 6.19. It is worth noting that the converse of Lemma 6.2 is also true: Given an oriented Riemannian 4-manifold (M, g) and a point $x \in M$, every pair of positive-oriented orthonormal bases of $\Lambda_x^+ M$ and $\Lambda_x^- M$ is of the form (6.10) for some positive-oriented orthonormal basis $e_1, \ldots, e_4$ of T_xM. The latter basis then is unique up to an overall change of sign.

In fact, multiplying each vector of either basis by $\sqrt{2}$, we obtain bases $\alpha_1, \alpha_2, \alpha_3$ of $\Lambda_x^+ M$ and $\beta_1, \beta_2, \beta_3$ of $\Lambda_x^- M$ which both satisfy the assertion of Corollary 6.5. Since each α_j commutes with each β_k (Corollary 6.3), the composites $F_j = \alpha_j \beta_j$ are all self-adjoint and satisfy the conditions $F_j^2 = \mathrm{Id}$, $j = 1, 2, 3$, and $F_1 F_2 = F_3$. According to Remark 3.2, T_xM is the direct sum of the subspaces $V_\pm = \operatorname{Ker}(F_3 \mp \mathrm{Id})$ and $\dim V_\pm = 2$ since the operator α_1 interchanges V_+ and V_- (as it anticommutes with F_3). The operators F_1, F_2 leave the space V_+ invariant (since they commute with F_3) and, their restrictions to V_+ coincide as $v = F_3 v = F_1 F_2 v$ and hence $F_1 v = F_1 F_1 F_2 v = F_2 v$ for every $v \in V_+$. Denoting $\Phi : V_+ \to V_+$ the common restriction of F_1 and F_2, we now have $\Phi^2 = 1$, which leads (cf. (3.2)) to a decomposition $V_+ = L_+ \oplus L_-$ with $\Phi = \pm\,\mathrm{Id}$ on $L_\pm$. However, since the operator α_3 commutes with F_3 and anticommutes with F_1, it leaves V_+ invariant and interchanges L_+ with L_-. Hence both $L_\pm$ are 1-dimensional. Picking a unit vector $e_1 \in L_+$ we now have $F_j e_1 = e_1$ for $j = 1, 2, 3$. Setting $e_2 = \alpha_1 e_1$, $e_3 = \alpha_2 e_1$ and $e_4 = \alpha_3 e_1$, and using the stated (anti)commutation properties of the α_j, we obtain formulae for $\alpha_j e_k$ and $\beta_j e_k$ for all j, k, which show (via (2.22)) that $e_1, \ldots, e_4$ have the required properties. Finally, uniqueness of $e_1, \ldots, e_4$ up to an overall sign change is clear since for $e_1, \ldots, e_4$ with these properties, e_1 must be a fixed point of all F_j, which as we saw determines e_1 up to a sign, while the other vectors then must be given by $e_2 = \alpha_1 e_1$, $e_3 = \alpha_2 e_1$ and $e_4 = \alpha_3 e_1$.

The following consequence of Lemma 6.2 will not be used until §17. In §19, we will need a reference to the following

Remark 6.20. Given a Riemannian 4-manifold (M, g) and a point $x \in M$, the following three conditions are equivalent:

(a) Equality $\operatorname{spec} W^+ = \operatorname{spec} W^-$ holds at x, for either local orientation; in other words, the self-adjoint operators $W^+(x) : \Lambda_x^+ M \to \Lambda_x^+ M$ and $W^-(x) : \Lambda_x^- M \to \Lambda_x^- M$ have the same eigenvalues (including multiplicities).

(b) There exists an orthonormal basis $e_1, \ldots, e_4$ of T_xM such that all the bivectors $e_j \wedge e_k$ with $1 \le j < k \le 4$ are eigenvectors of $W(x)$. (In other words, the Weyl tensor at x is a *pure curvature operator* in the sense of Maillot, 1974).

(c) There exists an orthonormal basis $e_1, \ldots, e_4$ of T_xM such that each of the three bivectors $e_1 \wedge e_2$, $e_1 \wedge e_3$, $e_1 \wedge e_4$, is an eigenvector of $W(x)$.

In fact, (a) implies (b): Choosing positive-oriented orthonormal bases of Λ_x^+M and Λ_x^-M consisting of eigenvectors of W^+ and, respectively, W^-, both for the same system $\lambda_1, \lambda_2, \lambda_3$ of eigenvalues, and then using Remark 6.19 to write those bases in the form (6.10), we obtain the orthonormal basis $e_1, \ldots, e_4$ of T_xM required in (b). On the other hand, (c) follows trivially from (b). Finally, assuming (c), that is, $W\beta_a = \lambda_a\beta_a$ with $\beta_a = e_1 \wedge e_a$, $a = 2, 3, 4$, we also have $W(*\beta_a) = \lambda_a * \beta_a$, since $*$ and W commute (Lemma 6.9(i); here $*$ corresponds to the orientation of T_xM represented by the basis $e_1, \ldots, e_4$. In view of (6.1), the bases (6.10) of Λ_x^+M and Λ_x^-M now consist of eigenvectors of W^+ and W^-, both with the same system of eigenvalues: $\lambda_1, \lambda_2, \lambda_3$. Thus, (c) implies (a).

An application of the following lemma can be found in §20.

Lemma 6.21. *Let $\alpha \in \Lambda_x^+M$ be a self-dual bivector at a point x in an oriented Riemannian 4-manifold (M, g), and let $u, v \in T_xM$ be any vectors tangent to M at x. Then*

$$u \wedge (\alpha v) - v \wedge (\alpha u) \in \Lambda_x^+M, \qquad \langle u \wedge \alpha v - v \wedge \alpha u, \alpha \rangle = 0. \tag{6.32}$$

If, in addition, $\langle \alpha, \alpha \rangle = 2$, and $\mathrm{pr} : [T_xM]^{\wedge 2} \to \mathcal{K}_x$ stands for the orthogonal projection onto the plane $\mathcal{K}_x \subset \Lambda_x^+M$ orthogonal to α, we have

$$2\,\mathrm{pr}\,(v \wedge w) = v \wedge w - (\alpha v) \wedge (\alpha w) \tag{6.33}$$

for any tangent vectors $v, w \in T_xM$. Finally, if $\alpha = \alpha_1$ for some basis $\alpha_1, \alpha_2, \alpha_3$ of Λ_x^+M satisfying the hypotheses of Corollary 6.5*, then*

$$u \wedge (\alpha v) - v \wedge (\alpha u) = \alpha_2(u, v)\,\alpha_3 - \alpha_3(u, v)\,\alpha_2\,. \tag{6.34}$$

Proof. Relations (6.32) are obvious since, by (2.25), the bivector $v \wedge (\alpha u) - u \wedge (\alpha v)$ is orthogonal both to α and to all bivectors in Λ_x^-M; in fact, by Corollary 6.6, they all commute with α. Assuming now that $\langle \alpha, \alpha \rangle = 2$, let us set

$$\gamma = v \wedge w - (\alpha v) \wedge (\alpha w)\,.$$

Since $\alpha^2 = -\,\mathrm{Id}$ (Lemma 6.1), applying (2.31) to v and $u = \alpha w$ we see that $\gamma \in \mathcal{K}_x$. Also, for any bivector $\beta \in [T_xM]^{\wedge 2}$, (2.25) with $u = \alpha w$ yields

$$\langle \gamma, \beta \rangle = \langle [\beta, \alpha]\alpha w, v \rangle\,. \tag{6.35}$$

In view of Corollary 6.4, elements of $\mathcal{K}_x$ are precisely those bivectors $\beta \in [T_xM]^{\wedge 2}$ which anticommute with α. Consequently, (6.35) along with $\alpha^2 = -\,\mathrm{Id}$ gives, for each $\beta \in \mathcal{K}_x$, $\langle \gamma, \beta \rangle = 2\langle \beta\alpha\alpha w, v \rangle = -\,2\langle \beta w, v \rangle$ which, by (2.20), equals $2\langle v \wedge w, \beta \rangle$. Now (6.33) follows.

Finally, to obtain (6.34), it suffices to note that, in view of (6.32), (2.20), (6.11) and (6.12), both sides of (6.34) are linear combinations of α_2 and α_3, and both give the same inner product with α_2 as with α_3. This completes the proof. ∎

The following consequences of Corollaries 6.5 and 6.3 will not be needed until §30.

Corollary 6.22. *Given a* 4*-dimensional oriented Riemannian manifold* (M,g)*, a point* x*, and a self-dual bivector* $\alpha\in\Lambda_x^{\pm}M$ *at* x *with* $\langle\alpha,\alpha\rangle=2$*, let* $\mathcal{N}_x$ *denote the subspace of* $\mathrm{Hom}\,(T_xM,T_xM)$ *spanned by* $[T_xM]^{\wedge 2}$ *and the identity operator* $\mathrm{Id}:T_xM\to T_xM$*, and let* $\mathcal{K}$ *be the orthogonal complement of* α *in* Λ_x^+M*. Denoting* $\mathrm{pr}:\mathcal{N}_x\to\mathcal{K}_x$ *the extension, with* $\mathrm{pr}(\mathrm{Id})=0$*, of the operator of orthogonal projection* $[T_xM]^{\wedge 2}\to\mathcal{K}_x$ *relative to the natural inner product* (2.17)*, we then have*

$$2\,\mathrm{pr}\,\beta\;=\;[\alpha,\beta]\,\alpha \tag{6.36}$$

for every $\beta\in\mathcal{N}_x$*, and*

$$\mathrm{pr}\,(\alpha\beta)\;=\;\alpha\,[\mathrm{pr}\,\beta] \tag{6.37}$$

for every β *in the subspace* $\mathcal{S}_x^+$ *of* $\mathcal{N}_x$ *spanned by* Id *and* Λ_x^+M*.*

Proof. Since $\mathrm{pr}(\mathrm{Id})=0$, both sides of (6.36) agree when $\beta=\mathrm{Id}$ or $\beta=\alpha$. Choosing $\alpha_1,\alpha_2,\alpha_3\in\Lambda_x^+M$ satisfying the hypotheses of Corollary 6.5 with $\alpha=\alpha_1$, we thus have $\alpha_2,\alpha_3\in\mathcal{K}_x$, and so, by (6.12), both sides of (6.36) also agree when $\beta=\alpha_2$ or $\beta=\alpha_3$. On the other hand, when $\beta\in\Lambda_x^-M$, the left-hand side of (6.36) vanishes due to the orthogonality relation in (6.4), and the right-hand side does so by Corollary 6.3. Since $\mathcal{N}_x$ is spanned by Id and the direct sum (6.4), this proves (6.36).

Finally, to establish (6.37), note that Id and $\alpha_1,\alpha_2,\alpha_3$ form an orthogonal basis of $\mathcal{S}_x^+$ and, by (6.12), the plane P spanned by Id and α_1, as well as its orthogonal complement $\mathcal{K}_x=\mathrm{Span}\,\{\alpha_2,\alpha_3\}$, are both invariant under the operator $\beta\mapsto\alpha\beta$ of the composition with $\alpha=\alpha_1$. Thus, both sides of (6.37) vanish when $\beta\in P$ while, for $\beta\in\mathcal{K}_x$, both sides equal $\alpha\beta$. This completes the proof. ■

Corollary 6.23. *Let* x *be a point in a four-dimensional oriented Riemannian manifold* (M,g)*, and let* $\alpha,\beta\in\Lambda_x^{\pm}M$ *be bivectors at* x *which are both self-dual or both anti-self-dual. Treated as skew-adjoint operators* $T_xM\to T_xM$*,* α *and* β *then satisfy the relations*

$$g(v,\alpha v)\;=\;0\,,\qquad 2\,g(\alpha v,\beta v)\;=\;\langle\alpha,\beta\rangle\,g(v,v)\,. \tag{6.38}$$

for any vector $v\in T_xM$ *tangent to* M *at* x*.*

Proof. The first equality is nothing else than skew-adjointness of α. As for the second, with a fixed v, it suffices to establish it when $\alpha=\beta=\alpha_1$, or $\alpha=\alpha_1$ and $\beta=\alpha_2$, with $\alpha_1,\alpha_2,\alpha_3$ as in Corollary 6.5. (This is clear since both sides are bilinear and symmetric in α and β.) The equality in question now is immediate from (6.12) and skew-adjointness of the α_j. ■

The remaining part of this section serves only as a source of background information and will not be applied anywhere else in the text, except for very brief (and inessential) references (at the beginning of Part I, and in §10).

Remark 6.24. Suppose that we are given fixed orientations of the tangent spaces T_xM, T_yM at points x,y in a Riemannian 4-manifold (M,g), such that the self-adjoint operators $W^{\pm}(x):\Lambda_x^{\pm}M\to\Lambda_x^{\pm}M$ and $W^{\pm}(y):\Lambda_y^{\pm}M\to\Lambda_y^{\pm}M$ have (for both choices of the sign) the same eigenvalues, including multiplicities. Then there exists a linear isomorphism $F : T_xM \to T_yM$ sending one selected orientation onto the other and $g(x)$ onto $g(y)$ and $W(x)$ onto $W(y)$. (To see this, apply Remark 6.19 to a pair of positive-oriented orthonormal bases of Λ_x^+M and Λ_x^-M which consist of eigenvectors of $W(x)$ and $W(y)$.) Combining this with (5.33) and Schur's Theorem 5.1, one easily verifies that any four-dimensional Riemannian Einstein manifold whose curvature operator acting on bivectors has constant eigenvalues must be curvature-homogeneous (as defined at the beginning of Part I).

Lemma 6.25 (Singer and Thorpe, 1969)**.** *For an oriented Riemannian* 4*-manifold* (M, g)*, the following three conditions are equivalent*:

(i) (M, g) *is Einstein.*
(ii) *The curvature tensor* R *and Hodge star* $*$ *of* (M, g) *commute when treated as bundle morphisms* $[TM]^{\wedge 2} \to [TM]^{\wedge 2}$.
(iii) $R(\Lambda^{\pm}M) \subset \Lambda^{\pm}M$.

Proof. The equivalence between (ii) and (iii) is clear since $\Lambda^{\pm}M$ are the eigenspace bundles of $*$. Furthermore, (i) implies (iii) in view of (6.21), since, by (5.5), the Einstein condition reads $\mathrm{E} = 0$. Finally, let us suppose that R and $*$ commute. The eigenspaces $\Lambda_x^{\pm}M$ of $*$ at any $x \in M$ then are invariant under R and so, by (6.21), $\mathrm{E}\alpha + \alpha\mathrm{E} = 0$ for each $\alpha \in \Lambda_x^{\pm}M$, and hence for all bivectors $\alpha \in [T_xM]^{\wedge 2}$. If $v, w \in T_xM$ now are eigenvectors of E for some eigenvalues λ, μ, we obtain $0 = g([\mathrm{E}\alpha + \alpha\mathrm{E}]v, w) = (\lambda+\mu)g(\alpha v, w)$. Choosing v, w to be unit and orthogonal and setting $\alpha = v \wedge w$, we have $g(\alpha v, w) = 1$ by (2.21). Thus, $\lambda + \mu = 0$ for any eigenvalues λ, μ of E that are either distinct, or equal and multiple. Since E is self-adjoint, this clearly implies that $\mathrm{E} = 0$, as required. ∎

Corollary 6.26 (Singer and Thorpe, 1969)**.** *A four-dimensional Riemannian manifold* (M, g) *is Einstein if and only if the sectional curvature* $K(P)$ *of any plane* $P \subset T_xM$*, tangent to* M *at any point* x*, is equal to the sectional curvature* $K(P^{\perp})$ *of the orthogonal complement of* P.

Proof. If (M, g) is Einstein, fixing an orientation in T_xM and choosing a positive-oriented orthonormal basis $e_1, \ldots, e_4$ of T_xM with $e_1, e_2 \in P$, we obtain $K(P^{\perp}) = \langle R(e_3 \wedge e_4), e_3 \wedge e_4\rangle = \langle R(*(e_1 \wedge e_2)), *(e_1 \wedge e_2)\rangle = \langle *(R(e_1 \wedge e_2)), *(e_1 \wedge e_2)\rangle = \langle R(e_1 \wedge e_2), e_1 \wedge e_2\rangle = K(P)$, in view of Lemma 6.25, (6.1) and the fact that, by (6.1), $*$ is inner-product preserving (as it sends the orthonormal basis of $[T_xM]^{\wedge 2}$, formed by the $e_j \wedge e_k$ with $j < k$, onto an orthonormal basis). Conversely, assuming that $K(P^{\perp}) = K(P)$ for all tangent planes P, we see that, for bivectors $\alpha, \beta \in [T_xM]^{\wedge 2}$, we have $\langle R(*\alpha), *\beta\rangle = \langle R\alpha, \beta\rangle$, since both sides are bilinear in α, β and coincide when α, β run through the basis $e_j \wedge e_k$ with $j < k$ (see (6.1)). On the other hand, $\langle R\alpha, \beta\rangle = \langle *(R\alpha), *\beta\rangle$ since $*$ preserves the inner product. Combining these two equalities we see that $R(*\alpha) - *(R\alpha)$ is orthogonal to $*\beta$ for each bivector β, and so $R* = *R$. Hence (M, g) is Einstein by Lemma 6.25, which completes the proof. ∎

§7. Jensen's theorem

It is fairly common that, among known examples of pseudo-Riemannian manifolds with a specific prescribed property, those easiest to construct are the (locally) homogeneous spaces. The reason is clearly their algebraic nature, i.e., the fact that such constructions, rather than involving partial differential equations, are reduced to solving problems in linear algebra. Of all locally homogeneous pseudo-Riemannian manifolds, the simplest ones are in turn the *locally symmetric spaces*, characterized by the condition $\nabla R = 0$.

In 1969 Gary Jensen proved that all locally homogeneous Riemannian Einstein 4-manifolds are locally symmetric. Since locally symmetric Riemannian Einstein 4-manifolds are very easy to classify (see §14), Jensen's theorem may be viewed as a nonexistence result, stating that local homogeneity in this case leads to nothing more than the obvious examples. This stands in marked contrast both with locally homogeneous Riemannian Einstein manifolds in higher dimensions, where the classification problem is much harder (see, e.g., Kerr, 1996, and references therein), and with the case of *indefinite* Einstein metrics in dimension four, some of which are locally homogeneous without being locally symmetric (see Proposition 8.5).

Jensen's 1969 proof was based on a case-by-case discussion of Lie algebras. In this section we derive Jensen's theorem from a statement with a weaker hypothesis and a more direct argument. Specifically, instead of local homogeneity, we only assume that the eigenvalues of the curvature operator (or the self-dual Weyl tensor) are all constant. Then, we have

Theorem 7.1. *Let (M, g) be a four-dimensional oriented Einstein manifold with a positive-definite metric, whose self-dual Weyl tensor W^+, acting on bivectors, has constant eigenvalues. Then W^+ is parallel.* ■

Before proving Theorem 7.1, let us list some of its consequences. First, applying Theorem 7.1 to both local orientations of M, we obtain

Corollary 7.2. *Let (M, g) be a four-dimensional Riemannian Einstein manifold whose curvature operator, acting on bivectors, has constant eigenvalues. Then (M, g) is locally symmetric.*

The word 'Riemannian' in the above statement raises an obvious question: Is an analogue of Corollary 7.2 valid for Einstein four-manifolds (M, g) with *indefinite* metrics? Note that the meaning of 'analogue' has to be chosen carefully, since, when the metric is indefinite, the self-adjoint operator $W(x) : [T_xM]^{\wedge 2} \to [T_xM]^{\wedge 2}$, $x \in M$, need not be diagonalizable (and may even fail to have any real eigenvalues; see the classification given in §39.) The most appropriate replacement of the constant-eigenvalues hypothesis in Corollary 7.2 therefore seems to be the condition of curvature-homogeneity, as defined at the beginning of Part I; in fact, in the Riemannian case, these two assumptions are equivalent (Remark 6.24).

With this clarification, the answer to the above question is 'no': A curvature-homogeneous indefinite Einstein metric in dimension four need not be locally symmetric (or even locally homogeneous). See Corollary 49.2 in §49.

As an immediate consequence of Corollary 7.2, we obtain Jensen's theorem.

Corollary 7.3 (Jensen, 1969). *Every locally homogeneous Riemannian Einstein 4-manifold is locally symmetric.* ■

Again, the assumption of positive-definiteness in Corollary 7.3 is essential, that is, the word 'Riemannian' cannot be replaced with 'pseudo-Riemannian'; see Proposition 8.5 in §8.

Proof of Theorem 7.1. Choose α_j, λ_j, ξ_j and u_j, $j = 1, 2, 3$, satisfying (6.24), (6.12), (6.26) and (6.28) on a nonempty open subset of M'. Defining the symmetric matrix B_{jk} of functions by $B_{jk} = \langle\alpha_j\xi_j\,,\alpha_k\xi_k\rangle$, $j,k\in\{1,2,3\}$, we have, from (6.12) and skew-symmetry of the α_j,

$$\text{(7.1)}\qquad \langle\alpha_j\xi_k,\xi_l\rangle \,=\, -\,\langle\alpha_j\xi_l,\xi_k\rangle \,=\, B_{lk} \,=\, B_{kl} \qquad \text{whenever} \quad \varepsilon_{jkl}=1\,.$$

Taking the inner products of both sides of (6.27) with α_j and using (2.20), (7.1) and (6.11), we now obtain

$$\text{(7.2)}\qquad \langle\alpha_j,d\xi_j\rangle \,=\, -\,B_{lk} \,+\, 2\lambda_j \,+\, \frac{\mathrm{s}}{6}\,, \qquad \text{if} \quad \varepsilon_{jkl}=1\,.$$

Using (4.46), (7.2), (7.1) and the relation $\operatorname{div}\alpha_j=\alpha_k\xi_l-\alpha_l\xi_k$, $\varepsilon_{jkl}=1$ (immediate from (6.26), (4.41) and (2.12)), we now easily verify that

$$\text{(7.3)}\qquad \operatorname{div}(\alpha_j\xi_j) \,=\, -\,B_{jk} \,-\, B_{jl} \,+\, B_{lk} \,-\, 2\lambda_j \,-\, \frac{\mathrm{s}}{6}\,, \qquad \varepsilon_{jkl}=1\,.$$

Since (M,g) is Einstein, we have $\operatorname{div}W^+=0$ (Lemma 6.14). Relation (6.30) along with the assumption that the λ_j are constant yields

$$\text{(7.4)}\qquad u_1 \,=\, u_2 \,=\, u_3 \,=\, u$$

for some vector field u on U.

Our assertion easily follows in the case where two (or all three) among the λ_j are equal; in fact, (6.28) and (7.4) then give (6.29), again due to constancy of the λ_j, and so $\nabla W^+=0$ by Lemma 6.18(a). Therefore, we from now on assume that

$$\text{(7.5)}\qquad \lambda_1\neq\lambda_2\neq\lambda_3\neq\lambda_1\,.$$

Setting

$$\text{(7.6)}\qquad v_j \,=\, (\lambda_l-\lambda_k)\xi_j \qquad \text{if} \quad \varepsilon_{jkl}=1\,,$$

we obtain, from (7.4), (6.28), (6.12) and skew-symmetry of the α_j,

$$\text{(7.7)}\qquad \alpha_ju \,=\, v_j\,, \qquad \langle v_j,v_k\rangle \,=\, \langle u,u\rangle\,\delta_{jk}\,, \qquad \langle u,v_j\rangle \,=\, 0$$

for $j,k\in\{1,2,3\}$, and hence, again by (6.12),

$$\text{(7.8)}\qquad \alpha_jv_k \,=\, -\,\alpha_kv_j \,=\, v_l\,, \qquad \alpha_jv_j \,=\, -\,u \qquad \text{if} \quad \varepsilon_{jkl}=1\,.$$

Relations (7.1) and (7.6) – (7.8) now imply

$$\text{(7.9)}\qquad (\lambda_j-\lambda_k)(\lambda_k-\lambda_l)B_{jl} \,=\, |u|^2 \qquad \text{whenever} \quad \varepsilon_{jkl}=\pm1\,.$$

On the other hand, defining the (constant) functions L_j by

$$\text{(7.10)}\qquad L_j \,=\, (\lambda_k-\lambda_l)(\lambda_j+\mathrm{s}/12) \qquad \text{if} \quad \varepsilon_{jkl}=1\,,$$

we obviously have

$$\text{(7.11)}\qquad \sum_{j=1}^{3}L_j \,=\, 0\,.$$

Let us also set

$$\phi \,=\, (\lambda_j-\lambda_k)(\lambda_k-\lambda_l)(\lambda_l-\lambda_j)\,, \qquad \varepsilon_{jkl}=1\,, \tag{7.12}$$

and

$$Y_j \,=\, 2(\lambda_k-\lambda_l)^2|u|^2 \,-\, 2\phi L_j \qquad \text{if} \quad \varepsilon_{jkl}=1\,. \tag{7.13}$$

Then

$$\phi\,\mathrm{div}\,u \,=\, Y_j\,, \qquad j=1,2,3\,. \tag{7.14}$$

In fact, $\mathrm{div}\,u=(\lambda_k-\lambda_l)\,\mathrm{div}\,(\alpha_j\xi_j)$, $\varepsilon_{jkl}=1$, in view of (6.28) and (7.4) (and since the λ_j are assumed constant). Now formula (7.14) is an easy consequence of (7.3) combined with (7.9), (7.13) and (7.10).

Furthermore,

$$|u| \qquad \text{is constant}\,. \tag{7.15}$$

To see this, note that, combining (7.13) with (7.10), (7.12) and the relation $Y_1=Y_2=Y_3$ (immediate from (7.14)), we can express $|u|^2$ as a specific rational function of the λ_j and s, unless $|\lambda_1-\lambda_2|=|\lambda_2-\lambda_3|=|\lambda_3-\lambda_1|$. However, in the latter case the identity $(\lambda_1-\lambda_2)+(\lambda_2-\lambda_3)+(\lambda_3-\lambda_1)=0$ leads to $\lambda_1=\lambda_2=\lambda_3$, contradicting (7.5). Therefore, (7.15) follows since the λ_j and s are constant.

In the case where $u\neq 0$ we can compute $\mathrm{div}\,u=\mathrm{Trace}\,\nabla u$ (see (4.42)) using the orthonormal frame consisting of $u/|u|$ and $v_j/|u|$, $j=1,2,3$ (cf. (7.7)). Whether $u=0$ or not, we thus have $|u|^2\,\mathrm{div}\,u=|u|^2\,\mathrm{Trace}\,\nabla u=\sum_j\langle v_j\,,\,\nabla_{v_j}u\rangle$. (Note that $\langle u\,,\,\nabla_u u\rangle=0$ by (7.15).) Relation (4.4) for $v=v_j$ and $w=u$ now yields $|u|^2\,\mathrm{div}\,u=\sum_j\langle v_j\,,\,[v_j\,,\,u]\rangle$, since $|v_j|=|u|$ is constant in view of (7.7) and (7.15). In other words, by (2.16) and (7.7),

$$|u|^2\,\mathrm{div}\,u \,=\, -\sum_{j=1}^{3}(dv_j)(v_j,u)\,. \tag{7.16}$$

However, $(dv_j)(v_j,u)=(\lambda_l-\lambda_k)^2\,(d\xi_j)(\xi_j,u)$ whenever $\varepsilon_{jkl}=1$, in view of (7.6) and constancy of the λ_j. Hence, by (6.27), (2.15), (2.12), (7.5) – (7.8) and (7.10), $(dv_j)(v_j,u)\,=\,|u|^2L_j$. Summing this over $j=1,2,3$, we obtain, from (7.11), $\sum_j(dv_j)(v_j,u)=0$. Hence, by (7.16), $|u|^2\,\mathrm{div}\,u=0$. This in turn implies that $u=0$. In fact, otherwise, by (7.15), $|u|$ would be a positive constant, and the last equality would give $\mathrm{div}\,u=0$. By (7.14), we would have $Y_j=0$ for $j=1,2,3$. Summing that relation over $j=1,2,3$ and using (7.13) and (7.11), we would in turn get $\lambda_1=\lambda_2=\lambda_3$, which is impossible in view of (7.5). This completes the proof. ■

§8. How Jensen's theorem fails for indefinite metrics

This section deals exclusively with indefinite metrics, and the reader interested just in the Riemannian case is advised to skip it.

An obvious question raised by Jensen's theorem (Corollary 7.3) is whether it remains valid for *pseudo-Riemannian* metrics. The answer turns out to be 'no' for each indefinite sign pattern of metrics in dimension four. Counterexamples, described in the proof of Proposition 8.5 below, go back to Petrov; see Petrov (1969), p. 185.

Throughout the whole section, our convention about indices is that

$$a, b, c, d \,=\, 2, \dots, n-1\,, \tag{8.1}$$

for any fixed dimension $n \geq 2$. As usual, we sum over repeated indices.

Lemma 8.1. *Given an integer $n \geq 2$ and a $(n-1) \times (n-1)$ matrix $[F_a^b]$, with indices as in* (8.1)*, there exists a manifold M of dimension n admitting C^∞ vector fields $u, e_2, \dots, e_n$ which are linearly independent at every point and satisfy the Lie-bracket relations*

$$[u, e_a] \,=\, F_a^b e_b\,, \qquad [e_a, e_b] \,=\, 0\,, \qquad \text{for} \quad a, b = 2, \dots, n\,. \tag{8.2}$$

Proof. For a fixed n-dimensional real vector space V with a basis $e_1, \dots, e_n$, let M be either connected component of $V \smallsetminus V'$, with $V' = \operatorname{Span}\{e_2, \dots, e_n\}$. We now define the vector field u on V by $u(x) = \varPhi x$, where $\varPhi : V \to V$ is the linear operator with $\varPhi e_1 = e_1$ and $\varPhi e_a = -\,F_a^b e_b$ for $a = 2, \dots, n$. This u and the basis vectors e_a, $a = 2, \dots, n$ (treated as constant vector fields on V) satisfy (8.2); in fact, (2.4) gives $[v, w] = d_v w - d_w v$ for vector fields v, w on a vector space V treated as functions $V \to V$, and so $[e_a, e_b] = 0$, while $[u, e_a] = -\,d_{e_a} u = -\,\varPhi e_a = F_a^b e_b$. Finally, for $x \in V \smallsetminus V'$, $\varPhi x$ is a combination of $e_1, \dots, e_n$ but not of $e_2, \dots, e_n$, so that the vectors $u(x) = \varPhi x$ and $e_2, \dots, e_n$ are linearly independent. This completes the proof. ∎

Lemma 8.2. *Let M be a manifold of dimension $n \geq 3$ with C^∞ vector fields $u, e_2, \dots, e_n$ which are linearly independent at each point, i.e., trivialize the tangent bundle TM, and satisfy* (8.2)*, where $[F_a^b]$ is an arbitrary $(n-1) \times (n-1)$ matrix of constants, and let g be the pseudo-Riemannian metric on M defined by*

$$g(u, u) \,=\, \varepsilon \,=\, \pm 1\,, \qquad g(u, e_a) \,=\, 0\,, \qquad g(e_a, e_b) \,=\, g_{ab} \tag{8.3}$$

for all $a, b = 2, \dots, n$, where $[g_{ab}]$ is any nonsingular, symmetric, $(n-1) \times (n-1)$ matrix of constants. If, moreover, the matrix $[F_{ab}]$ with $F_{ab} = F_a^c g_{cb}$ is symmetric:

$$F_{ab} \,=\, F_{ba} \qquad \text{for} \quad a, b = 2, \dots, n\,, \tag{8.4}$$

then the Levi-Civita connection ∇ of g, its curvature tensor R, and Ricci tensor Ric *satisfy the relations*

$$\nabla_u u \,=\, \nabla_u e_a \,=\, 0\,, \quad \nabla_{e_a} u \,=\, -\,F_a^b e_b\,, \quad \nabla_{e_a} e_b \,=\, \varepsilon F_{ab} u\,, \tag{8.5}$$

$$\begin{aligned} &R(e_a, e_c) e_b \,=\, \varepsilon \left[F_{cb} F_a^d - F_{ab} F_c^d \right] e_d\,, \\ &R(u, e_a) u \,=\, -\,F_a^c F_c^b \, e_b\,, \quad R(e_a, e_b) u \,=\, 0\,, \end{aligned} \tag{8.6}$$

$$\operatorname{Ric}(e_a, e_b) \,=\, -\,\varepsilon F_c^c F_{ab}\,, \quad \operatorname{Ric}(u, u) \,=\, -\,F_c^d F_d^c\,, \quad \operatorname{Ric}(u, e_a) \,=\, 0\,, \tag{8.7}$$

and

$$(8.8) \qquad (\nabla_{e_b} R)(u, e_a)u \;=\; -2\,\varepsilon F_a^c F_c^d F_{db} u\,.$$

for $a, b, c = 2, \ldots, n$.

Proof. In view of (8.2), (8.3) and (8.4), the connection ∇ *defined* by (8.5) is both torsionfree and compatible with g, and hence it must coincide with the Levi-Civita connection of g (see Remark 4.1). Now (8.6) is immediate from (4.23) along with (8.2), (8.5) and (8.4). On the other hand, by (4.37), Ric satisfies $\mathrm{Ric}\,(u, v) = g^{bc} g(R(u, e_b)v, e_c)$ for any tangent vector v, with $[g^{ab}] = [g_{ab}]^{-1}$. Applying this to $v = u$ and $v = e_a$, and using the analogous relation $\mathrm{Ric}\,(e_a, e_b) = \varepsilon g(R(u, e_a)u, e_b) + g^{cd} g(R(e_c, e_a)e_d, e_b)$ we now obtain (8.7). Finally, (8.8) follows from (8.6) and (8.5) along with $(\nabla_v R)(u, u')w = \nabla_v[R(u, u')w] - R(\nabla_v u, u')w - R(u, \nabla_v u')w - R(u, u')\nabla_v w$. This completes the proof. ■

Remark 8.3. Any metric obtained as in Lemma 8.2 is obviously curvature-homogeneous (in view of (8.3), (8.6) and the fact that g_{ab}, F_{ab} and F_a^b are all constant). However, all such metrics are also locally homogeneous; see Remark 17.23 in §17 below.

Lemma 8.4. *Let* C^∞ *vector fields* $u, e_2, \ldots, e_n$ *on a manifold* M *with* $\dim M = n \geq 3$ *be linearly independent at each point and satisfy* (8.2)*, and let* $[F_a^b]$ *and* $[g_{ab}]$ *be* $(n-1) \times (n-1)$ *matrices of constants with* $\det[g_{ab}] \neq 0$*,* $g_{ab} = g_{ba}$ *and* (8.4)*, where* $F_{ab} = F_a^c g_{cb}$*, with indices as in* (8.1)*. Furthermore, let* $\mathcal{X}$ *be the* $(n-1)$*-dimensional real vector space of vector fields spanned by* $e_2, \ldots, e_n$*, and let the inner product* $\langle\,,\rangle$ *in* $\mathcal{X}$ *and the linear operator* $F : \mathcal{X} \to \mathcal{X}$ *be characterized by* $\langle e_a, e_b \rangle = g_{ab}$ *and* $F e_a = F_a^b e_b$*. Finally, let* g *be the pseudo-Riemannian metric on* M *given by* (8.3)*.*

(i) *If* $\mathrm{Trace}\, F = \mathrm{Trace}\, F^2 = 0$*, then* g *is Ricci-flat.*
(ii) *If the operator* $F^3 : \mathcal{X} \to \mathcal{X}$ *is nonzero, then* g *is* not *locally symmetric.*

In fact, the assumption in (i) means that $F_c^c = F_c^d F_d^c = 0$, and so (i) is immediate from (8.7). As for (ii), it follows from (8.8), since $F^3 \neq 0$ amounts to $F_a^c F_c^d F_{db} \neq 0$ for some a, b. ■

It is now easy to obtain counterexamples to Jensen's theorem for indefinite metrics.

Proposition 8.5. *There exist pseudo-Riemannian metrics in dimension four representing any given indefinite sign pattern, which are Ricci-flat and locally homogeneous, but not locally symmetric.*

Proof. It suffices to exhibit a 3-dimensional real vector space $\mathcal{X}$ with an inner product $\langle\,,\rangle$ of each possible indefinite sign pattern, along with a self-adjoint linear operator $F : \mathcal{X} \to \mathcal{X}$ with $\mathrm{Trace}\, F = \mathrm{Trace}\, F^2 = 0$ and $F^3 \neq 0$. In fact, self-adjointness of F amounts to (8.4) (with F_{ab} as in Lemma 8.4 for any fixed basis $e_2, \ldots, e_n$ of $\mathcal{X}$), so that the metric g given by (8.3) then will exist (by Lemma 8.1), and have the required properties as a consequence of Lemma 8.4 and Remark 8.3. Note that the sign pattern of $\langle\,,\rangle$, coupled with an arbitrarily chosen sign of ε in (8.3), then realizes any prescribed indefinite sign pattern in dimension 4.

To this end, let us set $\mathcal{X} = \mathbf{R} \times \mathbf{C}$, $\langle (r, z), (r', z') \rangle = \pm\, rr' + \mathrm{Re}\,(zz')$ with any fixed sign $\pm$, and $F(r, z) = (r, \omega z)$, for $r, r' \in \mathbf{R}$, $z, z' \in \mathbf{C}$, where $\omega =$

$e^{2\pi i/3} = (\sqrt{3}\,i - 1)/2$ is a fixed nonreal cubic root of unity. Thus $F^2(r,z) = (r,\,\omega^2 z)$, $F^3(r,z) = (r,z)$, and hence (as $\omega^2 = \omega^{-1} = \overline{\omega}$), $\operatorname{Trace} F = \operatorname{Trace} F^2 = 0$, $F^3 = \mathrm{Id}$. Also, $\langle F(r,z),(r',z')\rangle = \langle (r,\,\omega z),(r',z')\rangle = \pm\, rr' + \operatorname{Re}(\omega z z')$, which is symmetric in (r,z) and (r',z'), i.e., F is self-adjoint. This completes the proof. ■

§9. Kähler manifolds

This section is a brief introduction to Kähler manifolds. As we will see later, discussing Kähler manifolds is quite relevant to our ultimate topic of interest, which is Einstein metrics; namely, some Einstein metrics can in turn be either found among Kähler metrics (see §23), or obtained from the latter via a conformal deformation (§18, §22).

Let M be a manifold with a fixed C^∞ bundle morphism $J : TM \to TM$ such that $J^2 = -\,\mathrm{Id}$. One says that J is an *almost complex structure on* M, or that M along with J forms an *almost complex manifold.* The tangent bundle TM then carries a natural structure of a *complex* vector bundle, for which J is the operator of multiplication by i in every fibre T_xM. (Cf. Remark 3.9.) As complex spaces, the T_xM are of dimension $n/2$, where $n = \dim M$ is necessarily even; to avoid confusion, we will often refer to n as the *real* dimension of the almost complex manifold M. Note that, according to Remark 3.6, the complex-space structure in each T_xM leads to a naturally distinguished orientation in T_xM. In other words, every almost complex manifold M carries a *canonical orientation*; when endowed with that orientation, M is said to be *canonically oriented.*

Let (M,g) now be a pseudo-Riemannian manifold. By an *almost complex structure on* M *compatible with* g we mean any C^∞ bivector field α on M which, treated as a skew-adjoint bundle morphism $TM \to TM$ with the aid of g, satisfies the condition

$$\alpha^2 \,=\, -\,\mathrm{Id}\,. \tag{9.1}$$

The triple (M,g,α) then is called a (pseudo-Riemannian) *almost Hermitian manifold.* By a (pseudo-Riemannian) *Kähler manifold* we mean an almost Hermitian manifold (M,g,α) such that the bivector field α is parallel. One then refers to α as the *Kähler form* of the Kähler manifold (M,g,α); this terminology reflects the fact that, using the metric g, one may regard α as a differential 2-form, that is, a twice-covariant skew-symmetric tensor field on M.

Remark 9.1. Since every almost Hermitian manifold (M,g,α) (and hence every Kähler manifold) is naturally an almost complex manifold, with J declared to α treated as a bundle morphism $TM \to TM$, the tangent bundles of almost Hermitian (and Kähler) manifolds may be treated as complex vector bundles, and all such manifolds are *canonically oriented.* Note that the orientation we choose for them is determined by the complex structure of TM (that is, α) as described in Remark 3.6. As an example, any oriented Riemannian surface (M,g) can naturally be turned into a Kähler manifold (M,g,α) whose canonical orientation coincides with the original one. For details, see Remark 18.7.

Remark 9.2. By a *Kähler metric* on a "real" manifold M we will mean a pseudo-Riemannian metric g on M such that, for *some* bivector field α on M, the triple (M,g,α) is a Kähler manifold. It should be emphasized that this usage is different from the case of a Kähler metric g on a *complex manifold* M, discussed in §23 and §36, where one requires (M, g, α) to be a Kähler manifold with α that is *partly fixed*, that is, corresponds via g to a fixed bundle morphism $J : TM \to TM$. Cf. Remark 23.4.

Due to skew-symmetry of bivectors, condition (9.1) imposed on a bivector field α in a pseudo-Riemannian manifold (M, g) is equivalent to the requirement that, for each $x \in M$, $\alpha(x) : T_xM \to T_xM$ preserve the inner product $g(x)$ in T_xM in the sense that

$$g(\alpha v, \alpha w) \,=\, g(v, w) \tag{9.2}$$

for all $x \in M$ and $v, w \in T_xM$.

Let (M, g, α) be an almost Hermitian manifold. Relation (9.2) implies, according to Remark 3.18, that g is the real part of a unique Hermitian complex-sesquilinear fibre metric $\langle\,,\rangle_{\mathbf{c}}$ in the complex vector bundle TM. Explicitly, by (3.35),

$$\langle v, w\rangle_{\mathbf{c}} \,=\, g(v, w) \,-\, i\, g(\alpha v, w) \tag{9.3}$$

for $x \in M$ and $v, w \in T_xM$. If, in addition, $\nabla\alpha = 0$ (that is, (M, g, α) is a pseudo-Riemannian Kähler manifold), the Levi-Civita connection ∇ of (M, g) is a connection in TM (treated as a complex vector bundle) and, obviously, it makes the fibre metric $\langle\,,\rangle_{\mathbf{c}}$ parallel; in other words, we have the *Leibniz rule*

$$d_u\langle v, w\rangle_{\mathbf{c}} \,=\, \langle\nabla_u v, w\rangle_{\mathbf{c}} \,+\, \langle v, \nabla_u w\rangle_{\mathbf{c}} \tag{9.4}$$

for any local C^1 vector fields u, v, w in M.

In the four-dimensional Riemannian case, assertion (c) of the following lemma provides an alternative description of the canonical orientation.

Lemma 9.3. *Let $n \ge 2$ be an even integer. Given a point x in an oriented n-dimensional Riemannian manifold (M, g) and a bivector $\alpha \in [T_xM]^{\wedge 2}$, the following two conditions are mutually equivalent*:

- (a) $\alpha^2 = -\operatorname{Id}$ *and the orientation in T_xM, determined as in* Remark 9.1 *by the complex structure α, coincides with the original orientation.*
- (b) *There exists a positive-oriented $g(x)$-orthonormal basis $e_1, \ldots, e_n$ of T_xM such that*

$$\alpha \,=\, e_1 \wedge e_2 \,+\, \ldots \,+\, e_{n-1} \wedge e_n\,. \tag{9.5}$$

Furthermore, (9.5) *holds for any $g(x)$-orthonormal bases $e_1, \ldots, e_n$ obtained by choosing an arbitrary $\langle\,,\rangle_{\mathbf{c}}$-orthonormal complex basis $e_1, e_3, \ldots, e_{2m-1}$ of T_xM with the complex structure α, where $\langle\,,\rangle_{\mathbf{c}}$ is given by* (9.3)*, and then setting $e_{2r} = \alpha e_{2r-1}$ for $r = 1, \ldots, n/2$.*

Finally, when $n = 4$, either of (a), (b) *is equivalent to*

- (c) *α is self-dual and of length $\sqrt{2}$; in other words, $\alpha \in \Lambda^+_xM$ and $\langle\alpha, \alpha\rangle = 2$.*

Proof. Let us assume (a), and choose $e_1, \ldots, e_n$ as described in the sentence following (b). We then have equality (9.5), since, by (2.22), both sides yield the same value when applied to any of the vectors $e_1, \ldots, e_n$. On the other hand, the basis $e_1, \ldots, e_n$ has the form (3.5) (with α playing the rôle of i), and so it is positive-oriented according to Remark 9.1. Thus, (a) implies (b). Conversely, if (b) holds, we have $e_{2r} = \alpha e_{2r-1}$ for $r = 1, \ldots, n/2$ (by (2.22)), and so, using (2.22) we obtain $\alpha^2 e_j = -e_j$, $j = 1, \ldots, n$, so that (a) follows (the statement about orientations being immediate from Remark 9.1). From now on, let $n = 4$. Now (b) implies (c) in view of (6.1) and (2.21). Finally, if (c) holds, using Lemma 6.1 with $r = 1$ we obtain $\alpha^2 = -\,\mathrm{Id}$ and (9.5) for some positive-oriented orthonormal basis $e_1, \ldots, e_4$; in view of (9.5) and (2.22), the basis $e_1, \ldots, e_4$ has the form (3.5) (with i replaced by α), so that (a) follows. This completes the proof. ■

Corollary 9.4. *For any canonically-oriented almost Hermitian Riemannian manifold (M, g, α) of real dimension 4, the bivector field α is a section of $\Lambda^+ M$.*

This is obvious from the '(a) implies (c)' assertion in Lemma 9.3. ■

Remark 9.5. Let (M, g) be an oriented Riemannian 4-manifold, and let $x \in M$. From Lemma 9.3 we obtain the following simple characterization of the space $\Lambda_x^+ M$ of self-dual bivectors at x: The sphere in $\Lambda_x^+ M$ of radius $\sqrt{2}$, centered at 0, coincides with *the set of all complex structures in $T_x M$ compatible with g and with the orientation*, that is, with the set of those skew-adjoint operators $\alpha : T_x M \to T_x M$ with $\alpha^2 = -\,\mathrm{Id}$ for which the original orientation is the same as the canonical orientation introduced by the complex structure in $T_x M$ whose operator of multiplication by i is α.

We now proceed to discuss those basic curvature properties of Kähler manifolds which follow just from the fact that the Kähler form α is parallel (rather than involving (9.1) as well). First, we have the following fundamental commutation formulae relating parallel bivector fields with the curvature and Ricci tensors.

Proposition 9.6. *Let α be a parallel bivector field on a pseudo-Riemannian manifold (M, g). Then*

$$[\mathrm{Ric}, \alpha] \,=\, 0\,. \tag{9.6}$$

and

$$[R(u, v), \alpha] \,=\, 0\,, \tag{9.7}$$

In other words, the skew-adjoint bundle morphism $\alpha : TM \to TM$ commutes both with the Ricci tensor of (M, g) and with the curvature operator $R(u, v)$ defined by (4.23) *for any given vectors or vector fields u, v tangent to M.*

Proof. Relation (9.7) is immediate from the Ricci identity (4.27) for $F = \alpha$. Contracting against g^{jm} the local-coordinate version $R_{jkl}{}^p \alpha_{pm} + R_{jkm}{}^p \alpha_{lp} = 0$ of (9.7) (cf. (4.29)), and using (4.37), we obtain

$$R_{jklp} \alpha^{pj} \,=\, R_k^p \alpha_{pl}\,. \tag{9.8}$$

However, in view of the algebraic symmetries (4.32) of R, $R_{jklp} \alpha^{pj} = R_{plkj} \alpha^{pj} = R_{jlkp} \alpha^{jp} = -\,R_{jlkp} \alpha^{pj}$, that is, $R_{jklp} \alpha^{pj}$ is skew-symmetric in k, l. Hence, according to (9.8), the composite bundle morphism $\alpha(\mathrm{Ric}) : TM \to TM$ is skew-adjoint and, as $(\mathrm{Ric})^* = \mathrm{Ric}$ and $\alpha^* = -\,\alpha$, we have $\alpha(\mathrm{Ric}) = -\,[\alpha(\mathrm{Ric})]^* = (\mathrm{Ric})\alpha$, which gives (9.6). This completes the proof. ■

Corollary 9.7. *Let (M,g) be an oriented Riemannian four-manifold admitting a parallel bivector field α which is self-dual, that is, a section of Λ^+M. For any point $x \in M$ and any bivector $\beta \in [T_xM]^{\wedge 2}$, we then have*

$$R[\alpha,\beta] = 0, \qquad W[\alpha,\beta] = -\frac{\mathrm{s}}{12}[\alpha,\beta], \tag{9.9}$$

where s is the scalar curvature, and $[\,,\,]$ denotes the commutator of bivectors treated, with the aid of g, as skew-adjoint operators $T_xM \to T_xM$.

In fact, $R[\alpha,\beta] = 0$ in view of Lemma 5.3 (applied to $A = R$) and (9.7). Now (6.20) with α replaced by $[\alpha,\beta]$ proves our assertion about W. (Note that $[\alpha,\beta] \in \Lambda^+_xM$ since, by Corollaries 6.6 and 6.3, Λ^+_xM is an ideal in the Lie algebra $[T_xM]^{\wedge 2} = \mathfrak{so}(T_xM)$.) ■

Relations (5.19) and (9.9) now lead to the following well-known

Proposition 9.8. *Given an orientable Riemannian four-manifold (M,g) admitting a nonzero parallel bivector field α, let us choose an orientation of M such that the Λ^+M component α^+ of α is nonzero. The eigenvalues of $W^+ : \Lambda^+M \to \Lambda^+M$, listed at each point with their multiplicities, then are*

$$\left\{\frac{\mathrm{s}}{6}, -\frac{\mathrm{s}}{12}, -\frac{\mathrm{s}}{12}\right\}, \tag{9.10}$$

where s is the scalar curvature, and the eigenvalue $\mathrm{s}/6$ corresponds to the eigenvector α^+. Consequently, we have $|W^+|^2 = \mathrm{Trace}\, W^2 = \mathrm{s}^2/24$.

Proof. Fixing a local orientation as above, then replacing α with α^+ and, finally, using a constant scale factor, we may assume that $\alpha = \alpha^+$ and $\langle \alpha, \alpha \rangle = 2$. Hence Λ^+_xM admits a basis α_j, $j = 1,2,3$, with $\alpha = \alpha_1$, that satisfies the assumptions of Corollary 6.5 and hence also the commutator relations (6.13) (Corollary 6.3). Thus, both α_2, α_3 are eigenvectors of W for the eigenvalue $\mathrm{s}/12$. This completes the proof. ■

Corollary 9.9. *Let (M,g,α) be a canonically-oriented Riemannian Kähler manifold of real dimension 4, and let s be the scalar curvature of g.*

(a) *The eigenvalues of the self-dual Weyl tensor $W^+ : \Lambda^+M \to \Lambda^+M$ at any point, with multiplicities, are $\{\mathrm{s}/6, -\mathrm{s}/12, -\mathrm{s}/12\}$.*
(b) *W^+ is parallel if and only if s is constant.*

In fact, (a) is obvious from Corollary 9.4 and Proposition 9.8. Denoting pr^+ and pr^α the bundle morphisms $[TM]^{\wedge 2} \to [TM]^{\wedge 2}$ of orthogonal projections onto Λ^+M and, respectively, onto the real-line subbundle spanned by α, we now have, by (a),

$$12\,W^+ = \mathrm{s}\left[3\,\mathrm{pr}^\alpha - \mathrm{pr}^+\right],$$

which clearly implies (b). ■

Corollary 9.10. *Let (M,g) be an oriented Riemannian four-manifold such that W^+ is parallel. Then*

(i) *One of the following two cases occurs:*
  a) *$W^+ = 0$ identically, or*
  b) *W^+ is parallel and nonzero, its eigenvalues are given by* (9.10), *and the conditions $W^+\alpha = \mathrm{s}\alpha/6$ and $\langle\alpha,\alpha\rangle = 2$ define, uniquely up to a sign at every point of M, a section $\pm\,\alpha$ of Λ^+M.*

(ii) *The unique section $\pm\,\alpha$ in* (b) *is parallel.*

Proof. Let $W^+ \neq 0$. Since $\operatorname{Trace} W^+ = 0$ (see formula (6.19)), W^+ then must have at least one *simple* eigenvalue λ. The corresponding local C^∞ eigenvector section α of $\Lambda^+ M$, normalized so as to satisfy $\langle \alpha, \alpha \rangle = 2$, is unique (at each point), up to a sign. Therefore, α is parallel (since so is W^+), and our assertion is immediate from Proposition 9.8. ∎

§10. The "algebraic" examples

The Einstein condition (0.1) is obviously satisfied by those pseudo-Riemannian manifolds (M, g) which are *Ricci-flat* in the sense that $\mathrm{Ric} = 0$ identically on M. For more on Ricci-flatness, see §15, §23, Remark 28.4, §33, §36, and Part IV.

In any given dimension greater than 3, there is an enormous wealth of examples of local-isometry types of Einstein metrics. (In fact, they form an *infinite-dimensional moduli space*; see Remark 49.3 in §49.) Thus, as long as no global constraints (such as compactness) are imposed, Einstein metrics are relatively easy to find and, in fact, there is an abundance of examples in the existing literature. (For particularly simple constructions of Ricci-flat indefinite metrics, see Corollary 15.10 in §15 and Corollary 41.2(b) in §41.) In this section, however, we discuss only several special classes of pseudo-Riemannian Einstein metrics, each characterized by having a curvature tensor R of some particular algebraic type.

To be specific, these algebraic types of R, in dimension four, are described by some explicit formulae, namely, (10.1) (for a function K), or (10.5) (for some functions λ, μ and a bivector field α with (9.1)) or, finally, (10.13) (with (10.14), (10.15)). In the Riemannian case, each of these types also has an equivalent, simple characterization in terms of the spectrum of the (anti)self-dual restrictions of the curvature operator, or the Weyl tensor, at any point; see Remark 10.11.

Our first observation is that all *flat manifolds* (M, g), characterized by $R = 0$, are Ricci-flat, and hence Einstein. More generally, the class of Einstein manifolds includes all *spaces of constant curvature,* that is, pseudo-Riemannian manifolds (M, g) satisfying the condition

$$R \,=\, K\, g \circledast g \tag{10.1}$$

for some constant K (notation as in (5.9) – (5.10)), which in local coordinates reads

$$R_{jklm} \,=\, K\,(g_{jl}g_{km} - g_{kl}g_{jm})\,. \tag{10.2}$$

In fact, contracting (10.1) we obtain the Einstein condition

$$\mathrm{Ric} \,=\, \frac{\mathrm{s}}{n}\, g\,, \qquad \text{with} \quad \mathrm{s} \,=\, n(n-1)\,K\,, \quad n = \dim M\,. \tag{10.3}$$

Note that, by Schur's Theorem 5.1, in dimensions $n \neq 2$ one needs only to assume (10.1) with a *function* K, as constancy of K then follows. Furthermore, by (5.10), an Einstein metric is of constant curvature if and only if its Weyl tensor W is identically zero. (In dimensions $n \geq 4$, condition $W = 0$ is known as *conformal flatness*. See §22 for details.)

Remark 10.1. In contrast with dimensions $n \neq 2$, in the case where $n = 2$ relations (10.1) – (10.3) not only fail to imply constancy of the function K, but actually hold, with some function K, for *every* pseudo-Riemannian surface (M,g). In fact, both sides of (10.2) share the algebraic symmetries (4.32) of R, and so, if $n=2$, they are uniquely determined by the component with the indices $(j,k,l,m)=(1,2,1,2)$; in other words, (10.2) will follow if we set $K=R_{1212}/[g_{11}g_{22}-g_{12}g_{21}]$. The function $\kappa=K=\mathrm{s}/2$ is called the *Gaussian curvature* of the pseudo-Riemannian surface (M,g). The use of the symbol κ for the Gaussian curvature is justified since (10.3) then reads $\mathrm{Ric}=\kappa g$, just as in (5.3).

Remark 10.2. Dimension 3 is also exceptional in regard to the meaning of relation (10.2). Specifically, denoting W the Weyl tensor (see (5.6)), we have, for any *three-dimensional* pseudo-Riemannian manifold (M,g),

(a) $W=0$ identically on M, and
(b) (M,g) is Einstein if and only if it is a space of constant curvature.

In fact, let us fix a point $x\in M$, choose orthonormal vectors $e_1,e_2,e_3\in T_xM$, and set $\varepsilon_j=g(e_j,e_j)=\pm 1$. Denoting $W_{jklm}=g(W(e_j,e_k)e_l,e_m)$ the components of the Weyl tensor, we have $W_{1213}=0$, as $\varepsilon_1 W_{1213}=\sum_{j=1}^{3}\varepsilon_j W_{j2j3}=0$ in view of (5.24) and (5.25). Permuting the vectors e_j, we thus see that $W_{jklm}=0$ unless $\{j,k\}=\{l,m\}$. To show that the remaining components of W also vanish, let us now set, for $j=1,2,3$, $\mu_j=\varepsilon_k\varepsilon_l W_{klkl}$, with k,l chosen so that $\{j,k,l\}=\{1,2,3\}$. Now, by (5.25), $\mu_j+\mu_k=0$ whenever $j\neq k$. Thus, if $\{j,k,l\}=\{1,2,3\}$, we have $\mu_j=-\mu_k=\mu_l=-\mu_j$, i.e., $\mu_j=0$ for $j=1,2,3$, as required. This proves (a). Now (b) is immediate from (5.9) (cf. (10.1)).

Example 10.3. Let V be a *pseudo-Euclidean vector space*, that is, a real vector space V with $\dim V<\infty$, equipped with a nondegenerate bilinear symmetric form $\langle\,,\rangle$ (the *inner product*). Then the pseudo-Riemannian manifold (M,g) formed by $M=V$ with the constant (translation invariant) metric $g=\langle\,,\rangle$, is flat. In fact, expressions (4.1), (4.25) all vanish in a linear coordinate system x^j, as the g_{jk} then are constant. (Equivalently, $R=0$ by (4.23), since $\nabla_v w=0$ for *constant* vector fields v,w.) In the case where $V=\mathbf{R}^n$ and $\langle\,,\rangle$ is the standard positive-definite inner product, one speaks of the *standard Euclidean space* $\mathbf{R}^n$.

Example 10.4. Let M be a connected component of a (nonempty) pseudosphere

$$S_c=\{v\in V:\langle v,v\rangle=c\}\,, \tag{10.4}$$

with a real $c\neq 0$, in a pseudo-Euclidean vector space V (Example 10.3), and let g be the restriction of the inner product $\langle\,,\rangle$ to TM. Then (M,g) is a space of constant curvature $K=1/c\neq 0$. (See Proposition 14.1 below.) The construction described here leads, in particular, to the ordinary "round" *spheres* (if $\langle\,,\rangle$ is positive definite and $c>0$), and the *real hyperbolic spaces* (obtained when $\langle\,,\rangle$ has the Lorentz sign pattern $-++\ldots+$, and $c<0$). This includes the case of the *standard sphere* S^n and the *standard hyperbolic space* H^n, obtained using $V=\mathbf{R}^{n+1}$ with the standard Euclidean (or, respecively, Lorentzian) inner product $\langle\,,\rangle$ and $c=1$ (or, respecively, $c=-1$).

Another interesting collection of examples can be found in the class of *Kähler-Einstein manifolds*, that is, those (pseudo-Riemannian) Kähler manifolds (M,g,α) (see §9) for which g is an Einstein metric. Specifically, by a *space of constant holomorphic sectional curvature* we mean a Kähler manifold (M,g,α) such that the curvature tensor R of the underlying pseudo-Riemannian manifold (M,g) can be written as

$$R = \lambda\,[3\,\alpha\otimes\alpha \,-\, \alpha\wedge\alpha] \,+\, \mu\, g\circledast g \tag{10.5}$$

for some C^∞ functions λ and μ (notation as in (5.9) – (5.8)), where, using the metric g, we treat α as an exterior 2-form. The local-coordinate version of (10.5) is

$$R_{jklm} \,=\, \lambda\,[\alpha_{jl}\alpha_{km} - \alpha_{kl}\alpha_{jm} + 2\alpha_{jk}\alpha_{lm}] \,+\, \mu\,[g_{jl}g_{km} - g_{kl}g_{jm}]\,. \tag{10.6}$$

A space of constant holomorphic sectional curvature is automatically a Kähler-Einstein manifold. More precisely, we have

Lemma 10.5. *Suppose that a C^∞ bivector field α on a pseudo-Riemannian manifold (M,g) of dimension $n\ge 4$ satisfies conditions* (9.1) *and* (10.5) *for some C^∞ functions λ and μ. Then*

(i) *(M,g) is an Einstein manifold with the Ricci tensor Ric and scalar curvature s given by*

$$\mathrm{Ric} \,=\, [3\lambda + (n-1)\mu]\,g\,, \qquad \mathrm{s} \,=\, 3n\lambda + n(n-1)\mu\,. \tag{10.7}$$

(ii) *At any point $x\in M$, the Weyl tensor W acting on bivectors satisfies*

$$\begin{aligned}(n-1)\,W\alpha \,&=\, (n^2-4)\,\lambda\alpha\,,\\ (n-1)\,W\beta \,&=\, -\,(n+2)\,\lambda\beta\,,\\ (n-1)\,W\gamma \,&=\, (n-4)\,\lambda\gamma\end{aligned} \tag{10.8}$$

whenever β,γ are bivectors at x such that $\alpha\beta = -\,\beta\alpha$ and $\langle\alpha,\gamma\rangle = 0$, $\alpha\gamma=\gamma\alpha$.

Proof. Contracting (10.6) in k,m and using (9.1), we get (10.7). On the other hand, by (5.13), (2.17), (10.5), (10.6) and (5.16), $R\beta \,=\, -\,\lambda\alpha\beta\alpha + 2\lambda\langle\alpha,\beta\rangle\alpha + \mu\beta$. Combining this with (9.1) and noting that condition $\alpha\beta = -\,\beta\alpha$ implies $\langle\alpha,\beta\rangle = 0$ (due to (2.17) and (3.1)), we now obtain $R\alpha = [(n+1)\lambda + \mu]\,\alpha$ and, for β and γ as in (ii), $R\beta = (\mu-\lambda)\,\beta$, $R\gamma = (\mu+\lambda)\,\gamma$. In view of (5.33) and the scalar curvature formula in (10.7), we now obtain (10.8). ■

Suppose that a a Lie group G acts by isometries on a pseudo-Riemannian manifold (N,h) in such a way that the quotient set $M=N/G$ consisting of all orbits of the action carries a structure of a manifold for which the natural projection $\mathrm{pr}\,:\,N\to M$ is a submersion. Moreover, let the restriction of the metric h to the tangent space of each orbit of G be nondegenerate. The quotient manifold $M=N/G$ then carries a unique pseudo-Riemannian metric g (called the *quotient metric*) which makes $\mathrm{pr}\,:\,N\to M$ into a *Riemannian submersion* in the sense that, for each $y\in N$, the differential $d\mathrm{pr}_y : T_yN \to T_xM$, with $x=\mathrm{pr}\,(y)$, restricted to the orthogonal complement of $\mathrm{Ker}\,[d\mathrm{pr}_y]$ in T_yN, is isometric (i.e., sends $h(y)$ onto $g(x)$). In fact, for any $x\in M$ we may fix $y\in N$ with $x=\mathrm{pr}\,(y)$, and then define $g(x)$ using this Riemannian-submersion property; its independence of the choice of y is clear since G acts by isometries.

One then refers to (M,g) as the *pseudo-Riemannian quotient manifold* of (N,h) relative to the isometric action of G.

Example 10.6. Let V be a pseudo-unitary vector space, that is, a complex vector space V with $\dim V < \infty$, carrying a fixed nondegenerate sesquilinear Hermitian form $\langle\,,\rangle$, and let (M^c, g^c) be the pseudo-Riemannian quotient manifold of a (nonempty) pseudosphere S_c given by (10.4) with a real $c \neq 0$, relative to the obvious isometric action of the circle group S^1 on S_c (through multiplications by complex numbers of modulus one). Then $(M, g) = (M^c, g^c)$ has constant holomorphic sectional curvature with $\lambda = \mu = 1/c$ in (10.5). (For a proof of this statement, see Proposition 14.3 in §14.) Hence, according to formula (10.10) below, its scalar curvature equals

$$\text{(10.9)} \qquad \mathrm{s} = \frac{n(n+2)}{c}\,, \qquad n = \dim M\,.$$

As a special case of this construction we obtain the *complex projective spaces* with the *Fubini-Study metrics* (when $\langle\,,\rangle$ is positive definite and $c > 0$) and the *complex hyperbolic spaces* (when $\langle\,,\rangle$ has the "complex Lorentzian" sign pattern $- + + \ldots +$, and $c < 0$). If, in addition, $V = \mathbf{C}^{q+1}$, while $\langle\,,\rangle$ is the standard positive-definite (or, complex Lorentzian), Hermitian inner product, and $c = 1$ (or, respecively, $c = -1$), one speaks here of the *standard complex projective space* $\mathbf{C}\mathrm{P}^q$ or, respectively, the *standard complex hyperbolic space*, which is sometimes denoted $(\mathbf{C}\mathrm{P}^q)^*$. Note that both $\mathbf{C}\mathrm{P}^q$ and $(\mathbf{C}\mathrm{P}^q)^*$ thus are manifolds of (real) dimension $n = 2q$.

As already mentioned in §7, by a *locally symmetric space* (*manifold*) we mean a pseudo-Riemannian manifold whose curvature tensor is parallel ($\nabla R = 0$).

Lemma 10.7. *Let* (M, g, α) *be a space of constant holomorphic sectional curvature, that is, a pseudo-Riemannian Kähler manifold satisfying conditions* (9.1) *and* (10.5) *for some* C^∞ *functions* λ *and* μ. *Then* (M, g) *is a locally symmetric Einstein manifold, while* μ *and* λ *are both constant and equal, and given by*

$$\text{(10.10)} \qquad \lambda = \mu = \frac{\mathrm{s}}{n(n+2)}\,, \qquad n = \dim M\,.$$

Proof. Since (M, g) is Einstein (Lemma 10.5(i)) and α is parallel, formula (5.20) gives $n(n-1)\,W\alpha = (n-2)\,\mathrm{s}\alpha$. Combined with the first equality of (10.8) and the scalar curvature formula in (10.7), this yields (10.10), and our assertion is immediate from Schur's Theorem 5.1. ■

Corollary 10.8. *All pseudo-Riemannian spaces of constant curvature and spaces of constant holomorphic sectional curvature, as well as products of locally symmetric Einstein manifolds having equal constant Ricci curvatures* κ *in* (5.3), *are locally symmetric and Einstein.*

This is an immediate consequence of (10.1) and (10.3) or, respectively, Lemma 10.7; the product case is obvious. ■

In dimension four, the conclusion of Lemma 10.7 holds even without assuming that α is parallel, provided that we require instead that λ or μ be constant:

Corollary 10.9. *Let* (M, g) *be an orientable Riemannian four-manifold such that conditions* (9.1) *and* (10.5) *are satisfied by a* C^∞ *bivector field* α *and some* C^∞

functions λ, μ *on* M*. Moreover, let one of* λ, μ *be constant. Then* (M, g) *is a locally symmetric Einstein manifold and, locally, for a suitably chosen orientation,* (M, g) *self-dual in the sense that*

$$W^- = 0\,. \tag{10.11}$$

Furthermore, unless (M, g) *is a space of constant curvature, we have*

$$\lambda = \mu = \frac{\mathrm{s}}{24}\,, \tag{10.12}$$

and α *must be parallel, so that the triple* (M, g, α) *is a Kähler-Einstein manifold.*

Proof. In view of (10.7) and Schur's Theorem 5.1, λ and μ are both constant. Thus, (10.8) and (5.33), combined with Corollary 7.2 and Lemma 10.5(i), imply that (M, g) is locally symmetric and Einstein. Furthermore, for a suitably chosen local orientation, α is a section of $\Lambda^+ M$, and bivector fields commuting with α and orthogonal to α are precisely the sections of $\Lambda^- M$. (See Corollary 6.8.) Thus, (10.11) follows from the last formula of (10.8). To establish (10.12), let us now assume that (M, g) is not a space of constant curvature. Then $W^+ \neq 0$ (by (10.11) and (5.10)) and so Corollary 9.10 shows that its eigenvalues are given by (9.10) while, by (10.8), these eigenvalues are $4\lambda, -2\lambda, -2\lambda$ (note that the triple eigenvalue 0 in (10.8) must correspond to $W^- = 0$). Hence $\lambda = \mathrm{s}/24$, and (10.12) is immediate from (10.7). Finally, in view of (10.8), the bivector field α appearing in (10.5) must coincide, up to a sign, with that described in Corollary 9.10 (due to the uniqueness assertion of Corollary 9.9(i)b)), and so, by Corollary 9.9(ii), α is parallel, which completes the proof. ■

Remark 10.10. It will be convenient for us to describe conditions similar to (10.1) or (10.5) that would characterize those pseudo-Riemannian Einstein 4-manifolds (M, g) which are Riemannian products of pseudo-Riemannian surfaces. As we will see later (Theorem 14.5(iii)), in a neighborhood U of any point of M, the curvature tensor R of such a product-of-surfaces Einstein 4-manifold (M, g) and its scalar curvature s satisfy the relation

$$R = \frac{\mathrm{s}}{4}\,[\delta\,\beta \otimes \beta + \varepsilon\,\gamma \otimes \gamma]\,, \tag{10.13}$$

for some C^∞ bivector fields β, γ on U and numbers δ, ε such that

$$\begin{aligned} &\langle \beta, \beta\rangle = \delta\,, \quad \langle \gamma, \gamma\rangle = \varepsilon\,, \quad \langle \beta, \gamma\rangle = 0\,, \quad \delta\,, \varepsilon \in \{1, -1\}\,, \\ &\beta\gamma = 0\,, \qquad \delta\,\beta^2 + \varepsilon\,\gamma^2 = -\,\mathrm{Id}\,, \qquad \nabla\beta = \nabla\gamma = 0\,, \end{aligned} \tag{10.14}$$

and

$$\beta = e_1 \wedge e_2\,, \qquad \gamma = e_3 \wedge e_4 \tag{10.15}$$

for some orthonormal C^∞ vector fields $e_1, \ldots, e_4$ on U.

Another curvature condition characterizing products of surfaces among pseudo-Riemannian Einstein 4-manifolds (M, g) is (see Theorem 14.5(iv))

$$R = \frac{\mathrm{s}}{4}\,[P \circledast P + Q \circledast Q]\,, \tag{10.16}$$

with symmetric twice-covariant C^∞ tensor fields P, Q such that

$$\begin{array}{lll} P^2 = P\,, & Q^2 = Q\,, & P + Q = \mathrm{Id}\,, \\ \nabla P = \nabla Q = 0\,, & \mathrm{rank}\, P = \mathrm{rank}\, Q = 2\,. & \end{array} \tag{10.17}$$

Note that, conversely, conditions (10.13) – (10.14) and, separately, (10.16) – (10.17), easily imply that (M, g) is a locally symmetric Einstein manifold.

Remark 10.11. Let (M, g) be an oriented Riemannian Einstein four-manifold, and let $\mathrm{spec}\, W^\pm$ and $\mathrm{spec}\, R^\pm$ denote the *spectra,* i.e., systems of eigenvalues (listed along with their multiplicities) of $W^\pm(x)$ and $R^\pm(x)$, which are the self-adjoint operators $\Lambda^\pm_x M \to \Lambda^\pm_x M$ at any point $x \in M$, obtained by restricting $W(x)$ and $R(x)$ to $\Lambda^\pm_x M$. (See (6.14) and Lemma 6.25(iii); the term 'system' used here when referring to such a spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ stands for an unordered system with well-defined multiplicities, so that the repetitions of some among the λ_j do matter, whereas their order does not.) For the special types of 4-dimensional Riemannian Einstein manifolds discussed in this section (namely, spaces of constant curvature or of constant holomorphic sectional curvature, as well as product-of-surfaces Einstein 4-manifolds), the spectra of $W^\pm(x)$ and $R^\pm(x)$ do not depend on x. More precisely, for each of the specific three special types, the spectra then are (with s standing for the scalar curvature): For spaces of constant curvature,

$$\mathrm{spec}\, W^+ = \mathrm{spec}\, W^- = \{0,\, 0,\, 0\}\,, \tag{10.18}$$

i.e.,

$$\mathrm{spec}\, R^+ = \mathrm{spec}\, R^- = \{\mathrm{s}/12,\, \mathrm{s}/12,\, \mathrm{s}/12\}\,. \tag{10.19}$$

For (suitably oriented) manifolds of constant holomorphic sectional curvature,

$$\mathrm{spec}\, W^+ = \{\mathrm{s}/6,\, -\mathrm{s}/12,\, -\mathrm{s}/12\}\,, \qquad \mathrm{spec}\, W^- = \{0,\, 0,\, 0\}\,, \tag{10.20}$$

that is,

$$\mathrm{spec}\, R^+ = \{\mathrm{s}/4,\, 0,\, 0\}\,, \qquad \mathrm{spec}\, R^- = \{\,\mathrm{s}/12,\ \mathrm{s}/12,\ \mathrm{s}/12\}\,. \tag{10.21}$$

Finally, for those products of oriented surfaces which are Einstein,

$$\mathrm{spec}\, W^+ = \mathrm{spec}\, W^- = \{\mathrm{s}/6,\, -\mathrm{s}/12,\, -\mathrm{s}/12\}\,, \tag{10.22}$$

i.e.,

$$\mathrm{spec}\, R^+ = \mathrm{spec}\, R^- = \{\mathrm{s}/4,\, 0,\, 0\}\,. \tag{10.23}$$

In fact, (10.1), (10.3) and (5.16) give (10.19), relations (9.10) and (10.11) imply (10.20) (which may also be obtained from Lemma 10.5(ii)), while (10.23) follows from (10.20) applied to either (local) orientation of M separately. This establishes three of the above six relations; the remaining three now are obvious in view of (5.33).

§11. Connections and flatness

As a preparation for proving some standard classification results in §14, it is convenient to first discuss flat connections in vector bundles.

A C^k mapping F of a rectangle $\Omega = [a,b] \times [c,d]$ into a manifold M may be referred to of as a *variation (family) of curves* $[a,b] \ni t \mapsto F^s(t) = F(t,s) \in M$, each of which corresponds to a fixed value of the *variation parameter* $s \in [c,d]$. When $F(a,s) = x$ and $F(b,s) = y$ for some $x, y \in M$ and all $s \in [c,d]$, F is also called a C^k *homotopy* with fixed endpoints between the curves F^c and F^d; if such a homotopy exists, one says that the curves F^c and F^d connecting x and y are *C^k-homotopic with fixed endpoints*.

Let $\mathcal{E}$ be a vector bundle over a manifold M, and let ϕ be a *section of $\mathcal{E}$ along a C^k mapping* $F : \Omega \to M$, where $\Omega = [a,b] \times [c,d]$, that is, an assignment of an element $\phi(t,s)$ of the fibre $\mathcal{E}_{F(t,s)}$ to each $(t,s) \in \Omega$. We say that ϕ is *of class* C^k if its components ϕ^a relative to any local trivialization e_a of $\mathcal{E}$ are C^k differentiable functions of (t,s). If $k \geq 1$, we can now define the *partial covariant derivatives* ϕ_t and ϕ_s relative to a fixed connection ∇ in $\mathcal{E}$ to be the C^{k-1} sections of $\mathcal{E}$ along F, obtained by covariant differentiation of ϕ treated as a section along the curve $F(\cdot, s)$ or $F(t, \cdot)$ (while s or t is kept fixed). Thus, ϕ_t and ϕ_s have the component functions

$$(11.1) \qquad \phi_t^a = \frac{\partial \phi^a}{\partial t} + (\mathit{\Gamma}_{jb}^a \circ F) \frac{\partial F^j}{\partial t} \phi^b, \qquad \phi_s^a = \frac{\partial \phi^a}{\partial s} + (\mathit{\Gamma}_{jb}^a \circ F) \frac{\partial F^j}{\partial s} \phi^b .$$

Taking in turn the partial covariant derivatives of ϕ_t and ϕ_s (when $k \geq 2$), we obtain the second-order partial covariant derivatives $\phi_{tt} = (\phi_t)_t$, $\phi_{ts} = (\phi_t)_s$, $\phi_{st} = (\phi_s)_t$ and $\phi_{ss} = (\phi_s)_s$. It is now easy to verify that, if $k \geq 2$,

$$(11.2) \qquad R^\nabla(F_t, F_s)\phi = \phi_{ts} - \phi_{st} ,$$

where both sides are C^{k-2} sections of $\mathcal{E}$ along F. (In fact, (11.1) and (4.53) yield $R_{jkb}{}^a (\partial F^j / \partial t)(\partial F^k / \partial s) \phi^b = \phi_{ts}^a - \phi_{st}^a$.)

A connection ∇ in a vector bundle $\mathcal{E}$ over M is called *flat* if $R^\nabla = 0$ everywhere. A (local) C^1 section ϕ of $\mathcal{E}$ is said to be *parallel* (relative to a fixed connection ∇) if $\nabla \phi = 0$, that is, $\nabla_v \phi = 0$ for all tangent vectors v.

Lemma 11.1. *Suppose that ∇ is a flat connection in a vector bundle $\mathcal{E}$ over a manifold M, while $x, y \in M$ and $F^0, F^1 : [a,b] \to M$ are C^2 curves in M that connect x to y. If F^0 and F^1 are C^2-homotopic with fixed endpoints, then they give rise to the same ∇-parallel transport $\mathcal{E}_x \to \mathcal{E}_y$.*

Proof. Choose a fixed-endpoints C^2 homotopy $F : [a,b] \times [0,1] \to M$ between F^0 and F^1. For any given $\psi \in \mathcal{E}_x$, let $\phi(t,s) \in \mathcal{E}_{F(t,s)}$ be the image of ψ under the parallel transport along the curve $[a,t] \ni t' \mapsto F^s(t') = F(t',s)$. Since $R^\nabla = 0$ and $\phi_t = 0$, (11.2) yields $\phi_{st} = 0$, i.e., ϕ_s is parallel in the t direction. Therefore $\phi_s = 0$, as $\phi_s(a,s) = 0$ (due to our initial conditions $F(a,s) = x$, $\phi(a,s) = \psi$). Setting $t = b$, we now obtain constancy of the curve $[0,1] \ni s \mapsto \phi(b,s) \in \mathcal{E}_y$. ■

The following basic classification result states that any flat connection looks, locally, like the standard flat connection in a product bundle:

Lemma 11.2. *Any flat connection* ∇ *in a vector bundle* $\mathcal{E}$ *over a manifold* M *admits, locally, a local trivialization* e_a *consisting of parallel sections. In other words, every point of* M *has a neighborhood* U *such that for each* $y \in U$ *and any* $\phi \in \mathcal{E}_y$ *there exists a unique parallel local section* ψ *of* $\mathcal{E}$*, defined on* U*, with* $\psi(y) = \phi$*.*

Proof. Fix $x \in M$ and identify a neighborhood U of x with an open convex subset of $\mathbf{R}^n$, $n = \dim M$. For any given $\phi \in \mathcal{E}_x$ we can construct a parallel section ψ of $\mathcal{E}$ restricted to U with $\psi(x) = \phi$ be defining $\psi(y)$, for $y \in U$, to be the parallel translate of ϕ along any C^2 curve connecting x to y in U; by Lemma 11.1, this does not depend on the choice of the curve, as two such curves admit an obvious fixed-endpoints C^2 homotopy due to convexity of U. Our e_a now may to be chosen to be the parallel sections of $\mathcal{E}$ restricted to U with $e_a(x)$ forming any prescribed basis of $\mathcal{E}_x$. ■

As a consequence, we obtain the *Poincaré Lemma* for C^∞-differentiable 1-forms:

Corollary 11.3. *Let* ξ *be a* 1*-form of class* C^∞ *on a manifold* M *such that* $d\xi = 0$ (*notation of* (2.16)). *Then, locally,* ξ *can be written as* $\xi = df$ *for some* C^∞ *function* f*.*

Proof. The connection ∇ in the product line bundle $\mathcal{E} = M \times \mathbf{R}$ given by $\nabla_v \psi = d_v \psi + \xi(v)\psi$ for real-valued functions ψ (i.e., sections of $\mathcal{E}$) is flat in view of (4.52) (or, (4.53)). Choosing a parallel local trivializing section e^{-f} (see Lemma 11.2), we now obtain $d_v f = \xi(v)$ for all tangent vectors v, as required. ■

Remark 11.4. Generalizing Corollary 11.3, we arrive at the following "traditional" (i.e., coordinate-and-trivialization dependent) interpretation of Lemma 11.2: Let us consider any system of first-order linear homogeneous partial differential equations with arbitrary C^∞ coefficient functions (which we choose to denote $-\Gamma^a_{jb}$), imposed on the unknown real or complex-valued functions ϕ^a, $a = 1, \ldots, q$, of n real variables x^j in an open subset U of $\mathbf{R}^n$, $j = 1, \ldots, n$, and let us assume that the system is "solved for the derivatives", i.e., has the form

$$\partial_j \phi^a \;=\; -\,\Gamma^a_{jb}\, \phi^b\,, \tag{11.3}$$

$\partial_j = \partial / \partial x^j$ being the partial derivatives. If we regard the q-tuple $\phi = (\phi^1, \ldots, \phi^q)$ as a section of the product bundle $\mathcal{E} = U \times \mathbf{K}^q$, with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, then equation (11.3) characterizes those sections which are ∇-parallel for the connection ∇ in $\mathcal{E}$ defined by (4.49). In view of (4.53), condition $R = 0$ is nothing else than the *consistency requirement* or *integrability condition* for (11.3), that is, the system of relations on the coefficient functions $-\Gamma^a_{jb}$ obtaining by applying ∂_k to (11.3) and then requiring that $\partial_k \partial_j \phi^a = \partial_j \partial_k \phi^a$. Now Lemma 11.2 states that this integrability condition is *not only necessary, but also sufficient* in order that, for any $x \in U$ and any prescibed initial values $\phi^a(x)$, $a = 1, \ldots, q$, there exist a solution $\phi = (\phi^1, \ldots, \phi^q)$ to (11.3) defined near x and realizing these initial data.

Remark 11.5. We will also need the Poincaré Lemma for C^∞-differentiable 2-forms, stating that such a form α with $d\alpha = 0$ has, locally, the form $\alpha = d\xi$ for some C^∞-differentiable 1-form ξ. To prove this, we use the following standard argument that can also easily be adapted to differential forms of any degree (see, e.g., Sulanke and Wintgen, 1972). In fact, fixing a suitable coordinate system, we may assume that α lives in a convex neighborhood U of $\mathbf{0}$ in $\mathbf{R}^n$; then, denoting α_{jk} the component functions of α, we may define $\xi = \xi_j\, dx^j$ through its component functions by setting $\xi_k(\mathbf{x}) = \int_0^1 tx^j\, \alpha_{jk}(t\mathbf{x})\, dt$ with $\mathbf{x} = (x^1, \dots, x^n)$. Relation $d\alpha = 0$ yields $\partial_k\alpha_{jl} - \partial_l\alpha_{jk} = \partial_j\alpha_{kl}$ and so, by (2.16), $(d\xi)_{kl} = \partial_k\xi_l - \partial_l\xi_k$. As $\partial_k\xi_l(\mathbf{x}) = \int_0^1 t\alpha_{kl}(t\mathbf{x})\, dt + \int_0^1 t^2x^j\, \partial_k\alpha_{jl}(t\mathbf{x})\, dt$ (since $d\alpha = 0$), this and skew-symmetry of the α_{kl} in k, l implies $(d\xi)_{kl} = \int_0^1 t\alpha_{kl}(t\mathbf{x})\, dt + \int_0^1 t^2x^j\, \partial_j\alpha_{kl}(t\mathbf{x})\, dt$. On the other hand, $\alpha_{kl} = \int_0^1 \frac{d}{dt}[t^2\alpha_{kl}(t\mathbf{x})]\, dt$, so $\alpha_{kl} = (d\xi)_{kl}$, as required.

We end this section with two more important consequences of Lemma 11.2. One characterizes those local trivializations of the tangent bundle which consist, locally, of the coordinate vector fields for a local coordinate system; the other is a local-structure theorem for flat torsionfree connections. They will not be needed until Part IV.

Corollary 11.6. *Let e_j be a local trivialization of the tangent bundle TM of a manifold M. Condition*

$$[e_j, e_k] \,=\, 0\,, \qquad \text{for all} \quad j, k\,, \tag{11.6}$$

then is necessary and sufficient in order that each point of the trivialization domain have a neighborhood with a local coordinate system x^j for which the e_j are the coordinate vector fields.

Proof. Let e^j be 1-forms forming at each point a basis of T_y^*M dual to the e_j, so that $e^j(e_k) = \delta_k^j$. Using (2.16) to evaluate $(de^j)(e_k, e_l)$ for all j, k, l, we now find that $de^j = 0$, that is, each e^j is closed. Let x^j be C^∞ functions (on a smaller version of U, if necessary) with $e^j = dx^j$. (They exist by Corollary 11.3.) In view of the inverse mapping theorem, the functions x^j form, in a neighborhood of any given point, a local coordinate system in M. Our assertion now follows from (2.3) along with $e^j(e_k) = \delta_k^j$.

Corollary 11.7. *Let ∇ be is a flat torsionfree connection in the tangent bundle TM of a manifold M. Then every point of M has a neighborhood U with a coordinate system x^j such that the corresponding component functions Γ_{jk}^l of ∇, characterized by* (4.2)*, are identically zero on U.*

In fact, since ∇ is torsionfree, ∇-parallel vector fields must commute by (4.4). Our assertion now follows from Lemma 11.2 and Corollary 11.6. ■

§12. Some constructions leading to flat connections

We now proceed to discuss some constructions of connections in vector bundles. In those case where the resulting connection are flat, Lemma 11.2 will guarantee solvability of a specific system partial differential equations (cf. Remark 11.4), which will in turn lead to classification theorems later in §14.

Lemma 12.1. *Let there be given a pseudo-Riemannian manifold (M, g), a symmetric twice-covariant C^∞ tensor field b on M and a number $\varepsilon = \pm 1$, and let $\mathcal{E} = TM \oplus [M \times \mathbf{R}]$ be the vector bundle over M obtained as the direct sum of TM and the product line bundle $M \times \mathbf{R}$. Also, let ∇ denote the Levi-Civita connection of (M, g). For C^1 vector fields v, u tangent to M and a C^1 function f, all defined in any given open subset of M, the symbol $b(v, \cdot)$ will stand for the* *vector field* w *with* $g(w,u) = b(v,u)$ *for all vectors* u*, i.e., the* 1*-form* $b(v,\,\cdot\,)$*, treated as a vector field with the aid of* g*; on the other hand, the pair* $\psi = (u,f)$ *is in this case a section of* $\mathcal{E}$*. The formula*

$$\mathrm{D}_v(u,f) \,=\, (\nabla_v u \,-\, \varepsilon f\, b(v,\,\cdot\,)\,,\; d_v f \,+\, b(v,u))\;, \tag{12.1}$$

with u,v,f *as above, then defines a connection* D *in* $\mathcal{E}$*. Furthermore,* D *is flat if and only if* b *and the curvature tensor* R *of* (M,g) *satisfy the conditions*

$$R \,=\, \varepsilon\, b \circledast b \tag{12.2}$$

with $\circledast$ *as in* (5.7)*, and*

$$db \,=\, 0\,, \tag{12.3}$$

where d *stands, as in* (5.28)*, for the exterior derivative of* b *treated as a* 1*-form valued in* 1*-forms.*

Proof. Computing the curvature tensor R^{D} of D via (4.52) (with the simplifications offered by Remark 4.4), we obtain, for any vector fields v,w on M and any section $\psi = (u,f)$ of $\mathcal{E}$, $R^{\mathrm{D}}(v,w)\psi = \big(R^{\mathrm{D}}_1(v,w)\psi\,,\, R^{\mathrm{D}}_1(v,w)\psi\big)$ with

$$\begin{aligned} R^{\mathrm{D}}_1(v,w)\psi \,&=\, R(v,w)u \,-\, \varepsilon\,[b(v,u)b(w,\,\cdot\,) - b(w,u)b(v,\,\cdot\,)] \\ &\quad +\, \varepsilon\, f\,([\nabla_v b]\,(w,\,\cdot\,) \,-\, [\nabla_w b]\,(v,\,\cdot\,))\,, \\ R^{\mathrm{D}}_2(v,w)\psi \,&=\, [\nabla_w b]\,(v,u) \,-\, [\nabla_v b]\,(w,u)\,, \end{aligned}$$

where R is the curvature tensor of (M,g). This completes the proof. ■

Remark 12.2. Relations (12.2) and (12.3) are known as the *Gauss* and *Codazzi equations*, respectively. Explicitly, they state that $g(R(v,w)v',w') = \varepsilon\,[b(v,v')b(w,w') - b(w,v')b(v,w')]$ and $[\nabla_v b]\,(w,u) = [\nabla_w b]\,(v,u)$ for all points $x \in M$ and vectors $u,v,w,v',w' \in T_xM$, while their local-coordinate versions are (cf. (5.37))

$$R_{jklm} \,=\, \varepsilon\,(b_{jl}b_{km} - b_{kl}b_{jm})\,, \qquad b_{jk,l} \,=\, b_{jl,k}\,. \tag{12.4}$$

Thus, contracting (12.2) twice in a row and using (4.36) and (4.40), we obtain

$$\varepsilon\,\mathrm{Ric} \,=\, (\mathrm{Trace}\,b)b \,-\, b^2\,, \qquad \varepsilon\mathrm{s} \,=\, (\mathrm{Trace}\,b)^2 \,-\, \mathrm{Trace}\,b^2\,, \tag{12.5}$$

where the coordinate form of the first equality is $\varepsilon\,R_{jl} = b^k_k b_{jl} - b^k_l b_{kj}$.

Remark 12.3. Relations (10.5) (i.e, (10.6)) and (10.10), which along with condition $\nabla\alpha = 0$ characterize pseudo-Riemannian spaces (M,g,α) of constant holomorphic sectional curvature, can be rewritten as follows, using the complex-sesquilinear fibre metric $\langle\,,\rangle_{\mathbf{c}}$ given by (9.3):

$$R(v,w)u \,=\, \lambda\,[\langle u,v\rangle_{\mathbf{c}} w \,-\, \langle u,w\rangle_{\mathbf{c}} v \,+\, 2\alpha(v,w)\,\alpha u]\,, \quad \lambda \,=\, \frac{\mathrm{s}}{n(n+2)}\,, \tag{12.6}$$

or, equivalently (cf. (2.19))

$$R(v,w)u \,=\, \lambda\,[\langle u,v\rangle_{\mathbf{c}} w \,-\, \langle u,w\rangle_{\mathbf{c}} w \,+\, \langle w,v\rangle_{\mathbf{c}} u \,-\, \langle v,w\rangle_{\mathbf{c}} u] \tag{12.7}$$

with $\lambda = \mathrm{s}/[n(n+2)]$, for all $x \in M$ and $v,w,u \in T_xM$.

Lemma 12.4. *Suppose that (M,g,α) is an almost Hermitian pseudo-Riemannian manifold and ξ is a differential 1-form of class C^∞ on M such that, for some real number $c\neq 0$,*

$$c\,d\xi \,=\, 2\alpha\,, \tag{12.8}$$

where α is treated as a differential 2-form, and $d\xi$ is given by (2.16). Let $\mathcal{E}=TM\oplus[M\times\mathbf{C}]$ be the complex vector bundle over M obtained as the direct sum of TM, with the complex structure introduced by α, and the product line bundle $M\times\mathbf{C}$. Sections of $\mathcal{E}$ then are pairs $\psi=(u,f)$ formed by a vector field u tangent to M and a complex–valued function f. Also, let ∇ denote the Levi-Civita connection of (M,g), and let $\langle\,,\rangle_{\mathbf{c}}$ be given by (9.3). The formula

$$\mathrm{D}_v(u,f)\,=\,\big(\nabla_{\!v}u\,+\,i\,\xi(v)u\,+\,fv\;,\quad d_vf\,+\,i\,\xi(v)f\,-\,c^{-1}\langle u,v\rangle_{\mathbf{c}}\big)\,, \tag{12.9}$$

for such C^1 sections $\psi=(u,f)$, and vectors v tangent to M, then defines a connection D in $\mathcal{E}$ regarded as a real *vector bundle. Furthermore,*

(i) *If $\nabla\alpha\,=\,0$, then the pseudo-Hermitian fibre metric $\langle\,,\rangle$ in $\mathcal{E}$ given by*

$$\langle(u,f),(w,h)\rangle\,=\,\langle u,w\rangle_{\mathbf{c}}\,+\,cf\overline{h} \tag{12.10}$$

is compatible with D and D is a connection in $\mathcal{E}$ treated as a complex *vector bundle. In other words, both $\langle\,,\rangle$ and the multiplication by i in $\mathcal{E}$ are D-parallel.*

(ii) *D is flat if and only if the following two conditions hold*:
 a) *$\nabla\alpha\,=\,0$, and*
 b) *The curvature tensor R of (M,g) satisfies* (10.5) *with $\lambda=\mu=1/c$.*
Thus, flatness of D implies that (M,g,α) is a nonflat space of constant holomorphic sectional curvature.

Proof. Assertion (i) is immediate from (9.4). On the other hand, since ∇ is torsionfree, relation (12.8) can also be rewritten as

$$[\nabla_{\!v}\xi]\,(w)\,-\,[\nabla_{\!w}\xi]\,(v)\,=\,2c^{-1}\alpha(v,w) \tag{12.11}$$

for all vectors v,w tangent to M. Combining formulae (4.52) and (9.3) (with the simplifications described in Remark 4.4), we now see that the curvature tensor R^{D} of D is given by $R^{\mathrm{D}}(v,w)(u,f)\,=\,\big(R_1^{\mathrm{D}}(v,w)(u,f)\,,\;R_2^{\mathrm{D}}(v,w)(u,f)\big)$, where

$$\begin{aligned} R_1^{\mathrm{D}}(v,w)(u,f)\,&=\,R(v,w)u\,+\,c^{-1}\left[\langle u,w\rangle_{\mathbf{c}}v\,-\,\langle u,v\rangle_{\mathbf{c}}w\,+\,2\alpha(w,v)\alpha u\right],\\ R_2^{\mathrm{D}}(v,w)(u,f)\,&=\,ic^{-1}\left[(\nabla_{\!w}\alpha)(u,v)\,-(\nabla_{\!v}\alpha)(u,w)\right], \end{aligned} \tag{12.12}$$

for any local C^2 vector fields u,v,w in M and any complex-valued C^2 function f. To prove (ii), let us first suppose that D is flat, i.e., $R_1^{\mathrm{D}}=R_2^{\mathrm{D}}=0$. Condition (ii)a) then follows as $\nabla_{\!w}\alpha)(u,v)=0$ for all u,v,w in view of the second relation in (12.12) and Lemma 3.1, while (ii)b) is immediate from (12.6). Conversely, conditions (ii)a) and (ii)b) imply flatness of D via (12.12). This completes the proof. ■

Lemma 12.5. *Suppose that P and Q are self-adjoint C^∞ bundle morphisms $TM \to TM$ in the tangent bundle TM of a pseudo-Riemannian 4-manifold (M,g), satisfying the conditions*

$$P^2 = P\,, \qquad Q^2 = Q\,, \qquad P + Q = \mathrm{Id}\,. \tag{12.13}$$

Let $\mathcal{E} = TM \oplus [M \times \mathbf{R}^2]$ denote the vector bundle over M obtained as the direct sum of TM and the product plane bundle $M \times \mathbf{R}^2$. Thus, $\mathcal{E}$ is the direct sum of the subbundles $\mathcal{P} = [\mathrm{Ker}\,Q] \oplus [M \times (\mathbf{R}^2 \times \{0\})]$ and $\mathcal{Q} = [\mathrm{Ker}\,P] \oplus [M \times (\{0\} \times \mathbf{R}^2)]$. Furthermore, let ∇ be the Levi-Civita connection of (M,g). Given a fixed real number $c \neq 0$, let us set $\varepsilon = \mathrm{sgn}\,c$. For a C^1 vector field u tangent to M and real-valued C^1 functions φ, χ on M, the triple (u,φ,χ) is a C^1 section of $\mathcal{E}$. Formulae

$$\langle (u,\varphi,\chi),(u',\varphi',\chi')\rangle = g(u,u') + \varepsilon\,(\varphi\varphi' + \chi\chi') \tag{12.14}$$

and

$$\mathrm{D}_v(u,\varphi,\chi) = \Big(\nabla_v u + |c|^{-1/2}[\varphi P v + \chi Q v]\,,\ d_v\varphi - \varepsilon|c|^{-1/2}P(v,u)\,,\ d_v\chi - \varepsilon|c|^{-1/2}Q(v,u)\Big)$$

for vector fields v tangent to M, then define a pseudo-Riemannian fibre metric $\langle\,,\rangle$ and a connection D in $\mathcal{E}$ which is compatible with $\langle\,,\rangle$, that is, $\langle\,,\rangle$ is D-parallel. Furthermore, the subbundles $\mathcal{P}$ and $\mathcal{Q}$ are mutually $\langle\,,\rangle$-orthogonal, and the following two conditions are equivalent:

(i) D *is flat and it makes the subbundles $\mathcal{P}$ and $\mathcal{Q}$ parallel.*
(ii) $\nabla P = \nabla Q = 0$ *and the curvature tensor R of (M,g) satisfies the following relation, with $\circledast$ as in (5.7):*

$$R = \frac{1}{c}\,[P \circledast P + Q \circledast Q]\,. \tag{12.15}$$

Proof. The assertions about $\langle\,,\rangle$ are immediate. On the other hand, by (12.13), P and Q are projections onto the summands $\mathcal{P}$ and $\mathcal{Q}$ of TM, and sections of $\mathcal{P}$ (or $\mathcal{Q}$) are precisely those sections (u,φ,χ) of $\mathcal{E}$ for which $Qu = 0$ and $\chi = 0$ (or, $Pu = 0$ and $\varphi = 0$). Furthermore, multiplying the last equality in (12.13) by P or Q, we obtain

$$PQ = QP = 0\,. \tag{12.16}$$

Thus, it is clear from the definition of D that $\mathcal{P}$ and $\mathcal{Q}$ are D-parallel if and only if $\nabla P = \nabla Q = 0$.

Let us now assume that $\nabla P = \nabla Q = 0$. In view of (4.52) and (12.16) (along with the simplification provided by Remark 4.4), the curvature tensor R^{D} of D then is given by

$$R^{\mathrm{D}}(v,w)(u,\varphi,\chi) = \big(R(v,w)u - c^{-1}[P(v,u)Pw - P(w,u)Pv + Q(v,u)Qw - Q(w,u)Qv]\,,\ 0\,,\ 0\big)\,,$$

which completes the proof. ■

Example 12.6. Let $\mathcal{X}$ be an n-dimensional vector space of C^∞ vector fields on an n-dimensional manifold M such that

(a) Each $v \in \mathcal{X}$ is either identically zero, or nonzero everywhere in M.
(b) $\mathcal{X}$ is closed under the Lie-bracket operation, that is, forms a Lie algebra of vector fields on M.

Note that, for a vector space $\mathcal{X}$ defined to be the span of C^∞ vector fields e_j, $j = 1, \ldots, n$, condition (a) means that the e_j trivialize the tangent bundle TM, i.e., at each $x \in M$ the $e_j(x)$ form a basis of the tangent space T_xM, while (b) says that there exist real constants c_{jk}^l such that $[e_j, e_k] = c_{jk}^l e_l$ for all $j, k \in \{1, \ldots, n\}$, with summation over l.

Such a Lie algebra $\mathcal{X}$ naturally distinguishes various flat connections in TM. Besides the most obvious one (which makes all $w \in \mathcal{X}$ parallel), we also have the connection D in TM uniquely characterized by

$$\mathrm{D}_v w \,=\, [v, w] \tag{12.17}$$

for all $v, w \in \mathcal{X}$. In other words, for any fixed basis e_j of $\mathcal{X}$, $\mathrm{D}_v w = v^j [d_j w^k + w^l c_{jl}^k]\, e_k$ with constants c_{jk}^l as above, where $v = v^j e_j$, $w = w^j e_j$ now are *arbitrary* C^1 vector fields and d_j stands for the directional derivative in the direction of e_j. Then

(i) D is flat.
(ii) The D-parallel local sections of TM are precisely those C^1 vector fields on open subsets of M which commute with every $v \in \mathcal{X}$.

In fact, (i) is obvious if one evaluates its curvature R^{D} using (4.23); namely, for $v, w, u \in \mathcal{X}$ we then obtain $R^{\mathrm{D}}(v, w)u = [w, [v, u]] - [v, [w, u]] + [[v, w], u] = 0$ in view of the Jacobi identity. As for (ii), it is immediate from (12.17).

The situation described in Example 12.6 characterizes, locally, the case where $\mathcal{X}$ is the Lie algebra of all left-invariant vector fields on any given n-dimensional Lie group $M = G$; the D-parallel sections of TM then coincide with the right-invariant vector fields on G.

See also Remark 17.23 in §17.

§13. Submanifolds

Given a C^1 mapping $F : M \to N$ between manifolds, by the *rank* of F we mean the integer-valued function $\operatorname{rank} F$ on M, which assigns to each x the number $\dim [dttF_x(T_xM)]$. The following classical result is known as *the rank theorem.*

Theorem 13.1. *Any C^∞ mapping $F : M \to N$ between manifolds of dimensions $m = \dim M$, $n = \dim N$, such that $\operatorname{rank} F$ is constant and equal to r in a neighborhood of a given point $z \in M$, has the form $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^r, 0, \ldots, 0)$, with $n - r$ zeros, in suitable local coordinate systems x^j for M, defined near z, and y^α for N, defined near $F(z)$. In other words, the component functions $F^\alpha = y^\alpha(F)$ of F in such coordinates are*

$$F^A \,=\, x^A \quad \text{for } A \le r\,, \qquad F^\lambda = 0 \quad \text{for} \quad \lambda > r\,. \tag{13.1}$$

Proof. Let us fix the following ranges for indices: $1 \le j, k \le m$, $1 \le \alpha, \beta \le n$, $1 \le A, B \le r$, $r < \lambda, \mu \le n$, and start from arbitrary local coordinates x^j and y^α, which we then modify in three successive steps (keeping the notation unchanged). First, by permuting the x^j and the y^α, we may assume that $\det[\partial_A F^B] \neq 0$. Second, we may require that $F^A = x^A$ and $F^\lambda = \Psi^\lambda(x^1,\dots,x^r)$ with some C^∞ functions Ψ^λ of r variables, by replacing $x^1,\dots,x^m$ with the new coordinates $F^1,\dots,F^r,x^{r+1},\dots,x^m$ defined near z (this is a coordinate system in view of the inverse mapping theorem). Then, in the new coordinates, $F^A = x^A$, and so $\partial_\mu F^\lambda = 0$ as $\operatorname{rank} F = r$. Finally, to achieve $F^A = x^A$ and $F^\lambda = 0$, let us replace the y^λ by $y^\lambda - \Psi^\lambda(y^1,\dots,y^r)$ with Ψ^λ as above, leaving the y^A unchanged. In the new coordinates, (13.1) holds, which completes the proof. ■

A C^∞ mapping $F: M \to N$ between manifolds is called an *immersion* if its differential $dF_x : T_xM \to T_{F(x)}N$ is injective at each point $x \in M$. Since that condition amounts to $\operatorname{rank} F = \dim M$ everywhere in M, it follows from Theorem 13.1 that any immersion must *locally injective*. By an *embedding* $F: M \to N$ we mean an immersion that is also *globally* injective. A *submanifold* of a manifold N is a manifold M such that the underlying set of M is a subset of N and the inclusion mapping $M \to N$ is an embedding (i.e., is both of class C^∞, and an immersion). Note that the manifold topology of M then *need not be* the subset topology inherited from N.

Corollary 13.2. *Suppose that $F: M \to N$ is a C^∞ mapping between manifolds and $P \subset M$ is a submanifold equipped with the subset topology. If F is an embedding, while $\dim M = \dim P$ and $F(M) \subset P$, then $F(M)$ is open as a subset of P, and F is a diffeomorphism of M onto $F(M)$ treated as an open submanifold of P.*

Proof. Set $m = \dim M = \dim P$ and $n = \dim N$, and fix $z \in M$. Choosing local coordinate systems x^j for M (near z) and y^α for N (with a coordinate domain U conataining $F(z)$) as in Theorem 13.1, we can make F appear as $(x^1,\dots,x^m) \mapsto (x^1,\dots,x^m,0,\dots,0)$, with $n-m$ zeros. Since the inclusion mapping $P \to N$ is a C^∞ immersion and the differentials $dy^1,\dots,dy^m$ are linearly independent in the cotangent space T_x^*N for any $x \in U$, it is clear that the intersection $P \cap U$ is open in P and the differentials of the restrictions of $y^1,\dots,y^m$ to $P \cap U$ (treated as functions on an open subset of P) are also linearly independent in T_x^*P for any $x \in P \cap U$. Since $\dim P = m$, we may therefore assume, replacing U with a smaller neighborhood of $F(z)$ if necessary, that the functions $y^1,\dots,y^m$ restricted to $P \cap U$ form a coordinate system for P (with its own manifold structure). Thus, F as a mapping $M \to P$ is locally diffeomorphic, since suitable local coordinates make it look like the identity mapping. This completes the proof. ■

Let $F: M \to N$ be a C^1 mapping between manifolds. Given a fixed pseudo-Riemannian metric h on N, we will say that F is *nondegenerate* (as a mapping of M into the pseudo-Riemannian manifold (N,h)), if the pullback tensor field F^*h given by (2.30) is nondegenerate at each point, that is, if $g = F^*h$ is a pseudo-Riemannian metric on M. A nondegenerate mapping F is necessarily an immersion, since, for $v \in T_xM$, $dF_x v = 0$ only if v is $(F^*h)_x$-orthogonal to all of T_xM. Conversely, if h is a *Riemannian* metric on N, every immersion $M \to N$ is a nondegenerate mapping into (N,h).

Let $F: M \to N$ now be a nondegenerate C^∞ mapping of a manifold M into a pseudo-Riemannian manifold (N,h). The *second fundamental form* of F is the object B associating with each $x \in M$ a bilinear symmetric mapping $B = B(x):$

$T_xM \times T_xM \to \mathcal{N}_x$ into the *normal space* of F at x (that is, the h-orthogonal complement $\mathcal{N}_x$ of $dF_x(T_xM)$ in $T_{F(x)}N$; in other words, B is a section of the vector bundle $[T^*M]^{\odot 2}\otimes\mathcal{N}$). Specifically, for $x \in M$ and $v, w \in T_xM$, let us choose a C^1 curve $t \mapsto x(t) \in M$ and a C^1 tangent vector field $t \mapsto w(t) \in T_{x(t)}M$ along it in such a way that, for some fixed parameter t_0, $x(t_0) = x$, $\dot{x}(t_0) = v$ and $u(t_0) = w$. This gives rise to the C^1 tangent vector field $t \mapsto \mathbf{w}(t) \in T_{\mathbf{x}(t)}N$ along the C^1 curve $t \mapsto \mathbf{x}(t) = F(x(t)) \in N$, given by $\mathbf{w}(t) = dF_{x(t)}[w(t)]$. Denoting D the Levi-Civita connection of (N, h), we then set

$$B(v,w) \,=\, [\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w}]^{\mathrm{norm}}\,. \tag{13.2}$$

In other words, we declare $B(v, w)$ to be the component of $\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w} \in T_{F(x)}N$ (at $t = t_0$) normal to $dF_x(T_xM)$, that is, the $\mathcal{N}_x$-part of $\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w}$ (at $t = t_0$) relative to the decomposition

$$T_{F(x)}N \,=\, dF_x(T_xM) \,\oplus\, \mathcal{N}_x\,. \tag{13.3}$$

For any local coordinates x^j in M and y^λ in N, defined near x and $F(x)$, respectively, and any normal vector $\mathbf{u} \in \mathcal{N}_x$, we then have

$$h(B(v,w),\,\mathbf{u}) \,=\, h_{\lambda\mu}u^\lambda v^j w^k \,\left(\partial_j\partial_k F^\mu \,+\, \varGamma^\mu_{\rho\sigma}[\partial_j F^\rho]\,\partial_k F^\sigma\right)\,, \tag{13.4}$$

where, as usual, the component functions $h_{\lambda\mu} = h(e_\lambda,\, e_\mu)$ of h and the second-order partial derivatives $\partial_j\partial_k F^\mu$ of the component functions F^μ of F are evaluated at x and $F(x)$, respectively, while $\mathbf{u} = u^\lambda\, e_\lambda$, and $\varGamma^\mu_{\rho\sigma}$ are the Christoffel symbols of (N, h) (cf. (4.1)). In fact, extending $\mathbf{u}$ to a C^1 normal vector field $t \mapsto \mathbf{u}(t) \in \left[dF_{x(t)}(T_xM)\right]^\perp \subset T_{\mathbf{x}(t)}N$ along the C^1 curve $t \mapsto \mathbf{x}(t) = F(x(t)) \in N$, we obtain $h(B(v, w),\, \mathbf{u}) = h(\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w},\, \mathbf{u})$, which equals the right-hand side of (13.4) since, according to (4.13), $\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w} = [dw^\mu/dt \,+\, \varGamma^\mu_{\rho\sigma}\dot{x}^\rho w^\sigma]\,e_\mu$, with $\dot{x}^\lambda = \dot{x}^j\,\partial_j F^\lambda$, $w^\lambda = w^j\,\partial_j F^\lambda$, while the vector field $[\dot{w}^j\,\partial_j F^\lambda]\,e_\lambda$ is tangent to the F-image of M and hence h-orthogonal to $\mathbf{u}$.

Formula (13.4) clearly shows that $B(v, w)$ is bilinear and symmetric in v and w, and also well-defined, that is, independent of the choices of $x(t)$ and $w(t)$. Due to symmetry, B is completely determined by its quadratic restriction, so that, choosing $w(t) = \dot{x}(t)$, we may characterize it by

$$B(\dot{x},\dot{x}) \,=\, [\mathrm{D}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}]^{\mathrm{norm}} \tag{13.5}$$

for any C^2 curve $t \mapsto x(t) \in M$ with the F-image $\mathbf{x}(t) = F(x(t))$. If, in addition, $\dim M = \dim N - 1$, we may choose (at least locally) a *unit normal vector field* for F, that is, a local C^∞ section $\mathbf{n}$ of the normal bundle $\mathcal{N}$ of F with $h(\mathbf{n}, \mathbf{n}) = \pm 1$. Then, for some symmetric twice-covariant C^∞ tensor field b in M, we have

$$B \,=\, b \otimes \mathbf{n}\,, \tag{13.6}$$

that is, $B_x(v, w) = [b_x(v, w)]\,\mathbf{n}(x)$ for all $x \in M$ and $v, w \in T_xM$. One also calls b the (real-valued) *second fundamental form* of F. Formula (13.4) now becomes the expression

$$b_{jk} \,=\, \varepsilon\, h_{\lambda\mu}u^\lambda\,\left[\partial_j\partial_k F^\mu \,+\, \varGamma^\mu_{\rho\sigma}(\partial_j F^\rho)\,\partial_k F^\sigma\right]\,,\quad h(\mathbf{n},\mathbf{n}) \,=\, \varepsilon \,=\, \pm 1\,, \tag{13.7}$$

for the component functions of b relative to any local coordinates x^j in M.

Lemma 13.3. *Suppose that F is a nondegenerate C^∞ mapping of a manifold M into a pseudo-Riemannian manifold (N,h), and let D and ∇ be the Levi-Civita connections of h and, respectively, of the pullback pseudo-Riemannian metric $g = F^*h$ on M. Given a C^1 tangent vector field $t \mapsto w(t) \in T_{x(t)}M$ along a C^1 curve $t \mapsto x(t) \in M$, let us set $\mathbf{x}(t) = F(x(t)) \in N$ and $\mathbf{w}(t) = dF_{x(t)}[w(t)] \in T_{\mathbf{x}(t)}N$. Then, for every t we have, at $x = x(t)$,*

$$dF_x(\nabla_{\dot{x}} w) \,=\, [\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w}]^{\mathrm{tang}} \tag{13.8}$$

$[\mathbf{v}]^{\mathrm{tang}}$ *being the $dF_x(T_xM)$-component of $\mathbf{v} \in T_{\mathbf{x}}N$ relative to the decomposition* (13.3).

Proof. Every point of M has a neighborhood U such that the immersion F restricted to U is an embedding, while U is the domain of both a local coordinate system x^j for M $(j = 1,\ldots,n = \dim M)$ and a local trivialization $\mathbf{u}_a$ of the normal bundle of F. From now on we will identify each $x \in U$ with $\mathbf{x} = F(x) \in F(U)$, i.e., treat U as a submanifold of N, for which F is the inclusion mapping. Using a suitable neighborhood U' of $(0,\ldots,0)$ in $\mathbf{R}^q$, we may define a mapping $\varPhi$: $U \times U' \to N$ by $\varPhi(x,\xi^1,\ldots,\xi^q) = \exp_x[\xi^a\mathbf{u}_a(x)] \in N$, where $q = \dim N - \dim M$ and $\exp$ denotes the geodesic exponential mapping of (N,h). From the inverse mapping theorem it follows that, if U and U' are chosen sufficiently small, $\varPhi$ is a diffeomorphism onto a neighborhood U'' of the given point in N. Using $\varPhi$ to identify U'' with $U \times U'$, we can now regard $x^1,\ldots,x^n,\xi^1,\ldots,\xi^q$ as a local coordinate system in N. The corresponding coordinate vector fields then have the property that, at each point of $U \subset N$, e_j are tangent to U and e^a_ξ are normal to U, i.e., the components of h relative to these coordinates satisfy

$$h_{jk} \,=\, g_{jk}\,, \qquad h_{ja} \,=\, 0 \tag{13.9}$$

for $j,k = 1,\ldots,n$ and $a = 1,\ldots,q$, where g_{jk} are the components of g relative to the coordinates x^j in M. Hence, by (4.1), the Christoffel symbols of h with all three indices j,k,l in the range $\{1,\ldots,n\}$ coincide with the corresponding Christoffel symbols $\varGamma^l_{jk}$ of g. Thus, by (4.13) and (13.9), $[\nabla_{\dot{\mathbf{x}}}\mathbf{w}]^j = [\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w}]^j$, $j = 1,\ldots,n$. This completes the proof. ■

Corollary 13.4. *Suppose that M is a nondegenerate submanifold of a pseudo-Riemannian manifold (N,h) and $\mathbf{w}$ is a parallel vector field on N which is tangent to M at each point of M. Then*

(a) *The restriction of $\mathbf{w}$ to M, treated as a tangent vector field on M, then is parallel relative to the Levi-Civita connection of the submanifold metric g that M inherits from V*

(b) *The geodesic $t \mapsto \mathbf{x}(t) \in N$ of (N,h), defined for t near 0 in $\mathbf{R}$ and satisfying the initial conditions $\mathbf{x}(0) = \mathbf{x} \in M$ and $\dot{\mathbf{x}}(0) = \mathbf{w}(\mathbf{x}(0))$, lies in M for all t sufficiently close to 0, and forms a geodesic of (M,g).*

In fact, (a) is obvious from (13.8), while (b) follows from the uniqueness-of-solutions theorem for ordinary differential equations applied to the geodesics on (N,h) and (M,g) satisfying the given initial conditions: Since $\mathbf{w}$ is parallel (in both M and N, its integral curve through $\mathbf{x}$ must coincide with both of these geodesics. ■

Remark 13.5. Let $F : M \to V$ be a nondegenerate C^∞ mapping of a manifold M into a pseudo-Euclidean vector space V with a constant metric. In the codimension-one case (i.e., when $\dim M = \dim V - 1$), the following traditional notations conveniently summarize the preceding discussion. First, let us use the dot symbol $\cdot$ (rather than $\langle\,,\rangle$) for the inner product of V. As before, let $\mathbf{n}$ be a fixed C^∞ unit normal vector field, so that $\mathbf{n}\cdot\mathbf{n} = \varepsilon$ with $\varepsilon = \pm 1$, and let b denote the real-valued second fundamental form of F (see (13.6)) relative to the unit normal field $\mathbf{n}$. Instead of F, one may use the traditional generic symbol $\mathbf{r}$. For any fixed local coordinate system x^j in M, one represents the partial derivatives $\partial/\partial x^j$ of V-valued functions (such as $\mathbf{n}$ or $\mathbf{r}$) by subscripts, as in $\mathbf{r}_j = \partial\mathbf{r}/\partial x^j$. Each $\mathbf{r}_j$ then is tangent to M, namely, it coincides with the coordinate vector field e_j in the direction of x^j (see (2.1)). The metric g on M obtained as the pullback under F of the constant metric in V is traditionally referred to as the *first fundamental form* of the immersion F, and its component functions g_{jk} can also be expressed as $g_{jk} = \mathbf{r}_j\cdot\mathbf{r}_k$. Applying (13.2) and (13.8) to $\mathbf{w} = \mathbf{r}_j$ along a curve $t \mapsto x(t) \in M$ which is a *coordinate line* (i.e., for some k, $x^k(t) = t$ and $x^l(t)$ is constant whenever $l \neq k$), we now obtain the second-derivative formula

$$\mathbf{r}_{jk} \,=\, \Gamma^l_{jk}\mathbf{r}_l \,+\, b_{jk}\mathbf{n}\,, \qquad g_{jk} \,=\, \mathbf{r}_j\cdot\mathbf{r}_k\,, \qquad b_{jk} \,=\, \varepsilon\,\mathbf{n}\cdot\mathbf{r}_{jk}\,, \tag{13.10}$$

where the Γ^l_{jk} are the Christoffel symbols of g (see (4.1)). A more general version of (13.10), for higher codimensions, may be obtained by choosing normal vector fields $\mathbf{n}_a$ that locally trivialize the normal bundle of F; then, (13.10) will remain valid if we replace the last term with $B^a_{jk}\mathbf{n}_a$, with suitable functions $B^a_{jk} = B^a_{kj}$.

Theorem 13.6 (Gauss, Codazzi, Bonnet)**.** *Let b be a symmetric twice-covariant C^∞ tensor field on a pseudo-Riemannian manifold (M, g), and let $\varepsilon = \pm 1$. The following two conditions then are equivalent*:

- (i) *The curvature tensor R of (M, g) along with b and ε satisfy the Gauss and Codazzi equations* (12.4)*, i.e.,* (12.2) *and* (12.3)*.*
- (ii) *Every point $x \in M$ has a neighborhood U that admits a codimension-one isometric embedding F into a pseudo-Euclidean vector space V with a constant metric $\langle\,,\rangle$ such that b is the real-valued second fundamental form of F relative to a C^∞ normal vector field $\mathbf{n}$, as in* (13.6)*, and $\langle\mathbf{n},\,\mathbf{n}\rangle = \varepsilon$.*

Proof. Necessity: If F with these properties exists, then the connection D in $\mathcal{E} = TM \oplus [M \times \mathbf{R}]$, given by (12.1), must be flat since, in view of (13.2) and (13.8), it is the pullback under F of the Levi-Civita connection of $(V, \langle\,,\rangle)$. Sufficiency (this part is known as *Bonnet's theorem*): Using flatness of D and Lemma 11.2, we may select, in a neighborhood of any point x of M, a local trivialization $e_a = (u_a, f_a)$ of $\mathcal{E}$ consisting of parallel sections. Since D is obviously compatible with the pseudo-Riemannian fibre metric in $\mathcal{E}$ (also denoted $\langle\,,\rangle$), which is obtained as the direct sum of g and $\varepsilon\,dt^2$ (where t is the standard coordinate in the fibre $\mathbf{R}$ of $M \times \mathbf{R}$), we may also require that the e_a be h-orthonormal. Then, with $\varepsilon_a = \langle e_a, e_a\rangle = \pm 1$,

$$g \,=\, \sum_a \varepsilon_a\, u_a \otimes\, u_a \tag{13.11}$$

in the sense that $g(v, w) = \sum_a \varepsilon_a g(v, u_a) g(w, u_a)$ for any vector fields v, w tangent to M near x. (In fact, as the e_a are $\langle\,,\rangle$-orthonormal, $g(v, w) = \langle(v, 0), (w, 0)\rangle =$

$\sum_a \varepsilon_a \langle(v,0),e_a\rangle\cdot\langle(w,0),e_a\rangle$.) On the other hand, in view of (12.1), relation $\mathrm{D}e_a = 0$ implies $\nabla_v w_a = \varepsilon f_a b(v,\,\cdot\,)$ for all vectors v tangent to M at points near x. Thus, symmetry of b yields $dw_a = 0$, where w_a is identified with the 1-form $g(\,\cdot\,,w_a)$ in M. Hence, by Corollary 11.3, near x we have $\langle\,\cdot\,,w_a\rangle = dF_a$ for some functions F_a, $a = 0,1,\dots,n$ $(n = \dim M)$. Setting $F = (F_0,F_1,\dots,F_n)$, we then obtain $g = \sum_a \varepsilon_a\, dF_a \otimes dF_a$ in view of (10.31), i.e., g is the pull-back under F of the standard inner product in $\mathbf{R}^{n+1}$ with the sign pattern $(\varepsilon_0,\dots,\varepsilon_n)$. This completes the proof. ■

Let B be the second fundamental form of a nondegenerate immersion $F : M \to (N,h)$. One says that F is *totally geodesic* if B is identically zero. More generally, F is called *totally umbilical* if there exists a section $\mathbf{u}$ of the normal bundle $\mathcal{N}$ of F with

$$B \,=\, g\otimes\mathbf{u} \qquad \text{with} \qquad g \,=\, F^*h \tag{13.12}$$

(notation as in (2.29)), in the sense that $B_x(v,w) = [g_x(v,w)]\,\mathbf{u}(x)$ for all $x\in M$ and $v,w\in T_xM$. This normal vector field $\mathbf{u}$ then is referred to as the *mean curvature vector* (field) of F.

We say that a submanifold M of a pseudo-Riemannian manifold (N,h) is *nondegenerate* if so is the inclusion mapping $F : M \to N$. Both the second fundamental form, and the properties of being totally umbilical (or, totally geodesic) thus make sense for nondegenerate submanifolds. In particular, by a nondegenerate subspace of a pseudo-Euclidean vector space V (see (3.22)) is nothing else than a vector subspace which is nondegenerate as a submanifold of V.

Lemma 13.7. *Given a finite-dimensional real or complex vector space V and C^1 curves $I\ni t\mapsto w_a(t)\in V$ of vectors in V, $a=1,\dots,m$, defined on an interval I and such that, for each $t\in I$, the vectors $w_1(t),\dots,w_m(t)$ are linearly independent, the following two conditions are equivalent*:

(a) *The subspace $\mathcal{W}=\mathcal{W}(t)$ spanned by $w_1(t),\dots,w_m(t)$ is the same for all $t\in I$.*

(b) *For some continuous functions $I\ni t\mapsto f_a^b(t)$, we have*

$$\dot w_a(t) \,=\, f_a^b(t)\,w_b(t)\,, \qquad a=1,\dots,m\,, \tag{13.13}$$

with a summation over $b=1,\dots,m$.

Proof. If $\mathcal{W}=\mathcal{W}(t)$ does not depend on t, (13.13) follows since the $w_b(t)$ form a basis of $\mathcal{W}$ and $\dot w_a(t)\in\mathcal{W}$. Conversely, let us assume (b) and fix $t_0\in I$. Solving (13.13) as a system of linear ordinary differential equations with the unknown functions w_a *valued in the space* $\mathcal{W}(t_0)$ *rather than* V, with the initial values $w_a(t_0)$ at $t=t_0$, we obtain a solution with $w_a(t)\in\mathcal{W}(t_0)$ for all a and t, and so (i) follows from the uniqueness-of-solutions theorem for ordinary differential equations. This completes the proof. ■

Lemma 13.8. *The totally geodesic nondegenerate submanifolds of any fixed dimension n in a given pseudo-Euclidean vector space V with the constant metric $h=\langle\,,\rangle$* (Example 10.3) *are precisely the open submanifolds of arbitrary cosets of n-dimensional nondegenerate vector subspaces of V.*

Proof. Let M be a nondegenerate submanifold of V, and let $t \mapsto \mathbf{w}(t) \in T_{\mathbf{x}(t)}M$ be a C^1 tangent vector field along a C^1 curve $t \mapsto \mathbf{x}(t) \in M$. Thus, at any t, $B(\dot{\mathbf{x}}(t), \mathbf{w}(t))$ equals the $[T_{\mathbf{x}(t)}M]^\perp$-component of $\mathrm{D}_{\dot{\mathbf{x}}}\mathbf{w} = d\mathbf{w}/dt$.

Suppose now that $M = x_0 + \mathcal{W}$ is a coset of a nondegenerate vector subspace $\mathcal{W}$ of V. Thus, we have $T_{\mathbf{x}(t)}M = \mathcal{W}$, and so $d\mathbf{w}/dt \in \mathcal{W}$ for all t, i.e., its $\mathcal{W}^\perp$-component is zero, which shows that $B = 0$.

Conversely, let $B = 0$, and let $\mathbf{w}_j(t)$, $j = 1, \dots, n$, be C^1 tangent vector fields along any C^1 curve $t \mapsto \mathbf{x}(t) \in M$, forming at each t a basis of at $T_{\mathbf{x}(t)}M$. Now, as $B = 0$, (13.2) implies that $d\mathbf{w}_j/dt \in T_{\mathbf{x}(t)}M$, i.e., $d\mathbf{w}_j/dt = f_j^k(t)\mathbf{w}_j(t)$ (summed over $k = 1, \dots, n$), with some continuous functions $t \mapsto f_j^k(t)$. By Lemma 13.7, the tangent space $T_{\mathbf{x}(t)}M$ of M is constant along the curve $t \mapsto \mathbf{x}(t)$ and, M is connected (by definition), $T_{\mathbf{x}}M$ is the same for all points $\mathbf{x} \in M$. Thus, the normal space of M is the same at all points; choosing its basis $\mathbf{u}_a$ ($a = n+1, \dots, m = \dim V$), we see that, for suitable constants c_a, every $\mathbf{x} \in M$ obeys the system of $m - n$ equations $\langle \mathbf{u}_a, \mathbf{x} \rangle = c_a$ (since $\langle \mathbf{u}_a, d\mathbf{x}/dt \rangle = 0$ for every C^1 curve $t \mapsto \mathbf{x}(t) \in M$). In view of Corollary 13.2, this completes the proof. ■

Lemma 13.9. *Given an n-dimensional nondegenerate submanifold M, $n \geq 2$, of a pseudo-Euclidean vector space V of any dimension, with the constant metric $h = \langle\, ,\rangle$* (Example 10.3)*, the following two conditions are equivalent*:

(i) *M is totally umbilical and its mean curvature vector $\mathbf{u}$ appearing in* (13.12) *is not null, that is, $\langle \mathbf{u}, \mathbf{u} \rangle \neq 0$ somewhere in M.*

(ii) *M is contained as an open submanifold in the intersection $\Sigma \cap (\mathcal{W} + \mathbf{o})$ of a pseudosphere*

$$\Sigma = \{\mathbf{x} \in V : \langle \mathbf{x} - \mathbf{o}, \mathbf{x} - \mathbf{o} \rangle = c\} \tag{13.14}$$

in V, with some center $\mathbf{o}$ and some real $c \neq 0$, and the coset through $\mathbf{o}$ of an $(n+1)$-dimensional nondegenerate vector subspace $\mathcal{W}$ of V.

Furthermore, for M, $\mathbf{o}$, c as in (ii)*, formula*

$$\mathbf{n}(\mathbf{x}) = |c|^{-1/2}(\mathbf{x} - \mathbf{o}), \qquad \mathbf{x} \in M, \tag{13.15}$$

defines a unit normal vector field $\mathbf{n}$ for M with $\langle \mathbf{n}, \mathbf{n} \rangle = \varepsilon$, where $\varepsilon = \operatorname{sgn} c = \pm 1$, and the second fundamental form B of M is given by

$$B_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = -\,\varepsilon |c|^{-1/2} \langle \mathbf{v}, \mathbf{w} \rangle\, \mathbf{n}(\mathbf{x}), \tag{13.16}$$

for all $\mathbf{x} \in M$ and $\mathbf{v}, \mathbf{w} \in T_{\mathbf{x}}M \subset V$.

Proof. Assume (ii). For any fixed C^2 curve $t \mapsto \mathbf{x}(t) \in M$, let us write $\mathbf{x}$ instead of $\mathbf{x}(t)$ and set $\mathbf{v} = \mathbf{v}(t) = c\ddot{\mathbf{x}} - \langle \ddot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle [\mathbf{x} - \mathbf{o}]$ with $\ddot{\mathbf{x}} = d^2\mathbf{x}/dt^2$. Since $T_{\mathbf{x}}M = (\mathbf{x} - \mathbf{o})^\perp \cap \mathcal{W}$ and $\mathbf{v} \in \mathcal{W}$ is clearly orthogonal to $\mathbf{x} - \mathbf{o}$, we have $\mathbf{v} \in T_{\mathbf{x}}M$ and, as $c\ddot{\mathbf{x}} = \mathbf{v} + \langle \ddot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle [\mathbf{x} - \mathbf{o}]$, we see that $\langle \ddot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle [\mathbf{x} - \mathbf{o}]$ is the component of $c\ddot{\mathbf{x}} = c\mathrm{D}_{\dot{\mathbf{x}}}\dot{\mathbf{x}}$ normal to M. Thus, by (13.5), $cB(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = \langle \ddot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle [\mathbf{x} - \mathbf{o}]$. However, applying d/dt twice in a row to the relation $\langle \mathbf{x} - \mathbf{o}, \mathbf{x} - \mathbf{o} \rangle = c$, we obtain $\langle \dot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle = 0$ and $\langle \ddot{\mathbf{x}}, \mathbf{x} - \mathbf{o} \rangle = -\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle$ for all t. Consequently, $cB(\dot{\mathbf{x}}, \dot{\mathbf{x}}) = -\langle \dot{\mathbf{x}}, \dot{\mathbf{x}} \rangle [\mathbf{x} - \mathbf{o}]$ and so (i) and (13.16) are immediate from symmetry of B.

Conversely, let us assume (i) and fix a local coordinate system x^j in M. Using the traditional notations described in Remark 13.5, we can express (13.12) as

$$\mathbf{r}_{jk} \,=\, \Gamma_{jk}^{l}\mathbf{r}_l \,+\, g_{jk}\mathbf{u}\,, \qquad g_{jk} \,=\, \mathbf{r}_j\!\cdot\!\mathbf{r}_k\,, \tag{13.17}$$

where the Γ_{jk}^{l} are the Christoffel symbols of g (see (4.1)), and $\mathbf{u}$ is the mean curvature vector field of M. Differentiating (13.17) and taking the component normal to M, we thus get $[\mathbf{r}_{jkl}]^{\mathrm{norm}} = (\Gamma_{jk}^{s}g_{sl} + \partial_l g_{jk})\,\mathbf{u} + g_{jk}[\mathbf{u}_l]^{\mathrm{norm}}$, as $[\mathbf{r}_l]^{\mathrm{norm}} = \mathbf{0}$. Hence $\mathbf{0} = [\mathbf{r}_{jkl} - \mathbf{r}_{jlk}]^{\mathrm{norm}} = [g_{jk}\,\mathbf{u}_l - g_{jl}\,\mathbf{u}_k]^{\mathrm{norm}}$. (Note that $\Gamma_{jk}^{s}g_{sl} + \partial_l g_{jk} - \Gamma_{jl}^{s}g_{sk} - \partial_k g_{jl} = 0$ in view of (4.1).) Contracting this against g^{jk}, we obtain $(n-1)[\mathbf{u}_l]^{\mathrm{norm}} = \mathbf{0}$.

Consequently, the partial derivatives $\mathbf{u}_j = \partial\mathbf{u}/\partial x^j$ are all tangent to M, i.e.,

$$\mathbf{u}_j \,=\, h_j^k\mathbf{r}_k \tag{13.18}$$

for some C^∞ functions h_j^k. In view of the second formula in (13.17), this gives $h_j^l g_{lk} = \mathbf{u}_j\!\cdot\!\mathbf{r}_k = -\,\mathbf{u}\!\cdot\!\mathbf{r}_{kj}$ (as $\mathbf{u}\!\cdot\!\mathbf{r}_k = 0$) and so the first relation in (13.17) implies $h_j^l g_{lk} = -\,[\mathbf{u}\cdot\mathbf{u}]\,g_{jk}$, i.e., $h_j^k = -\,[\mathbf{u}\cdot\mathbf{u}]\,\delta_j^k$. Now, using (13.18) and the equality $(\mathbf{u}\!\cdot\!\mathbf{u})_j = 2\,\mathbf{u}_j\!\cdot\!\mathbf{u}$, we obtain

$$\mathbf{u}_j \,=\, -\,(\mathbf{u}\!\cdot\!\mathbf{u})\,\mathbf{r}_j\,, \qquad (\mathbf{u}\!\cdot\!\mathbf{u})_j \,=\, 0\,, \tag{13.19}$$

with the subscripts still standing for partial derivatives.

On the other hand, in view of (13.17) and (13.18), we can apply Lemma 13.7 to $\mathbf{r}_1,\ldots,\,\mathbf{r}_n$ and $\mathbf{u}$ ($n = \dim M$) along any C^1 curve in M, concluding that the vector space $\mathcal{W} \subset V$ spanned by $\mathbf{r}_1,\ldots,\,\mathbf{r}_n$ and $\mathbf{u}$ is the same at all points of M. Choosing a basis $\mathbf{w}_a$ of $\mathcal{W}$ ($a = n+2,\ldots,m = \dim V$), we see that M is contained in a coset of $\mathcal{W}$ consisting of all $\mathbf{x}$ with $\mathbf{w}_a\!\cdot\!\mathbf{x} = c_a$ for some constants c_a (since $\mathbf{w}_a\!\cdot\!\dot{\mathbf{x}} = 0$, with $\dot{\mathbf{x}} = d\mathbf{x}/dt$, for every C^1 curve $t \mapsto \mathbf{x}(t) \in M$).

Furthermore, according to (13.19), $\mathbf{u}\!\cdot\!\mathbf{u}$ is constant on M. The assumption that $\mathbf{u}$ is not null (which we have not used yet) now allows us to define real numbers $c \neq 0$ and ε by $\mathbf{u}\cdot\mathbf{u} = 1/c$ and $\varepsilon = \operatorname{sgn} c = \pm 1$, and a unit normal vector field $\mathbf{n}$ for M with $\mathbf{n}\!\cdot\!\mathbf{n} = \varepsilon$, by $\mathbf{n} = |c|^{-1/2}\,\mathbf{u}$. The first relation in (13.19) now states that the V-valued function $\mathbf{r} - c\,\mathbf{n}$ is constant on M. Denoting $\mathbf{o}$ its constant value, we thus see that M is contained in the pseudosphere Σ with (13.14). Also, as $\mathbf{o} = \mathbf{r} - c\,\mathbf{n}$, the coset of $\mathcal{W}$ containing M must contain $\mathbf{o}$, i.e., is nothing else than the set $\mathcal{W} + \mathbf{o}$. Hence M is a subset of $\Sigma \cap (\mathcal{W} + \mathbf{o})$. Finally, openness of M in Σ follows from Corollary 13.2. This completes the proof. ■

Remark 13.10. In the case where the inner product $\langle\,,\rangle$ of V is positive definite, condition (i) in Lemma 13.9 states that M is totally umbilical, but not totally geodesic. However, for spaces V with indefinite inner products $\langle\,,\rangle$, there exist non-totally geodesic, totally umbilical submanifolds which are not of the type described in Lemma 13.9(ii). Namely, let us choose a degenerate subspace $\mathcal{W}$ of W such that $\mathcal{W} \cap \mathcal{W}^\perp = \mathbf{R}\mathbf{u}$ for some nonzero vector $\mathbf{u} \in V$ (cf. (3.22)) and select a subspace $\mathcal{X}$ of $\mathcal{W}$ which is complementary to $\mathbf{R}\mathbf{u}$ (that is, $\mathcal{W} = \mathcal{X} \oplus [\mathbf{R}\mathbf{u}]$). Furthermore, let $f : \mathcal{X} \to \mathbf{R}$ be any quadratic polynomial function whose second-degree homogeneous part is $\langle\,,\rangle$ (i.e., $f(\mathbf{x}) = \langle \mathbf{x} - \mathbf{o}, \mathbf{x} - \mathbf{o}\rangle + a$ for all $\mathbf{x} \in \mathcal{X}$, with some fixed $\mathbf{o} \in \mathcal{X}$ and $a \in \mathbf{R}$). Then the *pseudo-paraboloid*

$$M \,=\, \{\mathbf{x} \,+\, f(\mathbf{x})\mathbf{u} : \mathbf{x} \in \mathcal{X}\} \tag{13.20}$$

(i.e., the "graph" of f realized in $\mathcal{W}$) is a nondegenerate totally umbilical submanifold of V, for which our $\mathbf{u}$ is its (constant) mean curvature vector field. Note that $\mathbf{u}$ is null, being both in $\mathcal{W}$ and orthogonal to $\mathcal{W}$.

The proof of Lemma 13.9 can be easily modified to show that for any non-totally geodesic, totally umbilical nondegenerate submanifold M of a pseudo-Euclidean vector space V, the mean curvature vector field is either non-null at every point, or null and *constant*, and in the latter case M must be contained as an open submanifold in a translation image of a pseudo-paraboloid of the type just described. In fact, without using the assumption that $\langle \mathbf{u}, \mathbf{u} \rangle \neq 0$, we found that M is contained in a coset of the subspace $\mathcal{W}$ of V spanned by $\mathbf{r}_1, \ldots, \mathbf{r}_n$ and $\mathbf{u}$, while (13.19) with $\mathbf{u} \cdot \mathbf{u} = 0$ shows that $\mathbf{u}$ is constant. Thus, up to a translation in V, M has the form (13.20) for some function $f : \mathcal{X} \to \mathbf{R}$. A fixed basis $\mathbf{e}_j$ of $\mathcal{X}$ leads to the coordinates x^j in M, for which the numbers $x^1, \ldots, x^n$, $n = \dim M$ associated with the point $\mathbf{r} = \mathbf{x} + f(\mathbf{x})\mathbf{u} \in M$ are characterized by $\mathbf{x} = x^j \mathbf{e}_j$. In these coordinates, $\mathbf{r}_j = \mathbf{e}_j$, and so $\mathbf{r}_{jk} = (\partial_j \partial_k f)\, \mathbf{u}$. Thus, in view of (13.17), f now must have constant second-order partial derivatives $\partial_j \partial_k f = g_{jk} = \mathbf{e}_j \cdot \mathbf{e}_k$, i.e., f has to be a quadratic polynomial with the required second-degree part.

§14. The simplest classification theorems

In this section we show that the examples described in §10 have the properties they were claimed to have and, in addition, are uniquely (up to a local isometry) characterized by them. The classification result in question is a special case of a theorem due to Cartan (see Cartan, 1926, and Helgason, 1978).

Proposition 14.1. *Given a real number $c \neq 0$, any connected component of a nonempty pseudosphere $\{v \in V : \langle v, v \rangle = c\}$ in a pseudo-Euclidean vector space V* (Example 10.3) *is a space of constant curvature $K = 1/c$.*

In fact, formula (13.16) gives (13.6) with $b = -\varepsilon |c|^{-1/2} g$, where g is the submanifold metric of the pseudosphere, b is its real-valued second fundamental form, and $\varepsilon = \operatorname{sgn} c = \pm 1$. Theorem 13.6 now implies the Gauss equation (12.2), which is nothing else than (10.1) with $K = 1/c$. ■

Theorem 14.2. *Every pseudo-Riemannian space (M, g) of constant curvature K is locally isometric to*

(i) *a pseudosphere with a metric obtained as in* Example 10.4*, if $K \neq 0$, or*
(ii) *a pseudo-Euclidean vector space with the constant metric provided by its inner product as in* Example 10.3*, if $K = 0$.*

Proof. Let $\varepsilon = \pm 1$ be either arbitrary (if $K = 0$), or equal to $\operatorname{sgn}(K)$ (if $K \neq 0$), and let us set, in both cases, $b = \sqrt{|K|}\, g$. We thus have (12.2) and (12.3) (that is, (12.4)) in view of (10.1) and the obvious relation $\nabla b = 0$. According to Bonnet's Theorem (i.e., the sufficiency part of Theorem 13.6), a connected neighborhood U of any point of (M, g) can be isometrically embedded as a codimension-one submanifold of a pseudo-Euclidean vector space for which b is the real-valued second fundamental form relative to a unit normal vector field $\mathbf{n}$ with $\langle \mathbf{n}, \mathbf{n} \rangle = \varepsilon$. Since b is a multiple of g, this submanifold is totally umbilical, and so it must be contained as a relatively open subset in an affine hyperplane (when $b = 0$; Lemma 13.8), or in a pseudosphere (when $b \neq 0$; Lemma 13.8). This completes the proof. ■

Proposition 14.3. *A pseudo-Riemannian manifold* (M^c, g^c) *obtained as in* Example 10.6*, for any* $c \neq 0$*, is a nonflat space of constant holomorphic sectional curvature. In particular, it satisfies* (10.5) *with a suitable* α *and with* $\lambda = \mu = 1/c$.

Proof. The construction in Example 10.6, leading to a manifold (M^c, g^c) of some (even) real dimension n, begins with a complex vector space V of complex dimension $(n/2) + 1$. We then denote $P(V) \approx \mathbf{C}\mathrm{P}^{n/2}$ the projective space of V, formed by all complex lines through 0 in V, and let $\mathrm{pr} : V \smallsetminus \{0\} \to P(V)$ be the natural projection. Next, we fix a pseudo-Hermitian inner product $\langle\,,\rangle$ in V and a real number $c \neq 0$. Denoting S_c the pseudosphere in V given by (10.4), i.e., $S_c = \{v \in V : \langle v, v\rangle = c\}$, we then define M^c to be the pr-image

$$M^c \,=\, \mathrm{pr}\,(S_c) \,\subset\, P(V) \tag{14.1}$$

of S_c. Obviously, M^c depends only on $\mathrm{sgn}\,(c)$, and is a manifold (namely, an open subset of $P(V)$). Furthermore, M^c carries the quotient pseudo-Riemannian metric g^c induced by the metric of S_c, as described in Example 10.6. Specifically, given $v \in S_c$ and $w \in T_v S_c$ (i.e., any vector $w \in V$ with $\mathrm{Re}\,\langle w, v\rangle = 0$), denoting $u = d\mathrm{pr}_v w$, we have $g^c(u,u) = \langle w', w'\rangle$, where w' is the component of w $\langle\,,\rangle$-orthogonal to v. Thus,

$$g^c(u,u) \,=\, \langle w, w\rangle \,-\, c^{-1}|\langle w, v\rangle|^2\,. \tag{14.2}$$

For any fixed $v \in S_c$, let us choose a codimension-one complex vector subspace V' of V with $v \notin V'$, and let M be the open set in V' formed by all y with $c\langle v + y, v + y\rangle > 0$. The mapping

$$M \ni y \mapsto \,\mathrm{pr}\,(v + y) \tag{14.3}$$

now equals the composite

$$M \,\to\, V' \,\to\, S_c \,\to\, M^c$$

(with $V' = \{x \in V : c\langle x, x\rangle > 0\}$) of the shift $y \mapsto y + v$ followed by the S_c-valued normalization mapping $w \mapsto (v + y)/|w|$ (with $|w| = \sqrt{|\langle w, w\rangle|}$) and then by the restriction of pr to S_c. Clearly, (14.3) is a diffeomorphism of M onto an open subset M' of M^c (in fact, it is the inverse of a standard projective coordinate system). Identifying M' with M via (14.3), we now obtain an almost Hermitian pseudo-Riemannian manifold (M, g, α) (with g corresponding, under (14.3), to g^c), and one easily verifies that (12.8) is satisfied by the 1-form ξ on M given by $[\xi(y)]w = i\,\mathrm{Im}\,\langle v + y, w\rangle/\langle v + y, v + y\rangle$. The complex vector bundle $\mathcal{E} = TM \oplus [M \times \mathbf{C}]$ over M now can be naturally identified with the product bundle $M \times V$ in such a way that the standard flat connection in $M \times V$ corresponds to the connection D in $\mathcal{E}$ given by formula (12.9). In fact, a direct computation show that this can be achieved if the identification in question is chosen to be $(\dot y, z) \mapsto \dot y + z(v + y) \in V$ for any $y \in M$, $z \in \mathbf{C}$ and any $\dot y \in V' = T_y M$. By Lemma 12.4(ii), (M, g) thus is a nonflat space of constant holomorphic sectional curvature, which completes the proof. ■

We have the following converse to Proposition 14.3, which is a local classification theorem for nonflat spaces of constant holomorphic sectional curvature. Note that *flat* pseudo-Riemannian manifolds are already classified by Theorem 14.2(ii).

Theorem 14.4. *Every nonflat pseudo-Riemannian space of constant holomorphic sectional curvature is locally isometric to one of the manifolds* (M^c, g^c) *described in* Example 10.6.

Proof. Let (M, g, α) be a nonflat pseudo-Riemannian space of constant holomorphic sectional curvature. Thus, (10.5) holds with $\lambda = \mu = \mathrm{s}/[n(n{+}2)]$, $n = \dim M$ (see (10.10)), and we may define a real number $c \neq 0$ by $c = 1/\lambda$. Furthermore, since the Kähler form α is parallel, we have $d\alpha = 0$ (see the paragraph following formula (4.22) in §4) and so, according to Poincaré's Lemma for 2-forms (Remark 11.5), any point of M has a neighborhood U' with a differential 1-form ξ satisfying (12.8). Thus, by According to Lemma 12.4(ii), the connection D given by (12.9) in the complex vector bundle $\mathcal{E} = TU' \oplus [U' \times \mathbf{C}]$ over U' is flat. Let us now choose, for any given point $x \in M$, a neighborhood U of x contained in U' and satisfying the assertion of Lemma 11.2 with $\nabla = \mathrm{D}$, and let us denote V the set of all D-parallel sections of $\mathcal{E}$, defined on U. Due to our choice of U, this V is a complex vector space of complex dimension $(n/2) + 1$. Moreover, V carries a pseudo-Hermitian inner product $\langle\,,\rangle$ given by formula (12.10) (which, by Lemma 12.4(i), yields a constant value for a pair of parallel sections (u, f) and (w, h)). Denoting S_c the pseudosphere in V given by $S_c = \{v \in V : \langle v, v\rangle = c\}$, we may now define a mapping $\varPhi : U \to S_c$ by declaring $\varPhi(y)$, for any $y \in U$, to be the unique parallel section (u, f) of $\mathcal{E}$ (notation of Lemma 12.4) defined on U and satisfying the initial conditions

$$u(y) = 0\,, \qquad f(y) \,=\, 1\,. \tag{14.4}$$

In view of the dependence-on-parameters theorem for ordinary differential equations, $\varPhi$ is of class C^∞. Furthermore, let $P(V) \approx \mathbf{C}\mathrm{P}^{n/2}$ be the projective space of V, formed by all complex lines through 0 in V, and let $\mathrm{pr} : V \smallsetminus \{0\} \to P(V)$ be the natural projection. We now define a mapping $F : U \to P(V)$ to be the composite

$$F \,=\, \mathrm{pr} \circ \varPhi\,. \tag{14.5}$$

Note that the pr-image $\mathrm{pr}\,(S_c)$ of the pseudosphere S_c, which depends only on $\mathrm{sgn}\,(c)$, is a manifold (being an open subset of $P(V)$), and it carries the quotient pseudo-Riemannian metric g^c induced by the metric of S_c, and described in Example 10.6.

Let us now consider any C^1 curve $I \ni t \mapsto y(t) \in U$, where $I \subset \mathbf{R}$ is an interval. Thus, for $y \in U$, $[\varPhi(y(t))](y) = (u(t, y), f(t, y))$ (notation of Lemma 12.5), with

$$u(t, y(t)) \,=\, 0\,, \qquad f(t, y(t)) \,=\, 1 \tag{14.6}$$

for all $t \in I$ (in view of (14.4)). Applying d/dt to (14.6), we see that

$$\partial u/\partial t \,=\, -\,\nabla_{\dot y} u \qquad \partial f/\partial t \,=\, -\,d_{\dot y} f \tag{14.7}$$

along the curve $(t, y(t))$. (To see this, use the chain rule and (4.12), which then yields $[\nabla_{\dot y} u]^j = u^j{}_{,k}\dot y^k\,\partial_k u^j$, as $u(t, y(t)) = 0$.) On the other hand, $\varPhi(y(t))$ is, for each t, a D-parallel section of $\mathcal{E}$. Therefore, formulae (12.9) and (14.6), (14.7) give, along the curve $t \mapsto (t, y(t))$, $\partial u/\partial t = \dot y$ and $\partial f/\partial t = i\xi(\dot y)$. Thus, at $y = y(t)$, we have $d\Phi_y \dot y = [\Phi(y)]^{\boldsymbol\cdot} = (\partial u/\partial t,\, \partial f/\partial t) = (\dot y, i\xi(\dot y))$. This shows that

$$[\Phi(y)](y) = (0,1)\,, \qquad [d\Phi_y w]\,(y) = (w, i\xi(w)) \tag{14.8}$$

for any $y \in U$ and $w \in T_y M$. (Note that $d\Phi_y w \in V$, as Φ is V-valued, while $\operatorname{Re}\langle d\Phi_y w, \Phi(y)\rangle = 0$ by (14.8), which expresses the fact that $d\Phi_y w$ is tangent to the pseudosphere S_c.)

To evaluate $g^c(dF_y w, dF_y w)$ for $y \in U$ and $w \in T_x M$, we may use the description of a quotient metric given in the paragraph preceding Example 10.6. Thus,

$$g^c(dF_y w, dF_y w) = \langle v', v'\rangle = \langle d\Phi_y w, d\Phi_y w\rangle - \langle v, v\rangle\,, \tag{14.9}$$

with $v' \in V$ uniquely characterized by the existence of a decomposition $d\Phi_y w = v + v' \in V$ such that v is an imaginary multiple of $\Phi(y)$ and $\langle v', \Phi(y)\rangle = 0$. Consequently, $v = c^{-1}\langle d\Phi_y w, \Phi(y)\rangle\Phi(y)$, and so, by (14.9) and (9.3), (12.10), $g^c(dF_y w, dF_y w) = \langle w, w\rangle_{\mathbf c} = g(w,w)$. In other words, F is isometric, i.e., $g = F^* g^c$, and so it is a nondegenerate immersion of U into V (cf. the paragraph following Corollary 13.2). Replacing U with a smaller neighborhood of x, we may thus assume that $F : U \to \operatorname{pr}(S_c)$ is an isometric embedding (see the beginning of §13). Thus, by Corollary 13.2, $F(U)$ is open as a subset of $\operatorname{pr}(S_c)$, while F is an isometry between U and $F(U)$. This completes the proof. ■

The next result is a local classification of those Einstein 4-manifolds which are Riemannian products of surfaces.

Theorem 14.5. *For any pseudo-Riemannian 4-manifold (M,g), the following four conditions are equivalent*:

(i) *Every point of M has a neighborhood isometric to the Riemannian product of two pseudo-Riemannian surfaces with equal constant curvatures.*
(ii) *(M,g) is an Einstein manifold and the tangent bundle TU of some neighborhood U of any point of M can be decomposed as a direct sum $TU = \mathcal P \oplus \mathcal Q$ of two mutually orthogonal plane bundles $\mathcal P$, $\mathcal Q$, which are parallel as subbundles of TU, i.e., invariant under parallel transports.*
(iii) *Every point of M has a neighborhood U on which* (10.13) *holds for some C^∞ bivector fields β, γ, numbers δ, ε, and orthonormal C^∞ vector fields $e_1, \dots, e_4$ with* (10.14) *and* (10.15).
(iv) *In a neighborhood U of any point $x \in M$ we have* (10.16) *with some symmetric twice-covariant C^∞ tensor fields P, Q satisfying* (10.17).

Proof. Condition (i) clearly implies (ii), with the subbundles $\mathcal P$, $\mathcal Q$ in (ii) chosen so as to be tangent to the factor surfaces in (i). (To see this, use (4.1) in a product coordinate system.) Let us now assume (ii) fix a point $x \in M$. We may choose orthonormal C^∞ vector fields $e_1, \dots, e_4$ defined on neighborhood U of x and such that $\mathcal P = \operatorname{Span}\{e_1, e_2\}$ and $\mathcal Q = \operatorname{Span}\{e_3, e_4\}$, and set, for $j = 1, \dots, 4$,

$$\varepsilon_j = g_{jj} = g(e_j, e_j) = \pm 1\,, \qquad \delta = \varepsilon_1\varepsilon_2\,, \qquad \varepsilon = \varepsilon_3\varepsilon_4\,. \tag{14.10}$$

Let $P, Q : TU \to TU$ now be the bundle morphisms of orthogonal projections onto $\mathcal P$ and $\mathcal Q$. Thus,

$$\nabla P = \nabla Q = 0\,. \tag{14.11}$$

Applying (4.27) to $F = P$ or $F = Q$ we see that, for any $j, k \in \{1,2,3,4\}$, the operator $R(e_j, e_k)$ commutes with both P and Q, and hence leaves the spans $\mathcal{P}$ of e_1, e_2 and $\mathcal{Q}$ of e_3, e_4 invariant. Denoting $R_{jklm} = g(R(e_j,e_k)e_l, e_m)$ the component functions of the curvature tensor R, and using skew-symmetry of R_{jklm} in l, m (see (4.32)), we thus see that $R_{jklm} = 0$ unless $\{l,m\} = \{1,2\}$ or $\{l,m\} = \{3,4\}$. Hence, by (4.33), $R_{1234} = -R_{1342} - R_{1423} = 0$. Moreover, $R_{jklm} = R_{lmjk}$ by (4.32), and so R_{jklm} must be zero unless $\{j,k\} = \{l,m\} = \{1,2\}$ or $\{j,k\} = \{l,m\} = \{3,4\}$. Let us now set $\lambda = \varepsilon_1\varepsilon_2 R_{1212}$ and $\mu = \varepsilon_3\varepsilon_4 R_{3434}$, with ε_j as in (14.10). The components $R_{jk} = \mathrm{Ric}(e_j, e_k)$ of the Ricci tensor Ric thus satisfy the relations $R_{jj} = \lambda g_{jj}$ if $j \in \{1,2\}$ and $R_{kk} = \mu g_{kk}$ if $k \in \{3,4\}$. Since (M,g) is Einstein, it follows that $\mu = \lambda$, $\mathrm{Ric} = \lambda g$, and the scalar curvature s is given by $\mathrm{s} = 4\lambda$. Formula

$$\beta = e_1 \wedge e_2\,, \qquad \gamma = e_3 \wedge e_4 \tag{14.12}$$

now defines bivector fields β, γ on U which are uniquely (up to a sign) determined by $\mathcal{P}$ and $\mathcal{Q}$, and hence parallel. Using (14.10), (2.21) and (2.27), we easily obtain (10.14) and (10.15). Furthermore, (10.13) holds, since both sides have the same components R_{jklm}. (Note that, by (14.10) and (2.21), the only nonzero components of β and γ are $\beta_{12} = -\beta_{21} = \varepsilon_1\varepsilon_2$ and $\gamma_{34} = -\beta_{43} = \varepsilon_3\varepsilon_4$.) Thus, (ii) implies (iii).

Let us now assume (iii) and define the vector subbundles $\mathcal{P}$, $\mathcal{Q}$ of TU by $\mathcal{P} = \mathrm{Span}\{e_1, e_2\}$ and $\mathcal{Q} = \mathrm{Span}\{e_3, e_4\}$ In view of (14.12) and (2.22), $\mathcal{P}$ and $\mathcal{Q}$ are the kernels of γ and β (as well as the images of β and γ), and so they are parallel as subbundles of TU. Setting $P = \varepsilon_1 e_1 \otimes e_1 + \varepsilon_2 e_2 \otimes e_2$ and $Q = \varepsilon_3 e_3 \otimes e_3 + \varepsilon_4 e_4 \otimes e_4$, we easily see that $P, Q : TU \to TU$ are the bundle morphisms of orthogonal projections onto $\mathcal{P}$ and $\mathcal{Q}$, so that (14.11) follows, and so P, Q satisfy (10.17), Furthermore, applying (5.39) to $A = \delta P$, $v = e_1$ and $w = e_2$, or $A = \varepsilon Q$, $v = e_3$ and $w = e_4$, we can rewrite (10.13) as (10.16). We thus showed that (iv) follows from (iii).

Finally, to prove that (iv) implies (i), let us suppose that (iv) holds. In the case where $\mathrm{s} = 0$, (M,g) is flat and (i) is obvious (cf. Theorem 14.2(i)). Thus, from now on we may assume that $\mathrm{s} \neq 0$. We thus have (12.15) with $c = 4/\mathrm{s}$, and so the connection D in the vector bundle $\mathcal{E}$ over M, introduced in Lemma 12.5, is flat. Let us now choose, for any given point $x \in M$, a neighborhood U of x satisfying the assertion of Lemma 11.2 with $\nabla = \mathrm{D}$. The set V of all D-parallel sections of $\mathcal{E}$, defined on U, thus is a six-dimensional real vector space. Moreover, V carries the pseudo-Euclidean inner product $\langle\,,\rangle$ given by (12.14) (which gives a constant value for a pair of parallel sections), and has a pair of mutually $\langle\,,\rangle$-orthogonal three-dimensional subspaces V', V'' consisting of those parallel sections on U which are valued in the parallel subbundle $\mathcal{P}$ (or, respectively, $\mathcal{Q}$). Let us now define a mapping $F : U \to V$ in such a way that, for $y \in U$, $F(y)$ is the unique parallel section (u, φ, χ) of $\mathcal{E}$ (notation as in Lemma 12.5), defined on U, with

$$u(y) = 0\,, \qquad \varphi(y) = \chi(y) = |c|^{1/2} \tag{14.13}$$

and let $F'(y)$, $F''(y)$ be the components of $F(y)$ relative to the direct-sum decomposition $V = V' \oplus V''$. Both $F'(y)$ and $F''(y)$ thus are parallel sections of

$\mathcal{E}$, defined on U, with the values at y equal to $(0,|c|^{1/2},0)$ and, respectively, $(0,0,|c|^{1/2})$. Evaluating the inner product (12.14) at the point y, we thus get $\langle F'(y),F'(y)\rangle = \langle F''(y),F''(y)\rangle = c$, i.e., the image $F(U)$ is contained in submanifold N of V given by

$$N \,=\, \{\psi' + \psi'' : \psi' \in V',\, \psi'' \in V'',\, \langle \psi',\psi'\rangle = \langle \psi'',\psi''\rangle = c\}\,. \tag{14.14}$$

Clearly, N with the submanifold metric induced by $\langle\,,\rangle$ is the Riemannian product of pseudospheres with the same constant curvature $K = 1/c$ (Proposition 14.1). Finally, for any $y \in U$ and $w \in T_yM$, we have

$$[dF_y w]\,(y) \,=\, (w,0,0)\,. \tag{14.15}$$

(Note that $dF_yw \in V$, as F is V-valued; thus, dF_yw is a parallel section of $\mathcal{E}$, defined on U, and so its value at y is an ordered triple, namely, an element of $T_yM \times \mathbf{R} \times \mathbf{R}$.) To establish (14.15), let us consider any C^1 curve $t \mapsto y(t) \in U$ and set $F(y) = (u,\varphi,\chi)$ (notation of Lemma 12.5), where $y = y(t)$ and, similarly, each of u, φ, χ also depends both on t and on $y \in U$. Applying d/dt to the relations

$$u(t,y(t)) \,=\, 0\,, \qquad \varphi(t,y(t)) \,=\, \chi(t,y(t)) \,=\, |c|^{1/2} \tag{14.16}$$

(immediate from (14.13)), we obtain

$$\dot u \,=\, -\nabla_{\dot y} u \qquad \dot\varphi \,=\, -\,d_{\dot y}\varphi\,, \qquad \dot\chi \,=\, -\,d_{\dot y}\chi \tag{14.17}$$

along the curve $(t,y(t))$, where we write $\dot u = \partial u/\partial t$ and similarly for φ and χ; in fact, this is clear from the chain rule and (4.12), (which then yields $[\nabla_{\dot y}u]^j = u^j{}_{,k}\dot y^k\,\partial_k u^j$, as $u(t,y(t)) = 0$). However, since $F(y) = (u,\varphi,\chi)$ is D-parallel, as a section of $\mathcal{E}$, for each fixed t, the definition of D in Lemma 12.5, combined with (14.17) and (14.16), gives, along the curve $(t,y(t))$, $\dot u = P\dot y + Q\dot y = \dot y$ (cf. (12.13)) and $\dot\varphi = \dot\chi = 0$. Thus, $dF_y\dot y = [F(y)]^{\cdot} = (\dot u,\dot\varphi,\dot\chi) = (\dot y,0,0)$, which proves (14.15).

Evaluating the (constant) inner product of parallel sections at the point y, we now get, from (14.15), $\langle dF_yv, dF_yw\rangle = g(v,w)$. In other words, F is isometric, i.e., $g = F^*h$, and so it is a nondegenerate immersion of U into V (cf. the paragraph following Corollary 13.2. Replacing U with a smaller neighborhood of x, we may thus assume that $F : U \to V$ is an isometric embedding (see the beginning of §13). As $F(U) \subset N$, Corollary 13.2 implies that $F(U)$ is open as a subset of $F(U)$ and F is an isometry between U and $F(U)$. This completes the proof. ■

Remark 14.6. The assertion of Theorem 14.5 can also be derived from de Rham's Theorem 4.10.

We can now prove the main result of this section.

Theorem 14.7 (Cartan, 1926)**.** *Let (M,g) be a Riemannian four-manifold which is both locally symmetric and Einstein. Then, every point in M has a neighborhood isometric to an open subset of (N,h), where (N,h) is either one of the manifolds listed in* Examples 10.3, 10.4 *and* 10.6*, or the Riemannian product of two Riemannian surfaces with equal constant curvatures, both obtained as in* Example 10.4.

Proof. If $W^+ = 0$ for both local orientations, we have $W = 0$, so that (5.10) implies (10.1), and the assertion follows from Theorem 14.2. Therefore, we may now assume that every of point M has a neighborhood U with an orientation such that $W^+ \neq 0$ everywhere in U. Choosing α_j, λ_j, ξ_j and u_j, $j = 1, 2, 3$, satisfying (6.24), (6.12), (6.26) and (6.28) on a nonempty open subset of U, let us first note that the λ_j cannot be all equal (as $\lambda_1 + \lambda_2 + \lambda_3 = 0$ by (6.19), while $W^+ \neq 0$). Let us now fix the values of j, k, l in such a way that $\{j, k, l\} = \{1, 2, 3\}$ and λ_j is a simple eigenvalue of W^+. Then let us set $\alpha = \alpha_j$. In view of Proposition 9.8 and Corollary 9.9(ii), α is parallel, $\lambda_j = \mathrm{s}/6$, and $\lambda_k = \lambda_l = -\mathrm{s}/12$.

The parallel tensor W^- may or may not be identically zero. If it is, W has the spectrum (10.20), with $\mathrm{s} \neq 0$ (as $W^+ \neq 0$) and so, by (5.33), the spectrum of R is given by (10.21), with the parallel bivector field $\alpha = \alpha_j$ corresponding to the eigenvalue $\mathrm{s}/4$. Since the curvature operator acting on bivectors via (5.13) uniquely determines the curvature tensor, the latter must equal (10.5) with λ and μ given by (10.10). Thus, the Kähler manifold (U, g, α) has constant holomorphic sectional curvature, and our assertion follows from Theorem 14.4.

Finally, let us suppose that $W^- \neq 0$. Applying the above argument to the opposite orientation we see that $\Lambda^- U$ admits, locally, a parallel section α^-, which can be normalized so that $\langle \alpha^-, \alpha^- \rangle = 2$. Since $\alpha^+ = \alpha$ and α^-, treated as skew-adjoint bundle morphisms $TU \to TU$, commute by Corollary 6.3, and $[\alpha^\pm]^2 = -\mathrm{Id}$ (see (6.7)), their composite $F = \alpha^+\alpha^-$ is a self-adjoint and satisfies $F^2 = \mathrm{Id}$. This gives rise to a direct-sum decomposition $TU = \mathcal{P}^+ \oplus \mathcal{P}^-$ with subbundles $\mathcal{P}^\pm$ of TU such that $F = \pm\mathrm{Id}$ on $\mathcal{P}^\pm$ (see Remark 3.2). Moreover, the subbundles $\mathcal{P}^\pm$ are parallel, since so if F (cf. Remark 4.7). Also, since $\alpha = \alpha_j$ anticommutes with α_k and α_l, for j, k, l as above (see (6.12)), while α^- commutes with them (Corollary 6.3), it follows that F anticommutes with α_k and so both eigenspace bundles $\mathcal{P}^\pm$ of F must have the same fibre dimension 2 (as α_k interchanges them). Finally, self-adjointness of F implies that the subbundles $\mathcal{P} = \mathcal{P}^+$ and $\mathcal{Q} = \mathcal{P}^-$ are mutually orthogonal (see Remark 3.17(i)). Thus, $\mathcal{P}$ and $\mathcal{Q}$ satisfy condition (ii) of Theorem 14.5. By Theorem 14.5, this leads to the product-of-surfaces case of our assertion, which completes the proof. ■

Remark 14.8. Due to their algebraic provenience, all of the examples of Einstein manifolds described in this section are (real) analytic. This is more than a coincidence: According to a result of DeTurck and Kazdan (1981), every Riemannian Einstein metric g on a manifold M is analytic in suitable local coordinate systems whose domains cover M. It follows that the C^∞ differentiable structure of M then contains a unique real-analytic structure that makes g analytic; in fact, transitions between the coordinate systems just mentioned are isometries between analytic Riemannian metrics, and as such they must be analytic (since, in normal geodesic coordinates, an isometry appears as a linear operator).

DeTurck and Kazdan's analyticity theorem cannot, however, be generalized to indefinite Einstein metrics; see Remark 15.15.

§15. Einstein hypersurfaces in pseudo-Euclidean spaces

In this section we classify, locally, those Einstein four-manifolds (M, g) which are isometric to hypersurfaces in 5-dimensional pseudo-Euclidean vector spaces. In the case where g is positive-definite or Lorentzian, there are no surprises: (M, g) then is necessarily a space of constant curvature (Proposition 15.6). However, there is a large class of other Ricci-flat metrics with this property, having the neutral sign pattern $- - + +$. See Example 15.14.

The results presented here are due to Fialkow (1938).

Codimension-one submanifolds of a pseudo-Euclidean vector space (with a constant metric) seem to be a natural place to look for further examples of Einstein manifolds (M, g). In view of Theorem 13.6, this amounts to imposing on an Einstein metric g the Gauss and Codazzi equations (12.4) with some (unknown) tensor field b. The resulting problem is easy to solve, at least in those cases of most interest to us: It turns out (see Proposition 15.6 below) that nothing new can be obtained in this way if we insist that $\dim M = 4$ and the metric g be positive definite.

Lemma 15.1. *Let* $A : \mathcal{T} \to \mathcal{T}$ *be a linear operator in a* 4*-dimensional real vector space* $\mathcal{T}$ *such that* $(\mathrm{Trace}\, A)A - A^2$ *is a multiple of* Id*, i.e, for some* $c \in \mathbf{R}$*,*

$$A^2 - pA + c = 0 \qquad \text{with} \quad p = \mathrm{Trace}\, A\,. \tag{15.1}$$

If $\mathrm{Spec}\, A$ *denotes the complex spectrum of* A*, that is, the family of all complex roots of its characteristic polynomial, listed with their multiplicities, then one of the following five cases occurs, with a suitable* real *number* μ:

(a) $\mathrm{Spec}\, A = \{\mu, 0, 0, 0\}$, $\mu \neq 0$, *while* $c = 0$, $p = \mathrm{Trace}\, A = \mu$ *in* (15.1)*, and* $\dim A(\mathcal{T}) = 1$, $\dim(\mathrm{Ker}\, A) = 3$, $A^2 = pA$.

(b) $\mathrm{Spec}\, A = \{\mu, \mu, \mu, \mu\}$, $\mu \neq 0$, *while* $c = 3\mu^2$, $p = \mathrm{Trace}\, A = 4\mu$ *and* $A = \mu \cdot \mathrm{Id}$, *so that* $A^2 = \mu^2 \cdot \mathrm{Id} = c \cdot \mathrm{Id}/3$.

(c) $\mathrm{Spec}\, A = \{\mu, \mu, -\mu, -\mu\}$, $\mu \neq 0$, *while* $c = -\mu^2$, $p = \mathrm{Trace}\, A = 0$ *and* $A^2 = \mu^2 \cdot \mathrm{Id} = -c \cdot \mathrm{Id}$.

(d) $\mathrm{Spec}\, A = \{\mu i, \mu i, -\mu i, -\mu i\}$, $\mu \neq 0$, *while* $c = \mu^2$ *and* $p = \mathrm{Trace}\, A = 0$ *and* $A^2 = -\mu^2 \cdot \mathrm{Id} = -c \cdot \mathrm{Id}$.

(e) $\mathrm{Spec}\, A = \{0, 0, 0, 0\}$, *while* $c = 0$, $p = \mathrm{Trace}\, A = 0$, *as well as*

$$A^2 = 0\,, \qquad \text{that is,} \quad A(\mathcal{T}) \subset \mathrm{Ker}\, A\,, \tag{15.2}$$

and three subcases are possible:

(i) $A = 0$.

(ii) $A(\mathcal{T}) \subset \mathrm{Ker}\, A$ *and* $\dim A(\mathcal{T}) = 1$, $\dim(\mathrm{Ker}\, A) = 3$.

(iii) $A(\mathcal{T}) = \mathrm{Ker}\, A$ *and* $\dim A(\mathcal{T}) = \dim(\mathrm{Ker}\, A) = 2$. ■

Remark 15.2. Any linear operator $A : \mathcal{T} \to \mathcal{T}$ in a finite-dimensional vector space $\mathcal{T}$, such that $\dim A(\mathcal{T}) \leq 1$ (see cases (a), (e)i) and (e)ii) of Lemma 15.1), must have the form

$$A = \xi \otimes v \tag{15.3}$$

for some vector $v \in \mathcal{T}$ and a linear function $\xi \in \mathcal{T}^*$, in the sense that $Aw = \xi(w)v$ for all $w \in \mathcal{T}$. To see this, just choose any v which spans $A(\mathcal{T})$.

Remark 15.3. It is an easy exercise to verify that A with (15.1) must be diagonalizable in cases (a), (b), (c), (e)i), and nondiagonalizable in cases (d), (e)ii) and (e)iii).

Proof of Lemma 15.1. Applying both sides of (15.1) to any eigenvector of A in the complexified space $\mathcal{T}^{\mathbf{C}} = \mathcal{T} + i\mathcal{T}$, we see that every complex root μ of the characteristic polynomial P of A satisfies the quadratic equation

$$\mu^2 - p\mu + c = 0\,, \qquad p = \operatorname{Trace} A\,. \tag{15.4}$$

Hence P may have at most two distinct complex roots.

If P it has just one (quadruple) root μ, then $p = \operatorname{Trace} A = 4\mu$, and so, by (15.4), $c = 3\mu^2$, i.e., (15.1) reads $(A-3\mu)(A-\mu) = 0$. Thus, the image of $A-\mu$ is contained in the kernel of $A-3\mu$. If, in addition, $\mu = 0$, this becomes (15.2) and, as as $\dim(\operatorname{Ker} A) + \dim A(\mathcal{T}) = \dim\mathcal{T} = 4$, $r = \dim A(\mathcal{T})$ satisfies $r \le 4-r$, that is, r equals 0, 1 or 2, which leads to the cases (e)i), (e)ii), (e)iii). On the other hand, in the case where the quadruple root μ of P is nonzero, $A-3\mu$ is injective (since 3μ is not an eigenvalue of A), and so the image of $A-\mu$ is $\{0\}$ (case (b)).

However, if P has two distinct complex roots μ and ν, we may order them so that μ has the lowest multiplicity. By (15.4), $\mu+\nu = p = \operatorname{Trace} A$. Consequently, two cases are possible, namely (I): $\operatorname{Spec} A = \{\mu,\mu,\nu,\nu\}$, and (II): $\operatorname{Spec} A = \{\mu,\nu,\nu,\nu\}$. Since $\operatorname{Spec} A$ is invariant under complex conjugation, in case (I) μ and ν are either both real, or both nonreal and mutually conjugate, while in case (II) they must both be real. Furthermore, in case (I), $\mu+\nu = \operatorname{Trace} A = 2\mu+2\nu$, i.e., $\mu+\nu = 0$, while in case (II), $\mu+\nu = \operatorname{Trace} A = \mu+3\nu$, i.e., $\nu = 0$. Thus, in case (I), (15.4) gives $p = \mu+\nu = 0$ and $c = \mu\nu = -\mu^2$, and hence (15.1) implies assertion (c) (when μ,ν are real), or assertion (d) (when they are not real). In the remaining case (II), let us choose $v \in \mathcal{T}$ with $v \ne 0$ and $Av = \mu v$. We have $\dim[\operatorname{Ker}(A-\mu)] = 1$; in fact, μ is a (real) eigenvalue of A, and the dimension of the eigenspace cannot be higher than 1, since μ is a *simple* root of P. (Note that $\mu \ne \nu = 0$.) Thus, $\dim A(\mathcal{T}) \le 1$, since $A(\mathcal{T}) \subset \operatorname{Ker}(A-\mu)$; namely, (15.4) yields $p = \mu+\nu = \mu$ and $c = \mu\nu = 0$, so that, by (15.1), $(A-\mu)A = 0$. Finally, $\dim A(\mathcal{T}) = 1$, since $A \ne 0$, as A has the eigenvalue $\mu \ne 0$. This completes the proof. ■

Remark 15.4. Let b be any symmetric twice-covariant tensor field on a pseudo-Riemannian manifold (M,g). Using the index-raising operation corresponding to g, we may treat b as a self-adjoint bundle morphism $TM \to TM$. Thus, if b is of class C^1, we can form, for any $x \in M$ and $v \in T_xM$, the composite $b\,\nabla_v b$ of b and $\nabla_v b$, viewed as operators $T_xM \to T_xM$. For the same reason, b sends any tangent vector field w to a vector field denoted bw, and has a well-defined trace, $\operatorname{Trace} b = g^{jk}b_{jk}$, which is a function $M \to \mathbf{R}$, as well as a square b^2, which is a symmetric twice-covariant tensor field with the local components $[b^2]_{jk} = b_{jl}b^l_k$. Finally, we can speak of $\operatorname{rank} b$, which at any $x \in M$ equals the dimension of the image $b(T_xM)$ (and coincides with the matrix rank of $[b_{jk}(x)]$). If $\operatorname{rank} b = r$ is constant, we can form the vector subbundles $b(TM)$ and $\operatorname{Ker} b$ of TM, of the respective fibre dimensions r and $\dim M - r$.

By a *Codazzi tensor field* on a pseudo-Riemannian manifold (M,g) we mean any symmetric twice-covariant tensor field b on M which is of class C^∞ and satisfies the Codazzi equation (12.4) (or (12.2)), that is, $b_{jk,l} = b_{jl,k}$.

Lemma 15.5. *With notations as in* Remark 15.4*, for any Codazzi tensor field b on a pseudo-Riemannian manifold (M,g) such that b^2 is parallel. Then*

(a) *We have $b\nabla_v b = 0$ for all points $x \in M$ and tangent vectors $v \in T_xM$;*

in other words,

$$b_j^s b_{sk,l} = 0\,. \tag{15.5}$$

(b) *If* b *is nondegenerate at some point of* M, *then* b *itself is parallel.*

(c) *If* $b^2 = ag$ *for some nonzero constant* a, *while* b *and the curvature tensor* R *of* (M,g) *satisfy the Gauss equation* $R_{jklm} = \varepsilon\,(b_{jl}b_{km} - b_{kl}b_{jm})$ *with* $\varepsilon = \pm 1$, *i.e.,* (12.2)*, then* $b = \mu g$ *with a constant* $\mu \neq 0$ *and* (M,g) *is a space of nonzero constant curvature.*

(d) *If* $n = \dim M$ *is even, while* $b^2 = 0$, $\operatorname{rank} b = n/2$ *everywhere in* M, *and* b *along with the curvature tensor* R *satisfy the Gauss equation as in* (c)*, then* $b(TM) = \operatorname{Ker} b$ *and in a neighborhood of any point* $x \in M$ *there exist* $n/2$ *linearly independent, mutually orthogonal, null parallel vector fields, which all are sections of the subbundle* $b(TM)$.

Proof. Taking the covariant derivative of b^2 and using (12.4), we obtain $0 = b_j^s b_{sk,l} + b_k^s b_{sj,l} = 0$. Thus, $b_j^s b_{sk,l}$ is skew-symmetric in j,k, while by (12.4) it is symmetric in k,l, and hence it must be zero (Lemma 3.1), which proves (a). Furthermore, if b is nondegenerate at some point, then both b^2 and b are nondegenerate at every point (as $\nabla[b^2] = 0$), and so $\nabla b = 0$ in view of (15.5), so that (b) follows. To establish (c), let us assume (12.4) with $b^2 = ag$ for a constant $a \neq 0$. Now (b) gives $b_{jk,l} = 0$, so that, combining (4.29) (for $A = b$) and (12.4), we obtain $0 = b_{lm,jk} - b_{lm,kj} = R_{jkl}{}^s b_{sm} + R_{jkm}{}^s b_{ls} = \varepsilon a\,[b_{jl}g_{km} - b_{kl}g_{jm} + b_{jm}g_{kl} - b_{km}g_{jl}]$, as $b_{js}b_k^s = ag_{jk}$. Contracting this against g^{jl} we see that $0 = \varepsilon a\,[pg_{km} - nb_{km}]$ with $p = \operatorname{Trace} b$ and $n = \dim M$, since $g^{jk}g_{jk} = n$. Therefore $b = \mu g$ with $\mu = p/n$, and so μ is constant as $\mu^2 = a$. Now (12.4) gives (10.2) with $K = \varepsilon\mu^2$, which proves (c). Finally, under the assumptions of (d), the subbundle $\mathcal{E} = b(TM)$ of TM satisfies $\mathcal{E} \subset \operatorname{Ker} b$ and so $\mathcal{E} = \operatorname{Ker} b$ since both bundles have the same fibre dimension $n/2$. Furthermore, for any local C^1 vector fields v, w in M, we have $\nabla_v(bw) = b(\nabla_v w) + [\nabla_v b]\,w$ and hence $b(\nabla_v(bw)) = 0$ as $b^2 = 0$ and, by (a), $b\nabla_v b = 0$. Thus, $\nabla_v(bw)$ is a local section of $\mathcal{E} = \operatorname{Ker} b$. Since the subbundle $\mathcal{E} = b(TM)$ is spanned, locally, by such bw, it follows that $\mathcal{E}$ is parallel, i.e., closed under taking covariant derivatives of its sections in all directions. (See Remark 4.7(i).) This gives rise to a connection ∇ in $\mathcal{E}$ obtained by restricting to local sections of $\mathcal{E}$ the ordinary covariant derivative operation for vector fields corresponding to the Levi-Civita connection of TM (also denoted ∇; cf. Remark 4.7(ii)). Denoting R^∇ and R the curvature tensors of this connection ∇ in $\mathcal{E}$ and, respectively, of (M,g), we have, from (4.52), $R^\nabla(v,w)u = R(v,w)u$ for any local vector fields u, v, w in M such that u is a section of $\mathcal{E}$. As $\mathcal{E} = \operatorname{Ker} b$, the Gauss equation (see (c)) now shows that $R^\nabla = 0$ identically. From Lemma 11.2 we thus obtain, locally, the existence of trivializations of $\mathcal{E}$ consisting of parallel sections. These sections must in turn be null and mutually orthogonal, since each fibre $\mathcal{E}_x = b(T_xM)$ of $\mathcal{E}$ is a null subspace of T_xM; in fact, as $b^2 = 0$, we have $g(bw, bw) = g(b^2w, w) = 0$ for all $w \in T_xM$. This completes the proof. ∎

Proposition 15.6 (Fialkow, 1938)**.** *Spaces of constant curvature are the only Riemannian or Lorentzian Einstein* 4*-manifolds that can be isometrically embedded into a* 5*-dimensional pseudo-Euclidean vector space* V *with a constant metric.*

This is an immediate consequence of Lemma 15.7 below. ∎

The remainder of this section (except for the proof of Lemma 15.7) deals with indefinite metrics, and can be skipped by the reader interested just in the Riemannian case.

Lemma 15.7. *Suppose that* (M, g) *is a pseudo-Riemannian Einstein* 4*-manifold which admits an isometric embedding* $F : M \to V$ *into a* 5*-dimensional pseudo-Euclidean vector space* V *with a constant metric. Then* (M, g) *must be a space of constant curvature, unless* g *is a Ricci-flat indefinite metric of the neutral sign pattern* $- - + +$ *and there exists a nonempty open set* $U \subset M$ *such that, denoting* b *the real-valued second fundamental form of* F *relative to a unit normal vector field, we have*

(i) $b^2 = (\operatorname{Trace} b)b$ *everywhere in* M*,*
(ii) $\operatorname{Trace} b = 0$*,* $b^2 = 0$ *and* $\operatorname{rank} b = 2$ *everywhere in* U*,*
(iii) $R(x) = 0$ *and* $\operatorname{rank} b(x) \le 1$ *at all points* x *with* $x \notin U$*.*

Proof. Let b denote the real-valued second fundamental form of F relative to a unit normal vector field $\mathbf{n}$ with $\langle \mathbf{n}, \mathbf{n} \rangle = \varepsilon = \pm 1$. By Theorem 13.6, b satisfies (12.4). Thus, (12.5) combined with the Einstein condition (0.1) shows that (15.1) holds for $\mathcal{T} = T_x M$ and $A = b(x)$, at any point $x \in M$, with

$$c = \varepsilon \,\mathrm{s}/4\,. \tag{15.6}$$

Therefore, by Schur's Theorem 5.1, c is constant, i.e., the same at all points of M.

Suppose first that $c \ne 0$. Then, at each $x \in M$, $A = b(x)$ satisfies one of assertions (b), (c) or (d) of Lemma 15.1 with $b^2 = cg/3$ (case (b)) or $b^2 = -cg$ (cases (c), (d)). By Lemma 15.5(c), b is a constant multiple of g, so that cases (c) and (d) cannot really occur at any point x, and (M, g) must have a nonzero constant curvature.

Let us now consider the remaining case, with $c = 0$. In view of (15.6), (M, g) then is Ricci-flat. At every $x \in M$, $A = b(x)$ and $\mathcal{T} = T_x M$ must satisfy one of assertions (a), (e) of Lemma 15.1. Denoting U the open set of all points $x \in M$ at which $\operatorname{rank} b(x) = 2$, i.e., $\dim A(\mathcal{T}) = 2$, we see that points $x \in U$ are characterized by condition (e)iii), while x with $x \notin U$ are those points satisfying (a), (e)i) or (e)ii). If U is nonempty, i.e., case (e)iii) occurs at some $x \in M$, then $A(\mathcal{T}) = \operatorname{Ker} A$ is a null plane in $\mathcal{T} = T_x M$, since $A(\mathcal{T})$ is orthogonal to $\operatorname{Ker} A$ due to symmetry of $A = b(x)$. This can happen only if g has the sign pattern $- - + +$ (see (3.27)). Our assertion (ii) now is immediate from Lemma 15.1(e). On the other hand, if a fixed x is not in U, that is, satisfies (a), (e)i) or (e)ii) in Lemma 15.1, then our assertion (iii) holds at x. In fact, according to Remark 15.2, we have, at x, $b_{jk} = \xi_j v_k$ and, as $b_{jk} = b_{kj}$, this becomes $b_{jk} = a v_j v_k$ for some $a \in \mathbf{R}$. From (12.2) and (5.39), we now obtain $R(x) = 0$. Finally, the rank condition $\dim A(\mathcal{T}) \le 1$ at x is obvious from Lemma 15.1(a), (e)i), (e)ii). This completes the proof. ∎

Let us now consider a pseudo-Euclidean inner product $\langle\,,\rangle$ in a real vector space V of dimension n. Following Law (1991), we will continue to refer to $\langle\,,\rangle$ as *neutral* if it is indefinite and has a sign pattern of the form (q, q) (q minuses, q pluses); then, $n = 2q$ is even. Similarly, we will speak of *neutral pseudo-Riemannian manifolds* (M, g) or *neutral indefinite metrics* g, in even dimensions, to indicate that $g(x)$ is neutral at every, or some, point $x \in M$.

Lemma 15.8. *Let a linear operator* $A : \mathcal{T} \to \mathcal{T}$ *in a finite-dimensional vector space* $\mathcal{T}$ *and a vector subspace* $\mathcal{W} \subset \mathcal{T}$ *satisfy the conditions* $A(\mathcal{T}) \subset \mathcal{W}$ *and* $A(\mathcal{W}) = \{0\}$. *Then* $\operatorname{Trace} A = 0$.

In fact, this is immediate if we evaluate $\operatorname{Trace} A$ in a basis of $\mathcal{T}$ containing a basis of $\mathcal{W}$. ■

The following proposition leads (via Corollary 15.10 and Example 15.14 below) to easy constructions of examples of Ricci-flat pseudo-Riemannian metrics of the neutral sign pattern ($n/2$ minuses, $n/2$ pluses), in any even dimension n.

Proposition 15.9. *For any integer* $q \geq 1$, *let* (M, g) *be any* $2q$*-dimensional pseudo-Riemannian manifold that admits* q *linearly independent, mutually orthogonal, null parallel vector fields. Then* g *is a neutral Ricci-flat metric.*

Proof. Let w_a, $a = 1, \dots, q$, be the vector fields in question. For any fixed point $x \in M$, let us set $\mathcal{T} = T_xM$, and define $\mathcal{W} \subset \mathcal{T}$ be the subspace spanned by all $w_a(x)$, $a = 1, \dots, q$. By (4.26), $R(u, v)w_a = 0$ for any tangent vectors $u, v \in \mathcal{T}$. The algebraic symmetries (4.32) of R now imply that $g(R(u, u')v, v') = 0$ whenever one of the four vectors $u, u', v, v' \in \mathcal{T}$ lies in $\mathcal{W}$. In particular, $R(u, u')v$ is always orthogonal to $\mathcal{W}$, and hence $R(u, u')v \in \mathcal{W}$. (Note that, since $\mathcal{W}$ is null and $\dim \mathcal{W} = \dim \mathcal{W}^\perp = \dim \mathcal{T}/2$, we have $\mathcal{W}^\perp = \mathcal{W}$ by (3.26).) Similarly, $R(u, u')v = 0$ if $u, v \in \mathcal{T}$ and $u' \in \mathcal{W}$.

Let us now fix arbitrary vectors $u, v \in \mathcal{T} = T_xM$ and define $A : \mathcal{T} \to \mathcal{T}$ to be the operator with $Au' = R(u, u')v$ for all $u' \in \mathcal{T}$. Thus, A, $\mathcal{T}$ and $\mathcal{W}$ satisfy the hypotheses of Lemma 15.8, and so $\operatorname{Trace} A = 0$, i.e., by (4.34), $\operatorname{Ric}(u, v) = 0$. Finally, g has the neutral sign pattern (q, q) in view of (3.27). This completes the proof. ■

Given a submanifold M of a vector space V with $\dim V < \infty$ and a vector subspace $\mathcal{W} \subset V$, we will say that M is $\mathcal{W}$*-ruled* if it is a union of cosets (translation images) of $\mathcal{W}$. This means that M is closed under all translations by vectors in $\mathcal{W}$, that is, denoting $M + \mathcal{W}$ the set $\{x + w : x \in M,\, w \in \mathcal{W}\}$, we have

$$M + \mathcal{W} = M. \tag{15.7}$$

On the other hand, if V carries a fixed pseudo-Euclidean inner product $\langle \, , \rangle$, we will say that a submanifold M of V is *nondegenerate* if it is a nondegenerate submanifold (see §13) of $N = V$ regarded as a pseudo-Riemannian manifold with the constant metric $h = \langle \, , \rangle$. Among nondegenerate submanifolds of V, we have those which happen to be vector subspaces of V. (Recall that nondegeneracy of a vector subspace $\mathcal{T}$ of V is equivalent to (3.22), as well as (3.24).)

Corollary 15.10. *Let* $\mathcal{W}$ *be a* q*-dimensional null vector subspace of a pseudo-Euclidean vector space* V *of any finite dimension* $\dim V \geq 2q + 1$, *and let* M *be any* $2q$*-dimensional nondegenerate* $\mathcal{W}$*-ruled submanifold of* V, *as defined above. Then, the pseudo-Riemannian metric* g *that* M *inherits from* V *is Ricci-flat and has the neutral sign pattern* (q, q).

Proof. The constant vector fields on V provided by any fixed basis of $\mathcal{W}$ are null, mutually orthogonal and tangent to M at each point of M, and so their restrictions to M are parallel vector fields on (M, g) in view of Corollary 13.4. Thus, g is neutral and Ricci-flat by Proposition 15.9, which completes the proof. ■

We can now give a complete local classification of Einstein hypersurfaces in 5-dimensional pseudo-Euclidean spaces.

Theorem 15.11 (Fialkow, 1938)**.** *Let (M, g) be a pseudo-Riemannian Einstein 4-manifold which admits an isometric embedding $F : M \to V$ into a 5-dimensional pseudo-Euclidean vector space V with a constant metric. Then, either (M, g) is a space of constant curvature, or g is a Ricci-flat indefinite metric of the neutral sign pattern $- - + +$ and every point $x \in M$ with $R(x) \neq 0$ has a neighborhood whose F-image coincides with an open subset of a $\mathcal{W}$-ruled submanifold of V for some 2-dimensional null vector subspace $\mathcal{W} \subset V$.*

Proof. Suppose that (M, g) is not a space of constant curvature. Thus, the open subset U' of M formed by all x with $R(x) \neq 0$ is nonempty. Let b now denote the real-valued second fundamental form of F restricted to U', relative to a (local) unit normal vector field $\mathbf{n}$ with $\langle \mathbf{n}, \mathbf{n} \rangle = \varepsilon$ for some $\varepsilon = \pm 1$. Now b and g satisfy the part of the assertion of Lemma 15.7 starting from the word 'unless', and so, by Lemma 15.7(iii), U' is contained in the set U appearing in Lemma 15.7. Thus, according to Theorem 13.6 and Lemma 15.7(ii), b satisfies the hypotheses of Lemma 15.5(d) with $n = 4$. Thus, the F-image $Y = F(U'')$ of a suitable neighborhood of any given point in U' is a nondegenerate 4-dimensional submanifold of V which, in view of Lemma 15.5(d), admits two tangent vector fields $\mathbf{u}$, $\mathbf{w}$ which are null, mutually orthogonal and linearly independent at each point of Y, as well as parallel in Y (relative to the Levi-Civita connection of the metric that Y inherits from V). Furthermore, also by Lemma 15.5(d), $\mathbf{u}$ and $\mathbf{w}$ are sections of $\operatorname{Ker} b$, i.e., $b(\,\cdot\,, \mathbf{u}) = b(\,\cdot\,, \mathbf{w}) = 0$. Applying (13.2) (with (13.6)) and (13.8) to $\mathbf{w}$ (or, $\mathbf{u}$) along any C^1 curve $t \mapsto \mathbf{x}(t)$ in Y, we see that $\mathbf{u}$ and $\mathbf{w}$ are *constant* as V-valued functions on Y. Therefore, Y is contained in a $\mathcal{W}$-ruled 4-dimensional submanifold Y' of V, where $\mathcal{W} \subset V$ is the null vector subspace spanned by $\mathbf{u}$ and $\mathbf{w}$. In fact, applying Corollary 13.4(b) to the parallel (constant) vector field on V given by any constant-coefficient combination of $\mathbf{u}$ and $\mathbf{w}$, we see that Y contains, along with any given point $\mathbf{x}$, all points $\mathbf{x} + \mathbf{v}$ with $\mathbf{v} \in \mathcal{W}$ sufficiently close to $\mathbf{0}$. Finally, openness of Y in Y' follows from Corollary 13.2. This completes the proof. ■

Lemma 15.12. *Let $\mathcal{W}$ be a q-dimensional null subspace of a pseudo-Euclidean vector space V with $\dim V = 2q + k$, $k \geq 1$, and let $\mathrm{pr} : V \to V/\mathcal{W}$ denote the quotient projection of V onto the quotient vector space $V/\mathcal{W}$. The image*

$$\mathrm{pr}\,(\mathcal{W}^\perp) \,=\, \mathcal{W}^\perp/\mathcal{W} \tag{15.8}$$

then is a subspace of $V/\mathcal{W}$ with

$$\dim\,[\mathcal{W}^\perp/\mathcal{W}] \,=\, k\,. \tag{15.9}$$

Moreover, for any integer p with $0 \leq p \leq k$ there exists a natural bijective correspondence between the set of those $(q+p)$-dimensional vector subspaces $\mathcal{T}$ of V which contain $\mathcal{W}$, and the set $\mathrm{Gr}_p(V)$ of all p-dimensional vector subspaces $\mathcal{Z}$ of the quotient space $V/\mathcal{W}$, given by $\mathcal{Z} = \mathrm{pr}\,(\mathcal{T}) = \mathcal{T}/\mathcal{W}$ or, equivalently, $\mathcal{T} = \mathrm{pr}^{-1}(\mathcal{Z}) = \bigcup \mathcal{Z}$, the last object being the union of all cosets of $\mathcal{W}$ forming $\mathcal{Z}$. For $\mathcal{T}$ and $\mathcal{Z}$ related in this manner, $\mathcal{T}$ is nondegenerate as a subspace of the pseudo-Euclidean space V if and $\mathcal{Z}$ and the space (15.8) *together span $V/\mathcal{W}$.*

Proof. Bijectivity of the assignment $\mathcal{T} \mapsto \mathcal{Z}$ is clear, while (15.8) and (15.9) follow from (3.26). The last assertion is immediate from the fact that nondegeneracy of a subspace $\mathcal{T} \subset V$ is equivalent to (3.24). This completes the proof. ■

Proposition 15.13. *Let $\mathcal{W}$ be a q-dimensional null subspace of a pseudo-Euclidean vector space V of dimension $2q+r$, $r \geq 1$, and let us choose a subspace $\mathcal{Y} \subset \mathcal{W}^\perp$, complementary to $\mathcal{W}$ in $\mathcal{W}^\perp$, and a subspace $\mathcal{X} \subset V$, complementary to $\mathcal{W}^\perp$ in V, so that $\dim \mathcal{Y} = r$, $\dim \mathcal{X} = q$, $\mathcal{W}^\perp = \mathcal{Y} \oplus \mathcal{W}$ and*

$$V = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{W}. \tag{15.10}$$

Then

(a) *For any open subset $\mathcal{U}$ of $\mathcal{X}$ and any C^∞ function $\Phi : \mathcal{U} \to \mathcal{Y}$, the mapping*

$$\mathcal{U} \times \mathcal{W} \ni (x,w) \mapsto x + \Phi(x) + w \in V = \mathcal{X} + \mathcal{Y} + \mathcal{W}, \tag{15.11}$$

is an embedding and its image is a $2q$-dimensional nondegenerate $\mathcal{W}$-ruled submanifold of V.

(b) *Conversely, any given $2q$-dimensional nondegenerate $\mathcal{W}$-ruled submanifold of V is, locally, obtained as in* (a) *for some $\mathcal{U}$ and Φ.*

Proof. The assignment (15.11) is obviously an embedding and its image M is a submanifold of V with $\dim M = 2q$, which is $\mathcal{W}$-ruled, that is, satisfies (15.7). Let us now consider any $\mathcal{W}$-ruled $2q$-dimensional submanifold M of V. Thus, M is the preimage, under the quotient projection $\mathrm{pr} : V \to V/\mathcal{W}$, of a q-dimensional submanifold Q of the quotient vector space $V/\mathcal{W}$. Relation (15.10) now leads to an identification $V/\mathcal{W} = \mathcal{X} \oplus \mathcal{Y}$, under the isomorphism obtained by restricting pr to $\mathcal{X} \oplus \mathcal{Y}$, and $\mathcal{Y}$ then becomes identified with r-dimensional subspace $\mathcal{W}^\perp/\mathcal{W}$ of $V/\mathcal{W}$. In view of Lemma 15.12, the requirement that M be nondegenerate amounts to the *transversality condition* $\mathrm{Span}\,(\mathcal{Y} \cup T_zQ) = \mathcal{X} \oplus \mathcal{Y}$ for every point $z \in Q$, which can also be rewritten as $\mathcal{Y} \cap T_zQ = \{0\}$, since $\dim [\mathcal{X} \oplus \mathcal{Y}] = q + r = \dim \mathcal{Y} + \dim Q$. The q-dimensional submanifolds Q of $\mathcal{X} \oplus \mathcal{Y}$ with this property are precisely those for which the projection operator $\mathcal{X} \oplus \mathcal{Y} \to \mathcal{X}$ restricted to Q is locally diffeomorphic. Such submanifolds thus are, locally, nothing else than the graphs $\{x + \Phi(x) : x \in \mathcal{U}\}$ of arbitrary C^∞ functions $\Phi : \mathcal{U} \to \mathcal{Y}$, $\mathcal{U}$ being an open subset of $\mathcal{X}$.

On the other hand, still treating Q as a submanifold of $\mathcal{X} \oplus \mathcal{Y}$, we clearly have

$$M = Q + \mathcal{W} = \{z + w : z \in Q,\, w \in \mathcal{W}\}.$$

In view of the graph representation of Q just mentioned, this completes the proof. ■

Example 15.14. The local parametrizations (15.11) of all possible $\mathcal{W}$-ruled $2q$-dimensional nondegenerate submanifolds M of V (with V, W as in Proposition 15.13) leads to an easy description of the curvature tensor R of the pseudo-Riemannian metric g that M inherits from V. Note that, by Corollary 15.10, g then *is automatically a Ricci-flat metric with the neutral sign pattern* (q minuses, q pluses). To simplify our description of R (obtained via the Gauss equation (12.2)), we now assume that $r = 1$ in Proposition 15.15, so that M is a *codimension-one* submanifold of V.

Specifically, let us choose V with its inner product $\langle\,,\rangle$ in such a way that $\dim V = 2q+1$ and $\langle\,,\rangle$ has the "almost neutral" sign pattern with $q+1$ minuses and q pluses (or vice versa). From now on we use the dot symbol $\cdot$, rather than $\langle\,,\rangle$, for the inner product in V, and fix the following ranges for indices:

$$j,k \in \{1,\ldots,q\}\,, \qquad a,c \in \{q+1,\ldots,2q\}\,.$$

Given a q-dimensional null subspace $\mathcal{W}$ of V, let us choose a basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2q}$ of V such that, for $j = 1,\ldots,q$ and $a = q+1,\ldots,2q$,

$$\mathbf{e}_a \in \mathcal{W}\,, \qquad \mathbf{e}_0\cdot\mathbf{e}_j \,=\, \mathbf{e}_0\cdot\mathbf{e}_a \,=\, 0\,, \qquad \mathbf{e}_0\cdot\mathbf{e}_0 \,=\, \varepsilon \,=\, \pm 1\,.$$

This can be done by starting from any basis $\mathbf{e}_a$ of $\mathcal{W}$, then finding $\mathbf{e}_0 \in \mathcal{W}^\perp$ with $\mathbf{e}_0\cdot\mathbf{e}_0 = \pm 1$ (such $\mathbf{e}_0$ always exists; otherwise, $\mathcal{W}^\perp$ would be null, contradicting (3.27)) and, finally, selecting the $\mathbf{e}_j$ so as to complete the $\mathbf{e}_a$ to a basis of $\mathbf{e}_0^\perp$. Let the subspaces $\mathcal{X}$ and $\mathcal{Y}$ in Proposition 15.13 now be given by $\mathcal{X} = \mathrm{Span}\,\{\mathbf{e}_1,\ldots,\mathbf{e}_q\}$ and $\mathcal{Y} = \mathbf{R}\mathbf{e}_0$. Thus, a $\mathcal{Y}$-valued function on an open subset $\mathcal{U}$ of $\mathcal{X}$ may be written as $\mathcal{U} \ni x \mapsto \Phi(x) = f(x^1,\ldots,x^q)\,\mathbf{e}_0 \in \mathcal{Y}$, where the x^j are the linear coordinates in $\mathcal{X}$ associated with the basis $\mathbf{e}_j$. Using the traditional notations described in Remark 13.5, we can rewrite the local parametrization (15.11) of M as

$$(x^1,\ldots,x^{2q}) \,\mapsto\, \mathbf{r} \,=\, x^j\mathbf{e}_j \,+\, f(x^1,\ldots,x^q)\,\mathbf{e}_0 \,+\, x^a\mathbf{e}_a\,, \tag{15.12}$$

and so its partial derivatives are $\mathbf{r}_j = \mathbf{e}_j + (\partial_j f)\,\mathbf{e}_0$, $\mathbf{r}_a = \mathbf{e}_a$, $\mathbf{r}_{jk} = (\partial_j\partial_k f)\,\mathbf{e}_0$, $\mathbf{r}_{ja} = \mathbf{r}_{ac} = \mathbf{0}$. The "mixed" components $g_{aj} = \mathbf{r}_j\cdot\mathbf{r}_a$ of the metric g thus are constant, with $g_{aj} = \mathbf{e}_j\cdot\mathbf{e}_a$, and the corresponding components of the g^{aj} reciprocal metric form the (constant) inverse matrix $[g^{aj}] = [g_{ja}]^{-1}$, as $g_{a0} = g_{0j} = 0$. A unit normal vector field $\mathbf{n}$ with $\mathbf{n}\cdot\mathbf{n} = \varepsilon$ now can be defined by

$$\mathbf{n} \,=\, \mathbf{e}_0 \,-\, \varepsilon\, g^{aj}(\partial_j f)\,\mathbf{e}_a\,.$$

From the last equality in (13.10) combined with the Gauss equation, we now obtain

$$\begin{aligned} &b_{jk} \,=\, \partial_j\partial_k f\,, \qquad b_{a0} \,=\, b_{0j} \,=\, b_{ac} \,=\, 0\,,\\ &R_{jklm} \,=\, \varepsilon\,(b_{jl}b_{km} - b_{kl}b_{jm})\,, \qquad R_{\ldots a\ldots} \,=\, 0\,, \end{aligned} \tag{15.13}$$

where the last equality states that all curvature components that involve at least one index a in the range $\{q+1,\ldots,2q\}$ are identically zero. Consequently, this construction provides examples of Ricci-flat codimension-one submanifolds M of V, in all even dimensions $n = \dim M$ starting from $n = 2$; and, if $n \geq 4$, many of those examples are not flat (and so, being Ricci-flat, they are not spaces of constant curvature). See Remark 15.15 below.

The assertion of Proposition 15.6 thus fails in the case of Einstein metrics with the neutral sign pattern $-\,-\,+\,+$.

Remark 15.15. In contrast with the Riemannian case (cf. Remark 14.8), an indefinite Einstein metric g on a manifold M is not always analytic in suitable local coordinates. Namely, in all dimensions $n \geq 4$, the construction summarized in Example 15.14 produces some neutral Ricci-flat metrics g whose curvature tensor vanishes on a nonempty open subset of M without being identically zero on M (while M is connected); in particular, the construction in question leads to some non-flat metrics.

To achieve this, note that f in (15.12) can be just any C^∞ function of the variables $x^1, \ldots, x^q$. Therefore, we may select f to be identically zero on $\mathcal{U}'$ and equal to x^1x^2 on $\mathcal{U}''$, where $\mathcal{U}'$ and $\mathcal{U}''$ are two suitably chosen, disjoint, nonempty open subsets of the (connected) set $\mathcal{U}$. We then have, from (15.13), $R = 0$ everywhere in U' and $R \neq 0$ on U'', where $U', U'' \subset M$ are the images of $\mathcal{U}'$ and $\mathcal{U}''$ under the parametrization (15.12).

§16. Conformal changes of metrics

This section deals with the question of what happens with curvature-related invariants when the metric in question undergoes a conformal change, with the ultimate goal of using such a procedure to construct Einstein metrics (in §18). The results presented here go back to Weyl (1918) and Schouten (1921).

Any two connections ∇, $\tilde{\nabla}$ in a given vector bundle $\mathcal{E}$ over a manifold M differ by a tensor. More precisely, we have

$$\tilde{\nabla} = \nabla + F, \tag{16.1}$$

where F is a section of $\mathrm{Hom}\,(TM, \mathrm{Hom}\,(\mathcal{E},\mathcal{E}))$. Thus, F associates with each $v \in T_xM$, $x \in M$, a linear operator $F_v : \mathcal{E}_x \to \mathcal{E}_x$, and relation (16.1) reads $\tilde{\nabla}_v\psi = \nabla_v\psi + F_v(\psi(x))$ for all $x \in M$, $v \in T_xM$ and local C^1 sections of $\mathcal{E}$ defined near x. To see that $F_v(\psi(x))$ really depends on ψ only through $\psi(x)$, note that, by (4.49), $(\tilde{\nabla}_v\psi)^a - (\nabla_v\psi)^a = F^a_{jb}\psi^b$, where Γ^b_{ja} and $\tilde{\Gamma}^b_{ja}$ are the component functions of ∇ and, respectively, $\tilde{\nabla}$, while F^a_{jb} (the component functions of F) are characterized by

$$\tilde{\Gamma}^b_{ja} = \Gamma^b_{ja} + F^b_{ja}\,. \tag{16.2}$$

Let us now suppose that, besides $\mathcal{E}$, ∇, $\tilde{\nabla}$ and F as above, we are also given a fixed torsionfree connection in TM. Combined with ∇, this torsionfree connection then induces a connection in $\mathrm{Hom}\,(TM, \mathrm{Hom}\,(\mathcal{E},\mathcal{E}))$ (for which we use the same symbol ∇). Any fixed vector w tangent to M thus gives rise to the covariant derivative ∇_wF. Using (4.52), it is easy to see that the curvature tensors R of ∇ and $\tilde{R}$ of $\tilde{\nabla}$ then are related by

$$\tilde{R}(v,w) = R(v,w) + (\nabla_wF)_v - (\nabla_vF)_w + [F_w, F_v] \tag{16.3}$$

for $v, w \in T_xM$, $x \in M$, the last term being the commutator of operators $\mathcal{E}_x \to \mathcal{E}_x$. The component version of (16.3) is (cf. (4.48), (4.53))

$$\tilde{R}_{jka}{}^b = R_{jka}{}^b + F^b_{ja,k} - F^b_{ka,j} + F^b_{kc}F^c_{ja} - F^b_{jc}F^c_{ka}\,. \tag{16.4}$$

Let g now be a pseudo-Riemannian metric on a manifold M. Any fixed C^∞ function $f : M \to \mathbf{R}$ gives rise to a new metric

$$\tilde{g} = e^{2f}g\,. \tag{16.5}$$

One then says that g and $\tilde{g}$ are *conformally related*. For objects naturally associated with metrics g (such as the Levi-Civita connection ∇, the curvature tensor R, Ricci tensor Ric and its components R_{jk}, the scalar curvature s, Weyl tensor W, and divergence operator div), we will use the self-explanatory symbols $\tilde{\nabla}$, $\tilde{R}$, $\tilde{\mathrm{Ric}}$, $\tilde{R}_{jk}$, $\tilde{\mathrm{s}}$, $\tilde{W}$ and $\tilde{\mathrm{div}}$ to denote the analogous objects corresponding to $\tilde{g}$. The symbols ∇ and $\tilde{\nabla}$ will, as usual, stand not only for the Levi-Civita connections of g and $\tilde{g}$, but also for their *gradient operators*.

Remark 16.1. Discussing various methods that lead to constructions of pseudo-Riemannian Einstein metrics in dimension 4, in §19 and §18 we will take a look at metrics on product manifolds $N \times N'$ which are conformally related to Riemannian-product metrics, that is, have the form

$$e^{-2f}\,[h\,+\,h']\,, \tag{16.6}$$

where $(N \times N',\, h + h')$ is the Riemannian product of two pseudo-Riemannian manifolds (N,h) and (N',h'), while $f : N \times N' \to \mathbf{R}$ is a C^∞ function. One particularly prominent special case of this situation is that of *warped-product metrics* (Kručkovič, 1957; Bishop and O'Neill, 1969), given by (16.6) with a function f that is constant in the direction of one of the factors (N or N'), i.e., is just a function on the remaining factor manifold. For instance, a surface metric is, locally, a warped product if and only if it admits a non-null Killing field (see Corollary 19.3 in §19). Cf. Gębarowski (1992).

Lemma 16.2. *Let there be given two conformally related pseudo-Riemannian metrics g and $\tilde{g} = e^{2f}g$ on an n-dimensional manifold M.*

(i) *The Christoffel symbols Γ^l_{jk} of g and $\tilde{\Gamma}^l_{jk}$ of $\tilde{g}$ satisfy*

$$\tilde{\Gamma}^l_{jk}\,=\,\Gamma^l_{jk}\,+\,F^l_{jk}\qquad\text{with}\quad F^l_{jk}\,=\,\delta^l_k\partial_j f\,+\,\delta^l_j\partial_k f\,-\,g_{jk}g^{ls}\partial_s f\,. \tag{16.7}$$

In other words, the Levi-Civita connections ∇ of g and $\tilde{\nabla}$ of $\tilde{g}$ are related by $\tilde{\nabla}_v w = \nabla_v w + (d_v f)w + (d_w f)v - g(v,w)\nabla f$, i.e.,

$$\tilde{\nabla}_v w\,=\,\nabla_v w\,+\,g(v,\nabla f)w\,+\,g(w,\nabla f)v\,-\,g(v,w)\nabla f\,, \tag{16.8}$$

for any C^1 vector fields v,w, where d_v is the directional derivative corresponding to v, and ∇f stands for the g-gradient of f. Also, for any C^1 bundle morphism $\alpha : TM \to TM$ and a tangent vector u,

$$\begin{aligned} g((\tilde{\nabla}_v\alpha)w,u)\,&=\,g((\nabla_v\alpha)w,u)\,-\,g(\alpha v,u)d_w f\,-\,g(\alpha w,v)d_u f\\ &\quad+\,g(v,w)g(\alpha(\nabla f),u)\,+\,g(v,u)g(\nabla f,\alpha w)\,. \end{aligned} \tag{16.9}$$

(ii) *For any once-contravariant, three-times covariant tensor field A of class C^1 on M such that the corresponding four-times covariant tensor with the components $A_{jklm} = g_{js}A^s{}_{klm}$ has the algebraic symmetries* (5.23) – (5.25) *of the Weyl tensor, the g-divergence and $\tilde{g}$-divergence of A, defined as in* (5.26)*, are related by*

$$\tilde{\mathrm{div}}\,A\,=\,\mathrm{div}\,A\,+\,(n-3)\,A(df,\,\cdot\,,\,\cdot\,,\,\cdot\,)\,. \tag{16.10}$$

In local coordinates, this reads $\tilde{\nabla}_j A^j{}_{klm} = \nabla_j A^j{}_{klm} + (n-3)f_{,j}A^j{}_{klm}$.

Proof. Relation (16.7) is clear from (4.1), and it easily leads to (16.8). The local-coordinate version

$$\tilde\nabla_l\alpha_k^j \,=\, \nabla_l\alpha_k^j \,-\, \alpha_l^j\partial_k f + \alpha_s^j g_{kl}g^{sp}\partial_p f + \alpha_k^s\delta_l^j\partial_s f - \alpha_k^s g_{ls}g^{jp}\partial_p f$$

of (16.9) is also immediate from (4.1). Finally, (16.10) then follows since, in terms of the Christoffel symbols Γ_{jk}^l of any metric g, we have $A^j{}_{klm,j} = \partial_j A^j{}_{klm} + \Gamma_{js}^j A^s{}_{klm} - \Gamma_{jk}^s A^j{}_{slm} - \Gamma_{jl}^s A^j{}_{ksm} - \Gamma_{jm}^s A^j{}_{kls}$ while, for F_{jk}^l as in (16.7), $F_{jk}^j = n\,\partial_k f$. ■

Lemma 16.3 (Weyl, 1918)**.** *Let g and $\tilde g = e^{2f}g$ be two conformally related metrics on an n-dimensional manifold M. Then W, viewed as a once-contravariant, three-times covariant tensor field, is a* conformal invariant *in the sense that*

$$\tilde W \,=\, W\,, \tag{16.11}$$

i.e., $\tilde W^j{}_{klm} = W^j{}_{klm}$. Moreover, treating $\operatorname{div} W$ as a three-times covariant tensor field and denoting $\tilde{\operatorname{div}}\,\tilde W$ the analogous object corresponding to $\tilde g$, we have

$$\tilde{\operatorname{div}}\,\tilde W \,=\, \operatorname{div} W \,+\, (n-3)\,W(df,\cdot,\cdot,\cdot)\,. \tag{16.12}$$

In other words, $\tilde\nabla_j\tilde W^j{}_{klm} = \nabla_j W^j{}_{klm} + (n-3)f_{,j}W^j{}_{klm}$. Also, if the relation between g and $\tilde g$ is written as $\tilde g = g/\varphi^2$, with $\varphi = \pm\, e^{-f}$, then

$$\tilde{\mathrm{Ric}} \,=\, \mathrm{Ric} \,+\, (n-2)\,\varphi^{-1}\nabla d\varphi \,+\, \left[\varphi^{-1}\Delta\varphi \,-\, (n-1)\varphi^{-2}g(\nabla\varphi,\nabla\varphi)\right] g\,, \tag{16.13}$$

where ∇f again denotes the g-gradient of f, i.e., in component form,

$$\tilde R_{jk} \,=\, R_{jk} \,+\, (n-2)\,\varphi^{-1}\varphi_{,jk} \,+\, \left[\varphi^{-1}\Delta\varphi \,-\, (n-1)\varphi^{-2}g(\nabla\varphi,\nabla\varphi)\right] g_{jk}\,. \tag{16.14}$$

Finally, for $g(W,W)$ given by $4\,g(W,W) = W_{jklm}W^{jklm}$, as in (5.32),

$$\tilde g(\tilde W,\tilde W) \,=\, \varphi^4\, g(W,W)\,. \tag{16.15}$$

Proof. Combining (16.7) with (16.4), we find that

$$\begin{aligned}\tilde R_{jkl}{}^m \,=\, R_{jkl}{}^m \,&+\, \delta_j^m\,[f_{,kl} - f_{,k}f_{,l} + g(\nabla f,\nabla f)g_{kl}] + \,g_{jl}\,(f_{,k}f_{,}{}^m - f_{,k}{}^m)\\ &-\, \delta_k^m\,[f_{,jl} - f_{,j}f_{,l} + g(\nabla f,\nabla f)g_{jl}] - \,g_{kl}\,(f_{,j}f_{,}{}^m - f_{,j}{}^m)\,.\end{aligned} \tag{16.16}$$

Contracting this, we obtain

$$\tilde R_{jk} \,=\, R_{jk} \,-\, (n-2)\,(f_{,jk} - f_{,j}f_{,k}) \,-\, [\Delta f + (n-2)g(\nabla f,\nabla f)]\,g_{jk} \tag{16.17}$$

and

$$\tilde{\mathrm{s}} = e^{-2f}\,[\mathrm{s} - 2(n-1)\Delta f - (n-1)(n-2)g(\nabla f,\nabla f)]\,. \tag{16.18}$$

Now we can easily verify (16.11) (using (5.8)). Consequently, (16.12) follows from (16.10), and (16.15) is obvious since $4\,g(W,W) = -\,g^{lr}g^{ms}W^j{}_{klm}W^k{}_{jrs}$. Finally, (16.13) and (16.14) are immediate from (16.17), as $f_{,j} = -\varphi^{-1}\varphi_{,j}$ and $f_{,jk} = \varphi^{-2}\,[\varphi_{,j}\varphi_{,k} - \varphi\varphi_{,jk}]$ whenever $f = -\log|\varphi|$. This completes the proof. ■

Remark 16.4. For oriented Riemannian 4-manifolds (M,g), each of the following objects/conditions is conformally invariant, i.e., remains the same (for objects), or remains satisfied (for conditions), whenever the metric g is replaced with a conformally related metric $\tilde g = e^{2f}g$:

(a) The Hodge star $*$ acting on bivectors;
(b) The self-dual and anti-self-dual Weyl tensors W^+ and W^-, both treated as once-contravariant, three-times covariant tensor fields;
(c) The subbundles Λ^+M and Λ^-M of $[TM]^{\wedge 2}$;
(d) The functions $\#\,\mathrm{spec}\,W^\pm : M \to \{1,2,3\}$ (see (20.1) in §20);
(e) Conditions such as (16.35) and (20.2) below.

In fact, conformal invariance of $*$ is clear from (6.1). Combined with (16.11), this establishes the conformal invariance in (b), and hence (c). On the other hand, cases (d), (e) are now obvious since, for W, W^+ and W^- treated as bundle morphisms $[TM]^{\wedge 2} \to [TM]^{\wedge 2}$ or $\Lambda^\pm M \to \Lambda^\pm M$, the conformal transformation rule is

$$\tilde W \,=\, e^{-2f}W\,, \qquad \tilde W^\pm \,=\, e^{-2f}W^\pm\,, \tag{16.19}$$

whenever $\tilde g = e^{2f}g$. (To see this, note that the corresponding components then are $W^{jk}{}_{lm} = g^{ks}W^j{}_{slm}$, and similarly for $W^\pm$.) Note that the formula for $\tilde W$ in (16.19) holds in all dimensions $n \ge 3$, whether or not M is orientable.

A pseudo-Riemannian manifold (M,g) is called *conformally flat* if g is locally conformally related to a flat metric, that is, if every $x \in M$ has a neighborhood U with a C^∞ function $f : U \to \mathbf{R}$ such that the metric $\tilde g = e^{2f}g$ is flat.

It will be useful to rewrite the identity (5.28), i.e., (5.29), in the form

$$\mathrm{div}\,W \,=\, (n-3)\,Z\,, \qquad n = \dim M \ge 3\,, \tag{16.20}$$

or, in local coordinates,

$$W^j{}_{klm,j} \,=\, (n-3)\,Z_{klm}\,, \tag{16.21}$$

where Z is the three-times covariant tensor field defined by

$$2(n-1)(n-2)\,Z \,=\, d\,[2(n-1)\,\mathrm{Ric} \,-\, \mathrm{s}\,g] \tag{16.22}$$

whenever $n = \dim M \ge 3$, that is,

$$(n-2)\,Z_{klm} \,=\, R_{km,l} \,-\, R_{kl,m} \,+\, \frac{1}{2(n-1)}\,(\mathrm{s}_{,m}g_{kl} \,-\, \mathrm{s}_{,l}g_{km})\,. \tag{16.23}$$

As an obvious consequence of (16.20) and (16.12), we obtain the transformation rule

$$\tilde Z \,=\, Z \,+\, (n-3)\,W(df,\,\cdot\,,\,\cdot\,,\,\cdot\,) \tag{16.24}$$

for Z under conformal changes of the metric with $\tilde g = e^{2f}g$ in all dimensions $n \ge 4$. (Relation (16.24) is in fact valid in dimension 3 as well; see Remark 16.6 below.)

Theorem 16.5 (Schouten, 1921). *A pseudo-Riemannian manifold (M,g) of any dimension $n \geq 4$ is conformally flat if and only if $W = 0$ identically on M, where W is the Weyl tensor of (M,g).*

Proof. Let $\mathcal{E} = TM \oplus [M \times \mathbf{R}^2]$ denote the vector bundle over M obtained as the direct sum of the tangent bundle TM of M and the product plane bundle $M \times \mathbf{R}^2$. Using the Levi-Civita connection ∇ of (M,g), we can now define a connection D in $\mathcal{E}$ by the formula

$$\begin{aligned}&\mathrm{D}_v(u,\varphi,\chi)\\ &= \left(\nabla_v u + \frac{\varphi\,\mathrm{Ric}\, v - \chi v}{n-2}\,,\ d_v\varphi - g(u,v)\,,\ d_v\chi + \mathrm{Ric}\,(u,v) - \frac{2\,\mathrm{s}\,g(u,v) + \varphi d_v \mathrm{s}}{2(n-1)}\right)\end{aligned}$$

for vectors v tangent to M, where, for any C^1 tangent vector field u on M and real-valued C^1 functions φ, χ on M, the triple (u,φ,χ) is treated as a C^1 section of $\mathcal{E}$. Using (4.52) and (5.8) (and taking advantage of the shortcuts suggested by Remark 4.4), we easily verify that the curvature tensor R^{D} of D is given by

$$R^{\mathrm{D}}(v,v')(u,\varphi,\chi) = (W(v,v')u + \varphi\, Z(\,\cdot\,,v',v)\,,\ 0\,,\ Z(u,v',v))\,, \tag{16.25}$$

with Z as in (16.22), (16.23).

Let us now suppose that (M,g) is conformally flat. We then have $W = 0$ in view of conformal invariance of the Weyl tensor (relation (16.11)) along with the fact that $W = 0$ whenever g is flat (by (5.6), (4.34), (4.40)). Conversely, let us assume that $W = 0$ everywhere. From (16.20) it then follows that $Z = 0$ and so, by (16.25), the connection D in $\mathcal{E}$ defined above is flat. Thus, as a consequence of Lemma 11.2, every point $x \in M$ has a neighborhood on which $\mathcal{E}$ admits a D-parallel section (u,φ,χ) that, in addition, may be chosen so as to realize any prescribed value $(u(x),\varphi(x),\chi(x))$ at x. On the other hand, according to the definition of D, (u,φ,χ) is D-parallel if and only if

$$\begin{aligned}(n-2)\,u_{j,k} &= \chi g_{jk} - \varphi\, R_{jk}\,, \qquad \varphi_{,j} = u_j\,,\\ 2(n-1)\,\chi_{,j} &= 2\,\mathrm{s}\,u_j + \varphi\,\mathrm{s}_j - 2(n-1)\,R_{jk}u^k\,.\end{aligned} \tag{16.26}$$

Let us choose the D-parallel section (u,φ,χ) so that $\varphi(x) = 1$ (while $u(x)$ and $\chi(x)$ are still arbitrary, and will be chosen later). The metric $\tilde{g} = g/\varphi^2$ then is well-defined in a neighborhood of x and has, by (16.13), the Ricci tensor $\tilde{\mathrm{Ric}} = \tilde{\kappa}\tilde{g}$ with $\tilde{\kappa} = \varphi\chi + \varphi\Delta\varphi - (n-1)g(u,u)$. (In fact, by (16.26), $\nabla\varphi = u$, while $(n-2)\nabla d\varphi = (n-2)\nabla u = \chi g - \varphi\,\mathrm{Ric}$.) In view of Schur's Theorem 5.1, $\tilde{\kappa}$ must be constant. We can now make $\tilde{\kappa}$ identically equal to zero by picking $u(x)$ and $\chi(x)$ for which $\tilde{\kappa}(x) = 0$. Specifically, let us note that, in view of (16.26), $(n-2)\Delta\varphi = u^k{}_{,k} = n\chi - \varphi\,\mathrm{s}$, so that our formula for $\tilde{\kappa}$ can be rewritten as $(n-2)\tilde{\kappa} = -2\varphi\chi + \varphi^2\mathrm{s} - (n-1)(n-2)g(u,u)$. Now $u(x)$ may be fixed arbitrarily; as $\varphi(x) = 1$, condition $\tilde{\kappa}(x) = 0$ then will be satisfied by a unique value of $\chi(x)$. Consequently, with this choice of (u,φ,χ), the metric $\tilde{g}$ is Ricci-flat.

On the other hand, by (16.11), $\tilde{g}$ also satisfies $\tilde{W} = 0$. Hence $\tilde{g}$ is flat in view of (5.9). This completes the proof. ■

Remark 16.6. In dimension 3 we always have $W = 0$ (Remark 10.2(a)). This does not mean that pseudo-Riemannian 3-manifolds are all conformally flat; specifically, a tensorial condition characterizing conformal flatness of 3-dimensional pseudo-Riemannian manifolds is $Z = 0$, where Z is given by (16.22) (i.e., (16.23)). In fact, a direct computation based on (16.7), (16.17), (16.18), (4.26), (4.39) and (5.8) shows that (16.24) holds in dimension 3 as well (which, as $W = 0$, then means that Z is a conformal invariant). With this additional information, the above proof of Theorem 16.5 obviously works in the 3-dimensional case as well.

Let (M, g) be a pseudo-Riemannian 4-manifold obtained as the Riemannian product of two pseudo-Riemannian surfaces with the Gaussian curvatures λ and μ. In product coordinates x^j, y^a the components of R and Ric satisfy the relations (cf. Remark 10.1)

$$R_{jklm} = \lambda (g_{jl} g_{km} - g_{kl} g_{jm}) , \qquad R_{abcd} = \mu (g_{ac} g_{bd} - g_{bc} g_{ad}) , \tag{16.27}$$

$$R_{jk} = \lambda g_{jk} , \qquad R_{ab} = \mu g_{ab} , \qquad R_{ja} = R_{aj} = 0 , \tag{16.28}$$

and they vanish unless all the indices are of the same kind. Let us denote $\mathcal{P}$ and $\mathcal{Q}$ the vector subbundles of TM which are tangent to the factor surfaces, and let $P, Q : TM \to TM$ be the self-adjoint bundle morphisms of orthogonal projections onto $\mathcal{P}$ and $\mathcal{Q}$. Using the index-lowering operation corresponding to g, we can also treat P and Q as symmetric twice-covariant tensor fields on M. Since $\mathcal{P}$, $\mathcal{Q}$ are parallel as subbundles of TM (i.e., invariant under covariant derivatives in all directions, cf. Remark 4.7(i)), P and Q are parallel tensor fields on M. In product coordinates x^j, y^a as above, the components of P and Q are $P_{jk} = g_{jk}$, $P_{aj} = P_{ja} = P_{ab} = 0$ and $Q_{ab} = g_{ab}$, $Q_{aj} = Q_{ja} = Q_{jk} = 0$. Thus, by (16.27), (16.28) and (5.7),

$$\begin{aligned} R &= \lambda P \circledast P + \mu Q \circledast Q , \qquad \mathrm{Ric} = \lambda P + \mu Q , \\ g &= \mathrm{Id} = P + Q , \qquad \mathrm{s} = 2 (\lambda + \mu) . \end{aligned} \tag{16.29}$$

Since $\circledast$ is bilinear and symmetric (see (5.7)), using (5.6) we now obtain the following *relation satisfied by the Weyl tensor W and scalar curvature s of the product of any two pseudo-Riemannian surface metrics*:

$$W = \mathrm{s} A , \qquad \text{with} \quad \nabla A = 0 \quad \text{and} \quad A \neq 0 , \tag{16.30}$$

where A is given by

$$6 A = P \circledast P + Q \circledast Q - P \circledast Q . \tag{16.31}$$

Lemma 16.7. *Let (M, g) be a pseudo-Riemannian 4-manifold obtained as the Riemannian product of two pseudo-Riemannian surfaces. Then the following three conditions are equivalent*:

(a) *(M, g) is conformally flat.*
(b) *The scalar curvature s of g is identically zero.*
(c) *The Gaussian curvatures λ, μ of the factor surfaces are both constant and $\lambda + \mu = 0$.*

This is clear from (16.30) and the last equality in (16.29). (Note that, if $\lambda+\mu=0$ identically on M, then λ and μ must both be constant, as one sees using an obvious separation-of-variables argument.) ∎

Before stating the next lemma let us recall that a bivector at a point x in a pseudo-Riemannian manifold (M,g) is called nondegenerate if it is an isomorphism when treated as a skew-adjoint operator $T_xM \to T_xM$. A parallel bivector field on (M,g) which is nondegenerate at one point must be nondegenerate everywhere.

Lemma 16.8. *Let W and s denote, as usual, the Weyl tensor and the scalar curvature of a given pseudo-Riemannian 4-manifold (M,g), and let (M,g) admit a parallel bivector field α which is nondegenerate. Suppose that U is a connected open subset of M such that $\mathrm{s} \neq 0$ everywhere in U, and $\varphi : U \to \mathbf{R}$ is any nowhere-zero C^∞ function, defined on U, and having the property that the conformally related metric $\tilde{g} = g/\varphi^2$ on U satisfies the condition*

$$\tilde{\operatorname{div}}\, \tilde{W} \,=\, 0\,. \tag{16.32}$$

Then φ must be a constant multiple of the scalar curvature s.

Proof. By (5.19), $6W\alpha = \mathrm{s}\alpha$. Taking the divergence of both sides of this relation, (i.e., applying ∇^j to its local-coordinate version $3\,W_{jklm}\alpha^{lm} \,=\, \mathrm{s}\alpha_{jk}$), we obtain

$$6\,[\operatorname{div} W]\,\alpha \,=\, \alpha(\nabla \mathrm{s})\,, \tag{16.33}$$

that is, $3\,W^j{}_{klm,j}\alpha^{lm} \,=\, \mathrm{s}_{,j}\alpha^j{}_k$. On the other hand, (16.12) for $n=4$ along with (16.32) yields $\operatorname{div} W \,=\, -\,W(df\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$, which, with $f \,=\, -\,\log|\varphi|$, becomes $\operatorname{div} W \,=\, \varphi^{-1}W(d\varphi\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$. Combining this with (16.33) and (5.19), we obtain $\varphi\,\alpha(\nabla \mathrm{s}) = 6\,\varphi[\operatorname{div} W]\,\alpha = 6\,[W\alpha](\nabla\varphi) = \mathrm{s}\alpha(\nabla\varphi)$, that is, $\alpha\xi = 0$ (in local coordinates, $\xi_j\alpha^{jk} = 0$), for the 1-form $\xi \,=\, \mathrm{s}\,d\varphi \,-\, \varphi\,d\mathrm{s} \,=\, \mathrm{s}^2 d(\varphi/\mathrm{s})$. Nondegeneracy of α now implies that $\xi = 0$, i.e., φ/s is constant, which completes the proof. ∎

Remark 16.9. The assertion of Lemma 16.8 holds for those Riemannian products (M,g) of two pseudo-Riemannian surfaces for which $\mathrm{s} \neq 0$ everywhere. In fact, locally in M, we can define a nondegenerate parallel bivector fields α^+ and α^- by

$$\sqrt{2}\,\alpha^\pm \,=\, e_1 \wedge e_2 \,\pm\, e_3 \wedge e_4\,, \tag{16.34}$$

where the $e_1, \ldots, e_4$ are fixed orthonormal C^∞ vector fields such that e_1 and e_2 are tangent to the first factor surface. However, in this case the converse statement holds as well: Namely, the metric $\tilde{g} \,=\, g/\mathrm{s}^2$ now actually must satisfy condition (16.32). This is an obvious consequence of relations (16.12) (with $f \,=\, -\,\log|\mathrm{s}|$) and (16.30).

Remark 16.10. Since we devote a whole section (§18 below) to Einstein metrics in dimension four that are *locally conformally related to products of two surface metrics*, it may be worth noting that, in the Riemannian case, the Einstein metrics with that property are characterized by the requirement that

$$\operatorname{spec} W^+ \,=\, \operatorname{spec} W^-\,, \qquad \#\operatorname{spec} W^+ \,\leq\, 2 \tag{16.35}$$

at every point, for either local orientation of the underlying manifold M (notation as in Remark 10.11 and Lemma 6.15; see also (20.1), (20.2) in §20 below). Note that (16.35) is an algebraic condition on the eigenvalues of the operators $W^\pm$ acting on bivectors. Since the characterization (16.35) will not be used in the sequel, the proof we give here will rely on de Rham's Theorem 4.10 the uses of which, as stated at the end of §4, we are trying to avoid. First, (16.35) is satisfied by all products of two surface metrics: The nonzero parallel bivector field $\alpha^\pm$ defined, locally, as in (16.34), is a section of $\Lambda^\pm M$ (Lemma 6.2), and so (16.35) is immediate from Proposition 9.8 applied to either local orientation. Furthermore, (16.35) is a conformally invariant property (Remark 16.4(e)). Consequently, (16.35) is necessary for *any* Riemannian metric (Einstein or not) to be locally conformally related to a product of two surface metrics. Conversely, let an oriented Riemannian Einstein 4-manifold (M, g) satisfy (16.35). To show that g is locally conformally related to a product of two surface metrics, let us first suppose that W^+ and W^- both vanish identically. Thus, $W = 0$, and so g is conformally flat (Theorem 16.5), as required. On the other hand, if one of W^+, W^- is nonzero somewhere, then, by Proposition 20.1(i) and (16.35), they are both nonzero everywhere and, in view of Proposition 22.3(iii) in §22, a metric $\tilde g$ conformally related to g admits nonzero parallel local sections of both $\Lambda^+ M$ and $\Lambda^- M$. (Note that, by (16.35) and (5.32) for W^+ instead of R, we have $|W^+| = |W^-|$). According to Lemma 6.1(ii) such sections, when suitably normalized, may be written as α^+ and α^- in (16.34), with some $\tilde g$-orthonormal local vector fields $e_1, \dots, e_4$ of class C^∞, defined on a neighborhood U of any given point in M. Treating $\alpha \pm \alpha^-$ as skew-adjoint bundle morphisms $TM \to TM$ (with the aid of $\tilde g$), we now obtain $\tilde g$-parallel vector subbundles $\mathcal P$ and $\mathcal Q$ of TU given by $\mathcal P = \operatorname{Ker}(\alpha^+ - \alpha^-)$ and $\mathcal Q = \operatorname{Ker}(\alpha^+ + \alpha^-)$. Thus, $\mathcal P_x = \operatorname{Span}\{e_1, e_2\}$ and $\mathcal Q_x = \operatorname{Span}\{e_3, e_4\}$ for all $x \in U$. We can now apply Theorem 4.10 to the $\tilde g$-parallel vector-bundle direct-sum decomposition $TU = \mathcal P \oplus \mathcal Q$.

§17. Killing fields

This section covers basis facts on Killing fields. For more details, see, e.g., Kobayashi and Nomizu (1963).

By a *Killing field* on a pseudo-Riemannian manifold (M, g) we mean any C^1 tangent vector field w on M such that the bundle morphism $\nabla w : TM \to TM$ is skew-adjoint at every point, that is, $[\nabla w](x) \in \mathfrak{so}(T_xM)$ for all $x \in M$ (with the form $\langle\,,\rangle = g(x)$ in T_xM). This means, in other words, that the bilinear form $(u, v) \mapsto \langle \nabla_u w, v\rangle$ on each tangent space T_xM is skew-symmetric or, equivalently, that

$$\langle \nabla_v w, v\rangle \,=\, 0 \qquad \text{for all} \quad x \in M \quad \text{and} \quad v \in T_xM\,. \tag{17.1}$$

Therefore, in local coordinates, Killing fields are characterized by

$$w_{j,k} \,+\, w_{k,j} \,=\, 0\,. \tag{17.2}$$

Suppose that w is any C^1 vector field on a pseudo-Riemannian manifold (M, g). Using (4.20) we obtain $w_{k,l} + w_{l,k} = \partial_l w_k + \partial_k w_l - 2\Gamma_{kls} w^s$, which, as $w_k = g_{ks} w^s$, becomes $w_{k,l} + w_{l,k} = g_{ks}\partial_l w^s + g_{ls}\partial_k w^s + [\partial_l g_{ks} + \partial_k g_{ls} - 2\Gamma_{kls}]\, w^s$. Therefore, by (4.6), we have

$$w_{j,k} \,+\, w_{k,j} \,=\, g_{js}\partial_k w^s \,+\, g_{ks}\partial_j w^s \,+\, w^s\,\partial_s g_{jk}\,. \tag{17.3}$$

The statements made in Examples 17.1 – 17.3 and Lemma 17.4 below are proved directly, but can also be obtained as obvious consequences of the flow interpretation of Killing fields described in Lemma 17.16.

Example 17.1. Let x^j be a local coordinate system in a pseudo-Riemannian manifold (M,g), with some coordinate domain U. Then, for any fixed index l, the coordinate vector field e_l is a Killing field on (U,g) if and only if all components of g are locally constant in the direction of x^l, that is, $\partial_l g_{jk} = 0$. In fact, by (2.1) and (17.3), the left-hand side of (17.2) for $w = e_l$ coincides with $\partial_l g_{jk}$.

Example 17.2. Any two given Killing fields w and w' on pseudo-Riemannian manifolds (N,h) and, respectively, (N',h'), may be combined, in an obvious manner, into a Killing field $w + w'$ on the Riemannian-product manifold $(M,g) = (N \times N',\, h + h')$.

Example 17.3. Let w be a Killing field on a pseudo-Riemannian manifold (M,g).

(a) If N is a nondegenerate submanifold of (M,g) (see §13) and w is tangent to N along N (that is, $w(x) \in T_xN$ for every $x \in N$), then the restriction w' of w to N is a Killing field on (N,h), where h is the submanifold metric of N.

(b) If $\tilde{g} = e^{2f}g$ is a metric conformally related to g (§16), with some C^∞ function f such that $d_w f = 0$, then w is a Killing field for $(M,\tilde{g})$ as well.

In fact, in case (a), let D and ∇ be the Levi-Civita connections of (M,g) and (N,h), respectively. Then w' satisfies (17.1) since, by (13.8), we have $\langle \nabla_v w', v\rangle = \langle [\mathrm{D}_v w]^{\mathrm{tang}}, v\rangle = \langle \mathrm{D}_v w, v\rangle = 0$ for any $x \in N$ and $v \in T_xN \subset T_xM$. As for (b), it follows immediately from (17.3) along with (17.2).

Lemma 17.4. *Let w be a Killing vector field on a pseudo-Riemannian manifold (M,g) and let s stand, as usual, for the scalar curvature of g. Then $d_w\mathrm{s} = 0$.*

Proof. On the other hand, skew-adjointness of ∇w implies $\operatorname{div} w = 0$ (see (4.42); in other words, $w^j,j = 0$ by (17.2). This in turn gives (by (4.39) or (4.45)) $\operatorname{Ric} w = \operatorname{div}(\nabla w)$, that is, $R_{jk}w^k = w^k{}_{,jk}$. Taking the divergences of both sides, we now obtain $\operatorname{div}(\operatorname{Ric} w) = \operatorname{div}\operatorname{div}(\nabla w)$, or $(R_{jk}w^k)^j = w^j{}_{k,j}{}^k$. However, applying (4.47) to $\alpha = \nabla w$ we see that the right-hand side of the last equality is zero; its left-hand side, however, equals $R^j{}_{k,j}w^k + R^{jk}w_{k,j}$ and, by (5.2), $2R^j{}_{k,j}w^k = w^k\mathrm{s}_{,k} = d_w\mathrm{s}$, while, from (4.38) and (17.2), $R^{jk}w_{k,j} = 0$. This completes the proof. ■

It follows from (17.2) that every Killing field w on a pseudo-Riemannian manifold (M,g) satisfies the relation

$$w_{j,kl} \;=\; R_{jklp}w^p\,. \tag{17.4}$$

In fact, (4.26) gives $w_{j,kl} - w_{j,lk} = R_{lkjp}w^p$ and, by permuting the indices, we also have $w_{l,kj} - w_{l,jk} = R_{jklp}w^p$, $w_{k,lj} - w_{k,jl} = R_{jlkp}w^p$. Adding these three equalities, we obtain (17.4) from (17.2) and the first Bianchi identity (4.33).

Remark 17.5. In view of (17.4) and (4.32), a Killing field w on (M,g) restricted to any geodesic $t \mapsto x(t) \in M$ becomes a Jacobi field, that is, satisfies the Jacobi equation (4.51). This is obvious since $[\nabla_{\dot{x}}\nabla_{\dot{x}}w]_j = w_{j,kl}\dot{x}^k\dot{x}^l$ (as $\nabla_{\dot{x}}\dot{x} = 0$.

Furthermore, relation (17.4) allows us to identify Killing vector fields on (M,g) with parallel sections of a specific vector bundle over M. See Remark 17.25.

Remark 17.6. Formula (4.51) has the following immediate consequences, for any pseudo-Riemannian manifold (M,g):

(i) There can be at most one Killing vector field w on a given connected neighborhood of any point $x\in M$ realizing any prescribed initial data $w(x)$ and $[\nabla w](x)$ at x. In other words, if we denote $\mathfrak{isom}(M,g)$ the vector space of all Killing fields on (M,g), then, for any fixed $x\in M$, the linear operator

$$\mathfrak{isom}(M,g)\ni w\,\mapsto\,(w(x),\,[\nabla w](x))\in T_xM\times\,\mathfrak{so}(T_xM) \tag{17.5}$$

is *injective.* Consequently, by (3.31),

$$\dim\,[\mathfrak{isom}(M,g)]\,\le\,\frac{n(n+1)}{2}\,,\qquad n\,=\,\dim M\,. \tag{17.6}$$

(ii) For any vector space $\mathcal{W}$ of Killing fields on a nonempty connected open subset U of M such that $\dim\mathcal{W}\ge n(n+1)/2$ (where $n=\dim M$), we necessarily have $\mathcal{W}=\mathfrak{isom}(U,g)$ and $\dim\,[\mathfrak{isom}(U,g)]=n(n+1)/2$.

(iii) Every Killing field is automatically of class C^∞.

(iv) Killing fields have a *unique continuation property*: Two Killing fields w,u on a pseudo-Riemannian manifold (M,g), such that $w=u$ on a nonempty open subset of M, must coincide everywhere in M.

In fact, (i) and (iii) are obvious from (4.51) combined with the uniqueness and regularity theorem for ordinary differential equations, and (ii) is immediate from (i). As for (iv), note that, by (i), the set of points at which $w=u$ and $\nabla w=\nabla u$ is both closed and open in M, while our manifolds are connected by definition.

Killing fields are actually even more “rigid” than Remark 17.6(iv) indicates; namely, a Killing field w on (M,g) is uniquely determined by its behavior along a codimension-one submanifold N of M (where w is not assumed tangent to N). More precisely, we have

Lemma 17.7. *Let w and w' be Killing vector fields on a pseudo-Riemannian manifold (M,g) such that, for some codimension-one submanifold N of M, we have $w(x)=w'(x)$ whenever $x\in N$. Then $w=w'$ everywhere in M.*

Proof. It suffices to show that a Killing field w on (M,g) with $w(x)=0$ for all $x\in N$ must vanish identically on M. To this end, let us note that, for any C^1 curve $t\mapsto x(t)\in N$ we obviously have $\nabla_{\dot x}w=0$ (by (4.13)), since w vanishes along the curve; here ∇ is the Levi-Civita connection of (M,g). Comparing (4.13) with (4.12), we see that $\nabla_v w=0$ for every $x\in N$ and every $v\in T_xN$. Let us now fix $x\in N$ and set $A=[\nabla w](x)$. Thus, $A:T_xM\to T_xM$ is a skew-adjoint linear operator vanishing on the codimension-one subspace $V=T_xN$. Choosing $u\in T_xM\smallsetminus V$ we now obtain $\langle Au,u\rangle=0$ and $\langle Au,v\rangle=-\langle u,Av\rangle=0$ for all $v\in V$, so that $Au=0$ and, consequently, $A=0$. Hence $w(x)=0$ and $[\nabla w](x)=0$, so that $w=0$ identically (see Remark 17.6(i)). This completes the proof. ■

For Killing fields w,u on any pseudo-Riemannian manifold (M,g) we have

$$\nabla[w,u]\,=\,[\nabla u,\nabla w]\,+\,R(u,w)\,, \tag{17.7}$$

as one sees using (17.4) and relation (4.4). On the other hand, given a pseudo-Riemannian manifold (M,g), a point $x \in M$, and vectors $v, w \in T_xM$, we always have

$$R(v,w) \in \mathfrak{so}(T_xM) \tag{17.8}$$

in view (4.32). Thus, the vector space $\mathfrak{isom}(M,g)$ of all Killing fields on (M,g), with the Lie-bracket operation, is a Lie algebra; cf. also Remark 17.6(iii).

Example 17.8. Let (M,g) be a pseudo-Euclidean vector space V with the flat constant metric g provided by its inner product $\langle\,,\rangle$ (Example 10.3). The Killing fields on any nonempty connected open subset U of $M = V$ then are precisely those vector fields w on U which, treated as mappings $U \to V$, have the form $w(x) = Ax + v$ with $A \in \mathfrak{so}(V)$ and $v \in V$. In fact, Killing fields w in any flat manifold satisfy $w_{j,kl} = 0$ in view of Lemma 3.1, since $w_{j,kl}$ then is symmetric in k,l (by (4.26)) and skew-symmetric in j,k (by (17.2)). In linear coordinates, this becomes $\partial_l\partial_k w^j = 0$, and so $w(x)$ must be a (possibly nonhomogeneous) linear function of x, as required. Consequently,

$$\dim\,[\mathfrak{isom}(U,g)] = n(n+1)/2\,, \qquad n = \dim M = \dim V\,. \tag{17.9}$$

Finally, the Lie bracket of two such Killing fields w and w' with $w(x) = Ax + v$, $w'(x) = A'x + v'$, is given by

$$[w,w'](x) \,=\, (A'A - AA')x \,+\, (A'v - Av')\,. \tag{17.10}$$

This is clear from (4.4) since, in this case, $(\nabla_u w)(x) = (d_u w)(x) = Au$.

Example 17.9. Let (M,g) be a space of constant curvature obtained as a (nonempty) pseudosphere $M = S_c = \{v \in V : \langle v,v\rangle = c\}$ in a pseudo-Euclidean vector space V (Example 10.4). Every skew-adjoint linear operator $A : V \to V$ may be regarded as a vector field on V, given by $V \ni x \mapsto Ax \in V = T_xV$. The vector field A then is tangent to $M = S_c$ at each $x \in M$, since $Ax \in x^\perp = T_xM$ due to skew-adjointness of A. The restriction of A to M thus constitutes a tangent vector field $w = w_A$ on M, which, according to Example 17.3, must be a Killing field for (M,g). Conversely, according to Remark 17.6(ii), every Killing field on any connected open subset U of (M,g) arises in this way from some $A \in \mathfrak{so}(V)$ and, as in (17.9), we have

$$\dim\,[\mathfrak{isom}(U,g)] = n(n+1)/2\,, \qquad n = \dim M = \dim V - 1\,. \tag{17.11}$$

(The assignment $A \mapsto w_A$ is clearly injective.) Moreover,

$$[w_A, w_B] \,=\, w_{BA-AB}\,, \tag{17.12}$$

as one easily sees using (13.8) and (4.4). Note that, in this case, $[\;]^{\mathrm{tang}}$ becomes redundant in formula (13.8), i.e., by (17.7), $\nabla_v w = \mathrm{D}_v A$, since $Ax \in x^\perp = T_xM$.

Remark 17.10. Let (M,g) be a Riemannian surface of constant Gaussian curvature $K = \kappa$ (cf. Remark 10.1), obtained as in Example 10.3 (with $\kappa = 0$) or Example

10.4 ($\kappa \neq 0$), using a suitable pseudo-Euclidean vector space V. The Lie algebra $\mathfrak{isom}(M,g)$ then has a basis w_1, w_2, w_3 satisfying the Lie-bracket relations

$$(17.13) \qquad [w_1, w_2] = \delta w_3\,, \quad [w_2, w_3] = w_1\,, \quad [w_3, w_1] = w_2\,,$$

where $\delta = \mathrm{sgn}\,(\kappa) \in \{-1, 0, 1\}$. In fact, when $\kappa = 0$, we may choose an orthonormal basis u, v of the Euclidean plane $M = V$ and set, for all $x \in M$, $w_1(x) = u$, $w_2(x) = v$ and $w_3(x) = Ax$, with $A \in \mathfrak{so}(V)$ characterized by $Au = -v$, $Av = u$. (See Example 17.8.) On the other hand, if $\kappa \neq 0$, then $\dim V = 3$, M is a pseudosphere in V, and V admits an orthonormal basis u, v, w with the sign pattern $+\,+\,\pm$ (up to an overall sign change), the last sign $\pm$ being that of $\delta = \pm 1$. We then set $w_j = w_{A(j)}$ (see Example 17.9), with with $A(j) = A_j \in \mathfrak{so}(V)$, $j = 1, 2, 3$, characterized by $A_1 v = w$, $A_1 w = -\delta v$, $A_2 u = w$, $A_2 w = -\delta u$, $A_3 u = v$, $A_3 v = -u$, and $A_1 u = A_2 v = A_3 w = 0$. According to Examples 17.8 and 17.9, the w_j then form, in both cases, a basis of $\mathfrak{isom}(M,g)$. Finally, relations (17.13) easily follow from (17.10) and, respectively, (17.12).

The next two results establish some natural relations between Killing vector fields and parallel bivector fields in pseudo-Riemannian manifolds (M,g). In both, we will use the metric g to treat a bivector field α on M as a (skew-adjoint) bundle morphism $TM \to TM$, as in (2.12).

For any C^2 function $f : M \to \mathbf{R}$ on a pseudo-Riemannian manifold (M,g), the *second covariant derivative* or *Hessian* of f is the symmetric twice-covariant tensor field ∇df, with the local components $f_{,jk} = \partial_j \partial_k f - \Gamma^l_{jk} \partial_l f$ (which is just (4.19) for $\xi = df$). Using the index-raising operation corresponding to g, we may regard ∇df as a self-adjoint bundle morphism $TM \to TM$ which, at any point $x \in M$, acts by

$$(17.14) \qquad T_x M \ni v \mapsto (\nabla df)v = \nabla_v (\nabla f) \in T_x M\,.$$

Lemma 17.11. *Let α be a parallel bivector field on a pseudo-Riemannian manifold (M,g), and let $f : M \to \mathbf{R}$ be a C^2 function such that α and the second covariant derivative ∇df commute as bundle morphisms $TM \to TM$. Then the vector field $w = \alpha(\nabla f)$ is a Killing field on (M,g), and its covariant derivative is the composite of α and ∇df:*

$$(17.15) \qquad \nabla w = \alpha(\nabla df) = (\nabla df)\alpha\,.$$

Proof. Since α is parallel, (17.14) gives $\nabla_v w = \alpha(\nabla_v(\nabla f)) = [\alpha(\nabla df)]v$ for any tangent vector v, which proves (17.15). On the other hand, the composite AB of two operators A, B in a Euclidean space, which satisfy $A^* = -A$, $B^* = B$ and $AB = BA$, is necessarily skew-adjoint, as $(AB)^* = (BA)^* = A^* B^* = -AB$. Hence, by (17.15), ∇w is skew-adjoint at every point, i.e., w is a Killing field, as required. ■

Proposition 17.12. *Let α be a parallel bivector field on a pseudo-Riemannian manifold (M,g). Any of the following five conditions then implies that α commutes, as a bundle morphism $TM \to TM$, with the covariant derivative ∇w of every Killing vector field w on (M,g):*

(a) *(M,g) admits no other parallel bivector fields except constant multiples of α, and α is non-null, i.e., $\langle \alpha, \alpha \rangle \neq 0$;*

(b) (M, g) *is four-dimensional, oriented and Riemannian,* α *is a section of* $\Lambda^+ M$*, and the only parallel sections of* $\Lambda^+ M$ *are constant multiples of* α;
(c) (M, g) *is four-dimensional, oriented and Riemannian,* α *is a section of* $\Lambda^+ M$*, and the scalar curvature* s *is nonzero everywhere*;
(d) (M, g, α) *is a nonflat space of constant holomorphic sectional curvature*;
(e) (M, g, α) *is a Riemannian Kähler manifold of real dimension* 4 *and its scalar curvature* s *is nonzero everywhere.*

Proof. We may assume that $n = \dim M \geq 2$. The commutator $\beta = [\alpha, \nabla w] = \alpha(\nabla w) - (\nabla w)\alpha$ is skew-adjoint (since so are α and ∇w) and, obviously, satisfies the anticommutator relation

$$\alpha\beta + \beta\alpha = [\alpha^2, \nabla w]\,. \tag{17.16}$$

Furthermore,

$$\langle \alpha, \beta \rangle = 0\,, \qquad \nabla\beta = 0\,. \tag{17.17}$$

In fact, the first equality is immediate from (17.16) since, by (2.17), $-4\langle \alpha, \beta \rangle = 2\,\mathrm{Trace}\,\alpha\beta = \mathrm{Trace}\,(\alpha\beta + \beta\alpha)$, which equals zero by (3.1). To show that $\nabla\beta = 0$, note that β has the twice-covariant components $\beta_{jk} = \alpha^s{}_k w_{s,j} - \alpha_j{}^s w_{k,s}$ (cf. (2.12)). Therefore, as $\nabla\alpha = 0$, we have $\beta_{jk,l} = \alpha^s{}_k w_{s,jl} - \alpha_j{}^s w_{k,sl}$. Consequently, (17.4) and (4.32) yield $\beta_{jk,l} = w^p[R_{sjlp}\alpha^s{}_k - R_{kslp}\alpha_j{}^s] = w^p[R_{plj}{}^s\alpha_{sk} + R_{plk}{}^s\alpha_{js}]$, which vanishes in view of (4.29) with $F = \alpha$.

The uniqueness assumption of (a), applied to β, now yields our assertion in view of (17.17).

On the other hand, in each of the remaining cases (b) – (e), α^2 is a multiple of Id (as a consequence of Lemma 6.1 or, respectively, (9.1)), and so (17.16) and (17.17) give

$$\alpha\beta + \beta\alpha = 0\,. \tag{17.18}$$

In case (b), (17.18) combined with Corollary 6.4 shows that β is a section of $\Lambda^+ M$, and so our assertion now follows from (17.17) and the uniqueness condition in (b).

Furthermore, either of assumptions (c), (e) implies (b) (and hence our assertion), as one sees using Proposition 9.8 or, respectively, Corollary 9.4.

Finally, in case (d), (17.18) along with Lemma 10.5(ii) shows that, unless $\beta = 0$, we would have $(n^2 - 4)\lambda = -(n+2)\lambda$ with $n = \dim M$ (since (5.20) applies to both α and β, cf. Lemma 10.5(i)). As $n = \dim M \geq 2$, this could happen only if $\lambda = 0$ which, via (10.10) and (10.5), would in turn imply that (M, g) is flat, contrary to (d). Thus, $\beta = 0$, which completes the proof. ∎

Remark 17.13. The assertion of Proposition 17.12 in cases (d), (e) can be rephrased as "every Killing vector field w in (M, g, α) is holomorphic". More precisely, any pseudo-Riemannian Kähler manifold (M, g, α) naturally constitutes a complex manifold (see Remark 23.5 in §23 below). On the other hand, given a vector field w and a bundle morphism $\alpha : TM \to TM$, both of class C^1, in a pseudo-Riemannian manifold (M, g), the commutator $\beta = [\alpha, \nabla w]$ is nothing else than the Lie derivative $\mathcal{L}_w\alpha$.

Example 17.14. Let (M,g) be the Riemannian product of two pseudo-Riemannian surfaces (Σ,h) and (Σ',h') with Gaussian curvatures λ and μ, such that the scalar curvature s of (M,g) is nonzero everywhere, i.e., $\lambda+\mu\neq 0$ (cf. (16.29)). Let us also fix a connected open subset U of M. We then have

$$\dim[\mathfrak{isom}(\Sigma,h)] + \dim[\mathfrak{isom}(\Sigma',h')] \le \dim[\mathfrak{isom}(U,g)] \le 6\,. \tag{17.19}$$

In fact, the first inequality is immediate from injectivity of the operator

$$\mathfrak{isom}(\Sigma,h)\times\mathfrak{isom}(\Sigma',h') \to \mathfrak{isom}(U,g) \tag{17.20}$$

that combines Killing fields w,w' on Σ, Σ' into a Killing field $w+w'$ on M (Example 17.2), which we then restrict to U (the latter step being injective by Remark 17.6(iv)). To establish the second inequality in (17.19), note that assumption (c) in Proposition 17.12 holds for (M,g) with either fixed orientation along with one or the other of the parallel bivector fields $\alpha^\pm$ defined, locally, as in (16.34) (since, by Lemma 6.2, $\alpha^\pm$ is a section of $\Lambda^\pm M$). The assertion of Proposition 17.12 then shows that, for any Killing field w defined on U, the skew-adjoint bundle morphism $\nabla w: TU\to TU$ leaves invariant the vector subbundles $\mathcal{P}=\mathrm{Ker}\,(\alpha^+-\alpha^-)$ and $\mathcal{Q}=\mathrm{Ker}\,(\alpha^++\alpha^-)$ of TU tangent to the factor surfaces. Due to ∇w-invariance of $\mathcal{P}$ and $\mathcal{Q}$, for each $x\in M$ the injective linear operator (17.5) takes values in the space $T_xM\times\mathfrak{so}(\mathcal{P}_x)\times\mathfrak{so}(\mathcal{Q}_x)$ of dimension at most six (with elements of $\mathfrak{so}(\mathcal{P}_x)$ acting trivially on $\mathcal{Q}_x$, and vice versa). Thus, (17.19) follows. Let us now suppose that, in addition, the factor surfaces have constant Gaussian curvatures and each of them is obtained as in Example 10.3 or 10.4. Then, for any fixed connected open subset U of M, we have

$$\dim[\mathfrak{isom}(U,g)] = 6\,, \tag{17.21}$$

and (17.20) is an isomorphism, i.e., every Killing field on U is of the form $w+w'$ as described above. This is immediate from (17.19), since, according to Examples 17.8 and 17.9, we have $\dim[\mathfrak{isom}(\Sigma,h)]=3$ for either factor surface (Σ,h).

Let V now be a finite-dimensional complex vector space endowed with a fixed sesquilinear Hermitian complex-valued form $\langle\,,\rangle$. We then denote $\mathfrak{u}(V)$ the (real) Lie subalgebra of $\mathfrak{gl}_{\mathbf{C}}(V)=\mathrm{Hom}_{\mathbf{C}}(V,V)$ consisting of all complex-linear operators $A:V\to V$ that are *skew-adjoint relative to* $\langle\,,\rangle$, i.e., satisfy (3.30), and use the symbol $\mathfrak{su}(V)$ for the ideal in $\mathfrak{u}(V)$ formed by all such A which, in addition, are (complex) traceless. We obviously have

$$\dim\mathfrak{u}(V) = \mathfrak{so}(V)\cap\mathfrak{gl}_{\mathbf{C}}(V) \subset \mathfrak{so}(V)\,, \tag{17.22}$$

the real-valued form needed to define $\mathfrak{so}(V)$ being $\mathrm{Re}\,\langle\,,\rangle$. If $\langle\,,\rangle$ is nondegenerate, we have, for reasons analogous to those in (3.31),

$$\dim\mathfrak{u}(V) = m^2\,,\qquad \dim\mathfrak{su}(V) = m^2-1\,,\qquad m=\dim_{\mathbf{C}}V\,. \tag{17.23}$$

The symbol $\mathfrak{u}(T_xM)$ also makes sense whenever x is a point in an almost Hermitian pseudo-Riemannian manifold (M,g,α). In fact, T_xM then is a complex vector space and carries the pseudo-Hermitian complex inner product $\langle\,,\rangle_{\mathbf{c}}$, the real part of which is $g(x)$. See Remark 3.18.

Corollary 17.15. *If a pseudo-Riemannian manifold* (M,g) *admits a non-null parallel bivector field* α *satisfying one of conditions* (d), (e) *in* Proposition 17.12*, then, for each* $x\in M$*, the injective linear operator* (17.5) *takes values in the subspace* $T_xM\times\mathfrak{u}(T_xM)$*, cf.* (17.22)*, and we have*

$$\dim\,[\mathfrak{isom}\,(M,g)]\,\le\,\frac{n(n+4)}{4}\,,\qquad n\,=\,\dim M\,. \tag{17.24}$$

Proof. For any Killing field w and any $x\in M$, we have $[\nabla w(x)]\in\mathfrak{u}(T_xM)$ in view of Proposition 17.12 and the equality in (17.22). Now (17.23) with $m=n/2$ gives (17.24), as required. ∎

We can now discuss the main reason why Killing fields are important for geometry. That reason lies in the relation between them and *local isometries* of the underlying manifold (M,g), that is, isometries between open submanifolds of (M,g). (By an *isometry* between pseudo-Riemannian manifolds (M,g) and (N,h) we mean here, as usual, any C^1 diffeomorphism $F:M\to N$ such that $F^*h=g$, cf. §2.)

Let us recall that the *flow* of a C^1 vector field w on a manifold M is the mapping $(t,x)\mapsto e^{tw}x\in M$ characterized in the paragraph following formula (2.31) in §2. Also, note that, according to (17.2), a C^1 vector field w in a pseudo-Riemannian manifold (M,g) is a Killing field if and only if

$$\mathcal{L}_w g\,=\,0\,, \tag{17.25}$$

where $\mathcal{L}_w g$ denotes the symmetric twice-covariant tensor field with the local components

$$[\mathcal{L}_w g]_{jk}\,=\,w_{j,k}\,+\,w_{k,j}\,. \tag{17.26}$$

It is also worth noting that, for a Killing field v and any C^1 vector fields u,w we have

$$d_v\langle u,w\rangle\,=\,\langle[v,u],w\rangle\,+\,\langle[v,w],u\rangle\,. \tag{17.27}$$

(This means that the *Lie derivative* given by $\mathcal{L}_v u=[v,u]$ and $\mathcal{L}_v f=d_v f$ for vector fields u and functions f satisfies, for Killing fields v, the Leibniz rule $\langle\mathcal{L}_v u,w\rangle+\langle u,\mathcal{L}_v w\rangle=0$.) In fact, by (4.4) and skew-adjointness of ∇v, $\langle[v,u],w\rangle+\langle[v,w],u\rangle=\langle\nabla_v u-\nabla_u v,w\rangle+\langle\nabla_v w-\nabla_w v,u\rangle=\langle\nabla_v u,w\rangle+\langle u,\nabla_v w\rangle=d_v\langle u,w\rangle$, as required.

Lemma 17.16. *A* C^2 *vector field* w *on a pseudo-Riemannian manifold* (M,g) *is a Killing field if and only if its flow consists of local isometries of* (M,g)*.*

Proof. Set $\phi(t,y)=\phi_t(y)=e^{tw}y$ for (t,y) in a suitable open subset of $\mathbf{R}\times M$. Our assertion is an obvious consequence of the relation

$$\frac{d}{dt}\,[\phi_t^*g]\,=\,\phi_t^*\,[\mathcal{L}_w g] \tag{17.28}$$

with $\mathcal{L}_w g$ given by (17.26). To prove (17.28), let us rewrite it in local coordinates, using (2.30):

$$\frac{d}{dt}\,\left[g_{jk}(\phi)\phi_a^j\phi_b^k\right]\,=\,[w_{j,k}(\phi)+w_{k,j}(\phi)]\,\phi_a^j\phi_b^k\,, \tag{17.29}$$

where, for any fixed $z \in M$ and $\tau \in \mathbf{R}$, y^a and x^j are local coordinates for M defined near z and $e^{\tau w}z$, respectively, while t varies near τ, $\phi_a^j = \partial\phi^j/\partial y^a$, and the composites $g_{jk}(\phi)$, $w_{j,k}(\phi)$ depend on t via $\phi(t,y)$. Thus, $w^j(\phi) = \partial\phi^j/\partial t$, and so, by the chain rule, $\partial\phi_a^j/\partial t = \partial\,[w^j(\phi)]/\partial y^a = \phi_a^k(\partial_k w^j)(\phi)$, while, again from the chain rule and (4.1), we have $d\,[g_{jk}(\phi)]/dt = w^l(\phi)(\partial_l g_{jk})(\phi) = (w^l\Gamma_{lj}^s g_{sk} + w^l\Gamma_{lk}^s g_{js})(\phi)$. Since, by (4.12), $w_{j,k} = g_{jl}(\partial_k w^l + \Gamma_{ks}^l w^s)$, equality (17.29) now follows from the product rule and symmetry of Γ_{jk}^l in j,k. ■

Every C^1 vector field w on a compact manifold M is complete (in the sense described in Remark 2.3), i.e., gives rise to a flow homomorphism (2.33). As a consequence of Lemma 17.16, in the case of a Killing field w on a compact Riemannian manifold (M,g), (2.33) is a group homomorphism

$$\mathbf{R} \ni t \mapsto e^{tw} \in \operatorname{Isom}(M,g)\,, \tag{17.30}$$

valued in the group $\operatorname{Isom}(M,g)$ of all isometries of (M,g) onto itself.

Example 17.17. Let (M^c,g^c) be a nonflat pseudo-Riemannian space of constant holomorphic sectional curvature obtained as in Example 10.6 using a complex vector space V with a pseudo-Hermitian complex inner product $\langle\,,\rangle$ and a real number $c \neq 0$, and let S_c be the (nonempty) pseudosphere $S_c = \{v \in V : \langle v,v\rangle = c\}$. Every traceless skew-adjoint complex-linear operator $A : V \to V$, restricted to S_c, is a tangent vector field w_A on S_c (Example 17.8, with the real inner product $\operatorname{Re}\langle\,,\rangle$). Since $A(zx) = zAx$ for all $x \in V$ and $z \in \mathbf{C}$, both w_A and its component orthogonal to the S^1 orbits in S_c is invariant under the multiplicative action on S_c of the circle S^1 of unit complex numbers. As a result, w_A is projectable onto a vector field $w_{(A)}$ tangent to $M^c = S_c/S^1$. Furthermore, $w_{(A)}$ is a Killing field in (M^c,g^c). To see this, note that the flow of the linear vector field A in V obviously coincides with the one-parameter family $t \mapsto e^{tA}$ of linear operators given by $e^{tA} = \sum_{r=0}^{\infty}[(tA)^n/n!]$. Restricted to S_c these operators are isometries, commuting with the S^1 action and forming the flow of w_A; therefore, they descend to M^c so as to become a one-parameter family of isometries, which obviously constitute the flow of $w_{(A)}$, and so $w_{(A)}$ is a Killing field in view of Lemma 17.16. Let us now fix any connected open subset U of M^c. Restricting each $w_{(A)}$ to U, we thus obtain a linear operator

$$\mathfrak{su}(V) \ni A \mapsto w_{(A)} \in \mathfrak{isom}(U,g^c)\,, \tag{17.31}$$

which is injective. (In fact, if $w_{(A)} = 0$ on U, then every vector in V is an eigenvector of A, so that A is a multiple of Id and, being traceless, it has to be zero.) The $w_{(A)}$ thus obtained now form a subspace $\mathcal{W}$ of $\mathfrak{isom}(U,g^c)$, namely, the image of (17.31), which, by (17.23), satisfies $\dim\mathcal{W} \geq n(n+4)/4$, where n is the real dimension of M^c (so that $m = (n/2)+1$ is the complex dimension of V). Thus, by (17.24), $\mathcal{W} = \mathfrak{isom}(U,g^c)$ and, for any connected open subset U of M^c,

$$\dim\,[\mathfrak{isom}(U,g^c)] \,=\, n(n+4)/4\,, \qquad n = \dim M^c\,. \tag{17.32}$$

Proposition 17.18. *Let (M,g) denote a Riemannian manifold of dimension n which is either obtained by one of the constructions described in* Examples 10.3, 10.4, 10.6*, or the Riemannian product of two such manifolds, and let us set $\mathfrak{m} = \dim[\mathfrak{isom}(U,g)]$, where U is any fixed connected open subset of M.*

(i) *If (M,g) is a space of constant curvature, then $\mathfrak{m} = n(n+1)/2$;*
(ii) *If (M,g) is a nonflat space of constant holomorphic sectional curvature, we have $\mathfrak{m} = n(n+4)/4$;*
(iii) *If $n=4$ and (M,g) is the Riemannian product of two Riemannian surfaces with equal nonzero constant curvatures, both obtained as in* Example 10.4*, then $\mathfrak{m} = 6$.*

In fact, this is nothing else than the dimension formulae (17.9), (17.11), (17.21) and (17.32). ∎

We will say that a pseudo-Riemannian manifold (M,g) is *infinitesimally homogeneous* if for every $x \in M$ we have $T_xM = \{v(x) : v \in \mathfrak{g}\}$, with $\mathfrak{g} = \mathfrak{isom}(M,g)$.

Example 17.19. Every locally symmetric Riemannian Einstein 4-manifold is infinitesimally homogeneous, as one easily verifies using the description of Killing fields in Examples 17.8, 17.9 and 17.14 along with the fact that these examples, locally, represent all possible isometry types of the manifolds in question (Theorem 14.7). For the same reason, in all dimensions, spaces of constant curvature, as well spaces of constant holomorphic sectional curvature, are all infinitesimally homogeneous. (Here we may use the classifications provided by Theorems 14.2 and 14.4.)

Lemma 17.20. *Any infinitesimally homogeneous pseudo-Riemannian manifold (M,g) is also locally homogeneous.*

Proof. Let $n = \dim M$. Given $x \in M$, let w_j, $j = 1,\ldots,n$, be Killing fields on a connected neighborhood U of x such that the $w_j(x)$ form a basis of T_xM, and let $\mathfrak{h}$ be the subspace of $\mathfrak{isom}(U,g)$ spanned by the w_j. The assignment $w \mapsto e^w x \in M$ (notation as in Remark 2.4) is defined on a neighborhood of 0 in $\mathfrak{h}$ and, in view of the inverse mapping theorem, it sends some neighborhood of 0 in $\mathfrak{h}$ diffeomorphically onto a neighborhood U' of x in M. Since the e^w are local isometries of (M,g) (Lemma 17.16), we have $x \sim y$ for all $y \in U'$, where $\sim$ is the equivalence relation introduced in Remark 2.1. The equivalence classes of $\sim$ thus are all open and hence all closed, so that, as M is connected (by definition), there is only one such class. This completes the proof. ∎

Corollary 17.21. *Every locally symmetric Riemannian Einstein manifold of dimension four is locally homogeneous.*

This is clear from Example 17.19 and Lemma 17.20. ∎

Remark 17.22. The assertion of Corollary 17.21 is well-known to be valid in all dimensions and sign patterns; cf. also Remark 42.7. (This is usually proved using the characterization of locally symmetric metrics via geodesic symmetries; see, e.g., Helgason, 1978.)

Remark 17.23. Let an n-dimensional Lie algebra $\mathcal{X}$ of C^∞ vector fields on an n-dimensional manifold M satisfy conditions (a), (b) in Example 12.6, and let $\langle\,,\rangle$ be a (possibly indefinite) inner product in the underlying vector space of $\mathcal{X}$.

A pseudo-Riemannian metric g on M now can be defined by requiring that, for any $u,v\in\mathcal{X}$, $g(u,v)$ be the constant function $\langle u,v\rangle$. (Given a basis e_j of $\mathcal{X}$, $j=1,\dots,n$, this amounts to setting $g(e_j,e_k)=g_{jk}$ for all j,k, with any fixed nonsingular symmetric matrix $[g_{jk}]$ of constants.) The resulting pseudo-Riemannian manifold (M,g) then is infinitesimally homogeneous, and hence (Lemma 17.20) also locally homogeneous. In fact, let $x\in M$ and $u\in T_xM$. According to assertions (i), (ii) in Example 12.6 and Lemma 11.2, there exists a C^∞ vector field w on a neighborhood U of x such that $w(x)=u$ and w commutes, on U, with every $v\in\mathcal{X}$. Thus, by (4.4) and (4.5), $2g(\nabla_v w,v)=2g(\nabla_w v,v)=d_w\langle v,v\rangle=0$ whenever $v\in\mathcal{X}$, and since such v realize every tangent vector at any point, (17.1) shows that w is a Killing field with the arbitrarily prescribed value $w(x)=u$, as required.

Metrics g as above are well-known to be, locally, nothing else than left-invariant metrics on n-dimensional Lie groups.

Remark 17.24. Property (i) of Killing fields in Remark 17.6 is closely related to the fact that, given pseudo-Riemannian manifolds (M,g) and (N,h), points $x\in M$, $y\in N$ and a linear operator $A:T_xM\to T_yN$, there can exist *at most one isometry* $F:M\to N$ with $F(x)=y$ and $dF_x=A$. In other words, for any fixed $z\in M$, an isometry F is uniquely determined by the initial data $F(z)$ and dF_z. In fact, since F sends geodesics onto geodesics, these data at any given point z determine what F is at every point that can be joined to z by a geodesic in (M,g). Thus, for two isometries $F,F':M\to N$, the set U of all points $z\in M$ with $F(z)=F'(z)$ and $dF_z=dF'_z$ is both closed and open in M, so that it must coincide with M or be empty.

Remark 17.25. Given a pseudo-Riemannian manifold (M,g), let us define the vector bundle $\mathcal{E}$ over M to be the direct sum $\mathcal{E}=TM\oplus\mathfrak{so}(TM)$ of the tangent bundle TM and the subbundle $\mathfrak{so}(TM)$ of $\mathrm{Hom}\,(TM,TM)$ with the fibres $\mathfrak{so}(T_xM)$, $x\in M$, defined as in §3 (see the paragraph preceding formula (3.31)). Sections of $\mathcal{E}$ thus are pairs (u,F) consisting of a vector field u on M and a bundle morphism $F:TM\to TM$, skew-adjoint at every point. Formula

$$\mathrm{D}_v(u,F)\,=\,(\nabla_v u\,-\,Fv\,,\;\nabla_v F\,+\,R(v,u))\,,\tag{17.33}$$

with (u,F) and v tangent to M, obviously defines a connection D in $\mathcal{E}$, such that the D-parallel sections are precisely those (u,F) formed by a Killing field u in (M,g) with $F=\nabla u$. (See (17.2) and (17.4).) Using formula (4.52) (with the simplifications described in Remark 4.4), along with (4.27) and the Bianchi identities (4.33) and (5.1), we now easily verify that the curvature tensor R^{D} of D can be expressed as

$$\begin{aligned}&R^{\mathrm{D}}(v,w)(u,F)\\&=\,(0\,,\;(\nabla_u R)(v,w)\,+\,[F,R(v,w)]\,+\,R(Fv,w)\,+\,R(v,Fw))\,.\end{aligned}\tag{17.34}$$

One might combine this formula with Lemma 11.2 trying to prove a local existence theorem for Killing fields; however, flatness of D in the bundle $\mathcal{E}$ as described here characterizes the case where (M,g) is a space of constant curvature, which is far too special to be of much interest. Instead, we may (in a suitable situation) try to find a vector subbundle $\mathcal{C}$ of $\mathcal{E}$ such that

(a) $\mathcal{C}$ is D-parallel, as defined in Remark 4.7, and
(b) $R^{\mathrm{D}}(v,w)(u,F)=0$ for (u,F) in $\mathcal{C}$ and any v,w tangent to M.

In fact, by (a), D has a natural restriction to a connection in $\mathcal{C}$ and, by (b) along with (4.52), that restriction is flat. We thus are free to use Lemma 11.2 to obtain existence of specific (local) Killing fields.

Using this above argument, we obtain the following result, some applications of which are described in Remark 42.6 (§42).

Proposition 17.26. *Let (M,g) be a locally symmetric pseudo-Riemannian manifold, and let $R([TM]^{\wedge 2})$ denote the subbundle of the bivector bundle $[TM]^{\wedge 2}$ whose fibre $R([T_xM]^{\wedge 2})$ over each $x \in M$ is the image of the curvature tensor R of (M,g) acting on bivectors at x. Suppose that for some, or any, point $x \in M$, any two elements of $R([T_xM]^{\wedge 2})$ commute when treated, with the aid of g, as a skew-adjoint operators $T_xM \to T_xM$. Furthermore, let $\mathcal{C}_x$ be the* centralizer *of $R([T_xM]^{\wedge 2})$ in $[T_xM]^{\wedge 2}$ at any $x \in M$, that is, the set of all $F \in [T_xM]^{\wedge 2} = \mathfrak{so}(T_xM)$ which commute with all elements of $R([T_xM]^{\wedge 2})$. Then every point of M has a connected neighborhood U such that, for any $x \in U$, any $v \in T_xM$, and any $F \in \mathcal{C}_x$, there exists a unique Killing vector field $w \in \mathfrak{isom}(M,g)$, defined on U, with $w(x) = v$ and $[\nabla w](x) = F$. In particular, (M,g) is infinitesimally homogeneous and, consequently, locally homogeneous.*

Proof. Since R is parallel, the $\mathcal{C}_x$ are the fibres of a parallel subbundle $\mathcal{C}$ of $[TM]^{\wedge 2}$ (see Remark 4.7). The restriction to $\mathcal{C}$ of the connection (17.33) is flat in view of (17.34). To see this, note that $\nabla_u R)(v,w) = 0$ as $\nabla R = 0$, while $[F, R(v,w)] = 0$ since the F in question commute with the image of R. Finally, for any tangent vectors u, u', $g(R(Fv,w)u + R(v,Fw)u, u') = g(R(u,u')Fv, w) + g(R(u,u')v, Fw) = g(R(u,u')[Fv] - F[R(u,u')v], w)$ (due to the identity $R_{jklm} = R_{lmjk}$ in (4.32) and skew-adjointness of F) which equals $g([R(u,u'),F]v, w)$, and hence is zero since F commutes with all $R(u,u')$. Our assertion is now immediate from Lemma 11.2 (along with Lemma 17.20), which completes the proof. ■

§18. Extremal metrics on surfaces

Most results presented here go back at least five decades, and seem to have been independently discovered by various authors (see Petrov, 1969, pp. 348 – 350). The only newer result is Theorem 18.14, due to Calabi (1982).

We will say that a pseudo-Riemannian metric g is *locally conformally Einstein* if every point of the underlying manifold has a neighborhood U with a C^∞ function $f : U \to \mathbf{R}$ such that the metric $\tilde{g} = e^{2f} g$ is Einstein.

In this section (and, later, in §19 and §22) we will study metrics in dimension four that are locally conformally Einstein and at the same time belong to other special classes (such as product or Kähler metrics). We begin here with products of surfaces. More precisely, our goal is to provide a local classification, at "generic" points, of those pseudo-Riemannian Einstein metrics in dimension four that are locally conformally related to Riemannian products of two surface metrics. (See the paragraph following Remark 18.10.) It will be useful to introduce, after the following general remark, the class of *extremal surface metrics*, which turn out to be the possible factors in such products.

Remark 18.1. Here and in sections 19 and 22 we deal with the question of classifying metrics of some specific "class X" that are also locally conformally Einstein. Such a question can be approached from two "ends". One, the *class X end*, consists in asking which class X metrics are in fact locally conformally Einstein; an answer to this part of the problem will lead to a specific construction of Einstein metrics. The other, *Einstein-metric end* centers around characterizing those Einstein metrics that are conformally related to class X metrics. A solution to the second problem will, if nothing else, enhance our understanding of Einstein metrics in general.

It turns out that in all three instances of different "classes X" a common pattern emerges: The Einstein metrics conformal to class X metrics turn out to have a very simple and natural characterization, while a description of class X metrics conformal to Einstein metrics turns out much less satisfactory, i.e., more difficult to understand.

Let (Σ, h) be a pseudo-Riemannian surface. Following Calabi (1982), we will say that the metric h is *extremal* if its Gaussian curvature function κ has the property that $\nabla d\kappa = \sigma h$ for some function σ. (See also Remark 18.8 below.) Contracting both sides, we then see that $2\sigma = \Delta\kappa$, i.e., extremal surface metrics h are characterized by

$$\nabla d\kappa = \sigma h\,, \qquad 2\sigma = \Delta\kappa \tag{18.1}$$

or, in local coordinates x^j,

$$\kappa_{,jl} = \sigma h_{jl}\,, \qquad 2\sigma = \kappa_{,j}{}^{j}\,. \tag{18.2}$$

We define the *classifying parameters* of an extremal pseudo-Riemannian metric h on a surface Σ to be the *first parameter* c and the *second parameter* p with

$$c = \Delta\kappa + \kappa^2\,, \qquad p = c\kappa - h(\nabla\kappa, \nabla\kappa) - \kappa^3/3\,. \tag{18.3}$$

Thus, by (18.1) and (18.3),

$$2\,\nabla d\kappa = (c - \kappa^2)\,h\,, \tag{18.4}$$

and

$$\Delta\kappa = c - \kappa^2\,, \qquad h(\nabla\kappa, \nabla\kappa) = c\kappa - p - \kappa^3/3\,. \tag{18.5}$$

Example 18.2. Any surface metric having a constant Gaussian curvature κ is obviously extremal, with the classifying parameters $c = \kappa^2$ and $p = 2\kappa^3/3$.

Lemma 18.3. *For any extremal pseudo-Riemannian metric h on a surface Σ, the classifying parameters c and p with* (18.3) *are constant as functions on Σ.*

Proof. In view of the contracted Ricci-Weitzenböck formula (4.39) and the relation $R_{jl} = \kappa h_{jl}$ (see Remark 10.1), we have $c_{,j} = (\Delta\kappa + \kappa^2)_{,j} = \kappa_{,s}{}^{s}{}_{j} + 2R_j{}^s\kappa_{,s} = \kappa_{,s}{}^{s}{}_{j} + 2(\kappa_{,sj}{}^{s} - \kappa_{,s}{}^{s}{}_{j}) = 2\kappa_{,sj}{}^{s} - \kappa_{,s}{}^{s}{}_{j}$. However, taking the divergences of both sides of (18.1), (i.e., applying ∇^l to (18.2)), we obtain $2\kappa_{,sj}{}^{s} = (\Delta\kappa)_{,j} = \kappa_{,s}{}^{s}{}_{j}$. Thus, c is constant. On the other hand, both sides of (18.1) may be treated as bundle morphisms $T\Sigma \to T\Sigma$ (with g corresponding to the identity). Applying them to $\nabla\kappa$ and using (18.3), we obtain $0 = (\Delta\kappa)_{,j} - 2\kappa_{,j}{}^{s}\kappa_{,s} = (c - \kappa^2)\kappa_{,j} - 2\kappa_{,j}{}^{s}\kappa_{,s}$, i.e., p in (18.3) is constant. This completes the proof. ■

Lemma 18.4. *Let* (M,g) *be a pseudo-Riemannian* 4*-manifold, obtained as the Riemannian product of two pseudo-Riemannian surfaces, and such that the scalar curvature* s *of* g *is nonzero everywhere. Then, the following two conditions are equivalent*:

(i) g *is locally conformally Einstein*;
(ii) *Both factor-surface metrics are extremal and have the same first classifying parameter* c*, defined as in* (18.3)*.*

Furthermore, if (i) *or* (ii) *is satisfied, then an Einstein metric* $\tilde{g}$ *conformally related to* g *is unique up to a constant factor and, up to a factor, must be given by*

$$\tilde{g} \,=\, 4\,g/\mathrm{s}^2 \,=\, g/(\lambda+\mu)^2\,, \tag{18.6}$$

and the scalar curvature $\tilde{\mathrm{s}}$ *of the metric* (18.6) *then equals*

$$\tilde{\mathrm{s}} \,=\, 3\,(p+q)\,, \tag{18.7}$$

where λ *and* μ *are the Gaussian curvatures of the factor surfaces, while* p *and* q *stand for their second classifying parameters.*

Proof. Assume (i) and let $\tilde{g} = g/\varphi^2$ be an Einstein metric. According to Lemma 5.2 and Remark 16.9, we can now apply Lemma 16.8, concluding that, φ is a constant multiple of $\mathrm{s} = 2(\lambda+\mu)$, where λ, μ are the Gaussian curvatures of the factor surfaces (cf. the last equality in (16.29)). Combining (16.14) (for $n=4$) with (16.28), in product coordinates x^j, y^a, we now see that, on each factor surface, the second covariant derivative of the Gaussian curvature must be equal to a function times the metric. In other words, both factor metrics then are extremal. Relations (18.4) and (18.5) for $\kappa=\lambda$ and $\kappa=\mu$ (with, possibly, different pairs (c,p) for the two factors), substituted into (16.14) with $\varphi = \mathrm{s}/2 = \lambda+\mu$, along with the requirements that $\tilde{R}_{jk} = \tilde{\kappa}\tilde{g}_{jk}$, $\tilde{R}_{ab} = \tilde{\kappa}\tilde{g}_{ab}$, (with the same $\tilde{\kappa}$ in both, cf. (5.3)), now easily imply equality between the first classifying parameters of both factor metrics and relation (18.7) (as $\tilde{\mathrm{s}} = 4\tilde{\kappa}$). This proves (ii) and the last part of our assertion.

Conversely, let us assume (ii). Using (16.28) and (16.14), in product coordinates (where $n=4$, $\varphi = \lambda+\mu$, and λ, μ again stand for the Gaussian curvatures of the factor surfaces), along with (18.4) and (18.5) (for $\kappa=\lambda$ or $\kappa=\mu$), we obtain $\widetilde{\mathrm{Ric}} = \tilde{\mathrm{s}}\,\tilde{g}/4$, with $\tilde{\mathrm{s}}$ given by (18.7). This completes the proof. ■

Remark 18.5. Lemma 18.4 offers a simple construction leading to Einstein metrics on products of surfaces. One might be tempted to try it for *compact* surfaces, with the purpose of finding easy new examples of compact Riemannian Einstein four-manifolds. Any such hopes are, however, quickly dashed by a theorem of Calabi (1982), presented at the end of this section.

Remark 18.6. Any oriented pseudo-Riemannian surface (Σ,h) carries a natural nonzero parallel bivector field α given by $\alpha(x) = e_1\wedge e_2$ for any positive-oriented orthonormal basis e_1, e_2 of $T_x\Sigma$. (See (3.34), where α is denoted vol.) Regarded as a skew-adjoint bundle morphism $T\Sigma \to T\Sigma$ (see (2.12), (2.19)), α then satisfies

$$\alpha e_1 \,=\, \varepsilon_1 e_2\,, \quad \alpha e_2 \,=\, -\,\varepsilon_2 e_1\,, \quad \text{with} \quad \varepsilon_j \,=\, h(e_j,e_j)\,, \tag{18.8}$$

and

$$\alpha^2 = -\varepsilon\cdot\mathrm{Id}\,, \tag{18.9}$$

where the number $\varepsilon = \pm 1$ is the *sign factor* of the pseudo-Riemannian surface metric h, defined to be 1 when h is positive or negative definite and to -1 when it is indefinite. Also, for vectors v, w tangent to Σ,

$$h(\alpha v, \alpha w) = \varepsilon\, h(v, w)\,, \tag{18.10}$$

and

$$v\wedge(\alpha v) = h(v, v)\,\alpha\,. \tag{18.11}$$

In fact, (18.8) is immediate from (2.22), and it in turn implies (18.9) as $\varepsilon_1\varepsilon_2 = \varepsilon$. Now (18.10) follows from (18.9) and skew-symmetry of α, while (18.11) can be easily verified by writing v as a combination of e_1 and e_2.

Remark 18.7. Note that, according to (18.9), any oriented *Riemannian* surface (M, g), along with parallel bivector field α introduced in Remark 18.6, forms a Kähler manifold (M, g, α).

Remark 18.8. Let h be an extremal pseudo-Riemannian metric on an oriented surface Σ, and let α be the parallel bivector field introduced in Remark 18.6. Since $\nabla d\kappa$, treated as a bundle morphism $T\Sigma \to T\Sigma$, equals a function times the identity, it commutes with α. As a consequence of Lemma 17.11 and (18.4), formula $w = \alpha(\nabla\kappa)$ then defines a Killing vector field w on (Σ, h). Furthermore, by (17.15), α also commutes with ∇w. If, in addition, h is Riemannian, the last property, according to Remark 17.13, can be rephrased as saying that w (and hence also $\nabla\kappa$) is a holomorphic vector field. This amounts to a special case (for oriented Riemannian surfaces) of the original definition of extremal Kähler metrics introduced by Calabi (1982).

The following lemma amounts to a local classification of extremal surface metrics; see also Remark 18.10 below.

Lemma 18.9. *Let* κ, c, p *and* $\varepsilon = \pm 1$ *be the Gaussian curvature function, the classifying parameters* (18.3) *and the sign factor, described in* Remark 18.6*, of an extremal pseudo-Riemannian metric* h *on a surface* Σ*, and let* Ψ *be the cubic polynomial*

$$\Psi(\kappa) = c\kappa - p - \kappa^3/3 \tag{18.12}$$

in the variable κ*. In a neighborhood of any point at which* $\Psi(\kappa) \neq 0$*, there exists a local coordinate system* θ, κ *in* Σ *such that* κ *is the second coordinate function and the metric* h *is given by*

$$h = \varepsilon\,\Psi(\kappa)\,d\theta^2 + \frac{1}{\Psi(\kappa)}\,d\kappa^2\,. \tag{18.13}$$

Conversely, given real numbers c, p, ε *with* $\varepsilon = \pm 1$*, let* h *be the metric defined by* (18.13)*, with* $\Psi(\kappa)$ *as in* (18.12)*, on any connected component of the open subset* *of* $\mathbf{R}^2$ *given by* $\varPsi(\kappa) \neq 0$ *in the Cartesian coordinates* θ, κ. *Then* h *is an extremal pseudo-Riemannian surface metric with the sign factor* ε *and the classifying parameters* c *and* p, *while the Gaussian curvature of* h *coincides with the coordinate function* κ.

Proof. Let h be an extremal pseudo-Riemannian surface metric with κ, ε, c and p as in the initial clause of the lemma, and let α be the parallel bivector field, described in Remark 18.6, on a neighborhood of any given point with a fixed orientation. The 1-form ξ with

$$\xi = \frac{\alpha(\nabla\kappa)}{\varPsi(\kappa)}, \qquad \text{i.e.,} \qquad \xi_j = \frac{1}{\varPsi(\kappa)}\,\kappa_{,s}\alpha^s{}_j \tag{18.14}$$

is closed, that is, $d\xi = 0$. In fact, using (2.16) and (18.4) we obtain $[\varPsi(\kappa)]^2 d\xi = \varPsi(\kappa)\alpha - (\nabla\kappa) \wedge [\alpha(\nabla\kappa)]$, which is zero in view of (18.11) along with (18.12) and (18.5). By Corollary 11.2 we have, locally, $\xi = d\theta$ for some function θ. Relations (18.12), (18.5), (18.14) and (18.10) now imply

$$h(\nabla\kappa, \nabla\kappa) = \varPsi(\kappa), \quad \varepsilon\, h(\nabla\theta, \nabla\theta) = 1/\varPsi(\kappa), \quad h(\nabla\theta, \nabla\kappa) = 0\,. \tag{18.15}$$

(Note that $\nabla\theta$ and $\nabla\kappa$ are orthogonal due to (18.14) and skew-symmetry of α.) Thus, locally, in the set $\varPsi(\kappa) \neq 0$, κ and θ form a coordinate system. In general, for functions f, coordinates x^j, and any pseudo-Riemannian metric g, we have $(\nabla f)^j = g^{jk}\,\partial_k f$. Applying this to coordinate functions themselves, we obtain $g(\nabla x^j, \nabla x^k) = g^{jk}$, with g^{jk} as in (2.8). In our case, with $g = h$ and $x^1 = \theta$, $x^2 = \kappa$, this gives, by (18.15), $h^{11} = \varepsilon/\varPsi(\kappa)$, $h^{22} = \varPsi(\kappa)$ and $h^{12} = 0$, so that $h_{11} = \varepsilon\,\varPsi(\kappa)$, $h_{22} = 1/\varPsi(\kappa)$ and $h_{12} = 0$, which proves (18.13).

Conversely, let h denote the metric (18.13) in a region of $\mathbf{R}^2$ with the Cartesian coordinates θ, κ satisfying $\varPsi(\kappa) \neq 0$ (with $\varPsi(\kappa)$ given by (18.12)), and let $t = t(\kappa)$ be any function of the variable κ such that $dt/d\kappa = |\varPsi(\kappa)|^{-1/2}$. Setting $\delta = \operatorname{sgn}[\varPsi(\kappa)] = \pm 1$, we can rewrite (18.13) as

$$h = \varepsilon\delta\,\varPhi^2 d\theta^2 + \delta\, dt^2\,, \qquad \varPhi = |\varPsi(\kappa)|^{1/2}\,. \tag{18.16}$$

Hence, in the new coordinates $x^1 = \theta$, $x^2 = t$, we have $\partial\kappa/\partial\theta = 0$ and, with $(\;)^{\dot{}} = d/dt$ or $(\;)^{\dot{}} = \partial/\partial t$,

$$\dot\kappa = \varPhi > 0\,, \quad \delta\dot\kappa^2 = \varPsi(\kappa) = c\kappa - p - \kappa^3/3\,, \quad 2\delta\ddot\kappa = c - \kappa^2\,. \tag{18.17}$$

In fact, the first two relations are obvious from our choice of t and (18.16), while applying d/dt to the second relation we obtain $2\delta\dot\kappa\ddot\kappa = (c - \kappa^2)\dot\kappa$, and the third one follows. These equalities now give $2\ddot\varPhi = 2\,\dddot\kappa = \delta(c - \kappa^2)^{\dot{}} = -\,2\kappa\varPhi$, and so

$$\ddot\varPhi = -\,\delta\kappa\varPhi\,. \tag{18.18}$$

(We treat κ here as a function of the coordinate t, without assuming that it coincides with the Gaussian curvature of h.) Let us now define the vector fields u, v, w to be the coordinate fields $w = e_1$ and $u = e_2$ in the directions of, respectively, θ and t, and the unit vector field $v = |h(w,w)|^{-1/2}w = w/\varPhi$ be obtained by normalizing w. Since, by (18.16), the components of h do not depend on θ, the coordinate vector field $w = e_1$ is a Killing field (see Example 17.1). Furthermore,

$$h(u,u) = \delta\,, \quad h(u,w) = h(u,v) = 0\,, \quad \nabla_u u = \nabla_u v = 0\,, \tag{18.19}$$

i.e., the coordinate vector field u is unit, orthogonal to w (and v), its integral curves $t \mapsto x(t)$ (given by $x^1(t) = \theta$, $x^2(t) = t$ with any fixed value of θ) are geodesics, and v is parallel along each of them. To see this, note that the first two equalities amount to the metric-component formulae $h_{22} = \delta$, $h_{12} = 0$ (immediate from (18.16)), while $\nabla_u u$ must vanish since it is orthogonal both to u (as u is unit) and w (as $h(\nabla_u u, w) = -\,h(u, \nabla_u w) = 0$ due to the fact that $h(u,w) = 0$ and ∇w is skew-adjoint at every point, i.e., w is a Killing field). Finally, $\nabla_u v = 0$ since $\nabla_u v$ is orthogonal to v (note that v is unit) and $h(\nabla_u v, u) = -\,h(v, \nabla_u u) = 0$ as u, v are orthogonal and $\nabla_u u = 0$.

Using the symbol K for the Gaussian curvature of h, we now have

$$\nabla_u \nabla_u w = -\,\delta K w\,. \tag{18.20}$$

In fact, according to Remark 17.5, the restriction of the Killing field w to any integral curve of u satisfies the Jacobi equation (4.51), so that $\nabla_u \nabla_u w = R(w,u)u$, which in turn equals $-\,\delta K w$ in view of formula (10.1) for $g = h$ (cf. Remark 10.1) along with (18.20). However, since $w = \Phi v$ and v is parallel along u (by (18.19)), equality (18.20) becomes $-\,\delta K \Phi v = \nabla_u \nabla_u (\Phi v) = \ddot{\Phi} v$ which, in view of (18.18), equals $-\,\delta \kappa \Phi v$. Thus, $K = \kappa$.

To show that $\nabla d\kappa$ is a functional multiple of h, note that, according to Remark 18.6, formula $\alpha = u \wedge v$ defines a nonzero parallel bivector field. On the other hand, $\nabla \kappa = \delta \Phi u$. In fact, $\nabla \kappa = f u$ for some function f, since $h(\nabla \kappa, w) = d_w \kappa = \partial \kappa / \partial \theta = 0$, while (cf. (18.19)), $\delta f = f h(u,u) = h(\nabla \kappa, u) = d_u \kappa = \dot{\kappa} = \Phi$ in view of (18.17). Since, by (18.8), $\alpha u = \delta v$, this implies $\alpha \nabla \kappa = \Phi v = w$ and, as α is parallel, $\nabla w = \alpha(\nabla d\kappa)$ (notation as in (17.15)). Since w is a Killing field, ∇w is skew-adjoint at every point, and so, $\nabla w = \sigma \alpha$ for some function σ. Now, by (18.9), $\varepsilon \sigma h = \alpha \nabla w = \alpha^2 (\nabla d\kappa) = \varepsilon\, \nabla d\kappa$, and (18.1) follows. This completes the proof. ■

Remark 18.10. Lemma 18.9, along with Theorem 14.2, provides a complete local classification, up to an isometry, of extremal pseudo-Riemannian metrics h on surfaces Σ, valid at all points of an open dense subset U of Σ. Specifically, given an extremal metric h on Σ, we may define U to be the set of all points $x \in \Sigma$ such that either

(a) $h(\nabla \kappa, \nabla \kappa) = 0$ identically in a neighborhood of x, or
(b) $h(\nabla \kappa, \nabla \kappa) \neq 0$ at x.

Note that U then is dense in Σ, since it obviously intersects every nonempty open subset of Σ. To see that the results mentioned actually yield a local classification of h, let us fix any $x \in U$. In case (b), $\Psi(\kappa) \neq 0$ at x (due to (18.5) and (18.12)), and so Lemma 18.9 gives (18.13) in a neighborhood of x. On the other hand, if $h(\nabla \kappa, \nabla \kappa) = 0$ on some nonempty open connected set $U' \subset \Sigma$, the Gaussian curvature κ of (Σ, h) must be constant (which, in turn, is a case classified by Theorem 14.2, as $\kappa = K$ according to Remark 10.1). In fact, κ is locally constant on U' since, by (18.5), the values assumed by κ in U' are roots of the polynomial

$\varPsi$ given by (18.12). Consequently, the Killing field $w = \alpha(\nabla\kappa)$ defined as in Remark 18.8 (up to a sign, since α depends on the choice of an orientation) vanishes on U'. As U' is nonempty, by Remark 17.6(iv) we have $w = 0$ everywhere and so κ is constant, as $\nabla\kappa = -\varepsilon\alpha w$ by (18.9).

In view of Lemma 18.4, one can also use Lemma 18.9 combined with Theorem 14.2 to obtain a local classification, at all points of an open dense subset U of M, of those pseudo-Riemannian Einstein four-manifolds (M, g) which are locally conformally related to Riemannian products of surfaces. Specifically, U is the union of the open sets U_x described as follows. First, let U' be the (obviously dense) open set in M consisting of all points x such that either $W = 0$ identically in a neighborhood U_x of x, or $W(x) \neq 0$ (where W is, as usual, the Weyl tensor). Given $x \in U'$, in the former case ($W = 0$ on U_x) it follows that (U_x, g) is a space of constant curvature, since (5.10) then implies (10.1) on U_x; hence, we can use Theorem 14.2. If, on the other hand, $W \neq 0$ at x (and hence in a neighborhood U'_x of x, on which g is conformally related to a product-of-surfaces metric $\tilde{g}$), the same property ($\tilde{W} \neq 0$ on U'_x) will, by (16.11), also hold for $\tilde{g}$. According to (16.30), the scalar curvature $\tilde{\mathrm{s}}$ of $\tilde{g}$ then is nonzero everywhere in U'_x. Applying Lemma 18.4 to $(U'_x, \tilde{g})$, we see that (up to a constant factor), $g = \tilde{g}/(\kappa_1 + \kappa_2)^2$, where κ_j is the Gaussian curvature of the factor surface $(\varSigma_j, h_j)$. (Note that the roles of g and $\tilde{g}$ have been switched here, compared to Lemma 18.4.) The phrase 'up to a constant factor' may actually be omitted, i.e., the "constant factor" just mentioned can always be made equal to 1 (or any given positive real number), since multiplying $\tilde{g}$ by a constant $a \neq 0$ results in multiplying $g = \tilde{g}/\tilde{\mathrm{s}}^2$ by a^3 (Remark 4.2). Lemma 18.4 also shows that the metrics of the factor surfaces $(\varSigma_1, h_1)$, $(\varSigma_2, h_2)$ of $(U'_x, \tilde{g})$ are both extremal (and have the same first classifying parameter c). Denoting U_j the open dense subset of $\varSigma_j$, $j = 1, 2$, defined as in Remark 18.10, we can now declare U_x to be the open dense subset of U'_x corresponding to $U_1 \times U_2$ under our fixed diffeomorphic identification $U'_x \approx \varSigma_1 \times \varSigma_2$.

By a *Kottler metric* we mean any pseudo-Riemannian Einstein metric on a 4-manifold M with the property that every every point of M has a neighborhood U with a C^∞ function $f : U \to \mathbf{R}$ such that the metric $\tilde{g} = e^{2f} g$ on U, conformally related to g, is isometric to the Riemannian product of two pseudo-Riemannian surface metrics, one of which has a constant Gaussian curvature. A Kottler metric will be called a *Schwarzschild metric* if it is Ricci-flat. See Schwarzschild (1916), Kottler (1918) and Petrov (1969).

Remark 18.11. The classes of Kottler and Schwarzschild metrics deserve a more detailed discussion, due to their significance both for geometry (in the context of mobility of Riemannian Einstein four-manifolds, §20) and for physical applications (since they provide interesting spacetime models in general relativity; see §48). The classification described in the preceding paragraph is valid at points of an open dense subset U of the manifold in question; here we will refer to such points as being *in general position.* That classification, when applied to the special case of Kottler metrics g on 4-manifolds M, can be spelled out as follows. First, let us simplify our discussion by assuming that all points of M are in general position, which is equivalent to replacing M with a connected neighborhood of any fixed point $x \in U$. Second, let us leave aside the case where the Kottler metric g is of constant curvature, which now amounts to requiring that $W \neq 0$ everywhere in

M (cf. (5.10) and (10.1)). As in the previous paragraph, we then have

$$g = \tilde{g}/(\kappa+\lambda)^2\,, \tag{18.21}$$

where $\tilde{g}$ is a pseudo-Riemannian metric obtained as the Riemannian product of two pseudo-Riemannian metrics h and h' on surfaces Σ and Σ' such that the Gaussian curvature κ of h satisfies (18.1), while h' is of constant Gaussian curvature λ. (Note that $\kappa+\lambda\neq 0$ everywhere in Σ, since the scalar curvature $\tilde{\mathrm{s}}$ of $\tilde{g}$ equals $\mathrm{s} = 2(\kappa+\lambda)$.) In view of Lemma 18.3, h also satisfies (18.5) with some constants c and p, while the corresponding constants for h' are λ^2 and $2\lambda^3/3$ (Example 18.2). If we now define the real-valued function r on M by $r = (\kappa+\lambda)^{-1}$, we obtain

$$\kappa = \frac{1}{r} - \lambda\,, \quad c = \lambda^2\,, \quad p = \frac{\mathrm{s}}{3} - \frac{2}{3}\lambda^3\,, \quad 3\,\Psi(\kappa) = \frac{3\lambda}{r^2} - \frac{1}{r^3} - \mathrm{s}\,, \tag{18.22}$$

where s is the scalar curvature of g and $\Psi(\kappa)$ is given by (18.12). (In fact, by (18.7), $\mathrm{s} = 3p + 2\lambda^3$.) Also, (18.21) now can be rewriten as

$$g = r^2(h + h')\,, \tag{18.23}$$

the symbol $h+h'$ being used for the Riemannian product of h and h' (since it is the sum of their pull-backs to the product manifold). To further rewrite formula (18.23), let us combine (18.13) with a local-coordinate expression for h':

$$h' = h^{[\lambda]}_{jk}(x^1,x^2)\,dx^j dx^k \tag{18.24}$$

with some C^∞ functions $h^{[\lambda]}_{jk}$ of the variables x^1,x^2. (Here we assume that Σ' is covered by a single coordinate system x^j, $j=1,2$, while the superscript $[\lambda]$ is to remind us that the Gaussian curvature of this surface metric is constant and equal to λ.) In addition, let us replace the local coordinates θ,κ in Σ, appearing in (18.13), by t,r with $r=(\kappa+\lambda)^{-1}$ (as above) and $t=\theta$. The reason why we now use the symbol t rather than θ is that, in physical applications, this function serves as a time coordinate. (The new t is *not* the same parameter t as in our proof of Lemma 18.9.) We thus have $d\kappa = -\,dr/r^2$, $d\theta = dt$, and so (18.13) yields the following expression for our Kottler metric g (with summation over $j,k=1,2$):

$$g = \varepsilon\left[\lambda - \frac{1}{3r} - \frac{\mathrm{s}}{3}r^2\right]dt^2 + \left[\lambda - \frac{1}{3r} - \frac{\mathrm{s}}{3}r^2\right]^{-1} dr^2 + r^2 h^{[\lambda]}_{jk}(x^1,x^2)\,dx^j dx^k\,,$$

where (18.24) is a surface metric of constant Gaussian curvature λ. This is an explicit description of g in the coordinate system t,r,x^1,x^2 for M, depending on three parameters

$$\mathrm{s}\in\mathbf{R}\,, \qquad \lambda\in\mathbf{R} \qquad \varepsilon = \pm 1\,, \tag{18.25}$$

namely, the (constant) scalar curvature of g, the (constant) Gaussian curvature of (18.24), and the sign factor of the other surface metric h. (A fourth discrete parameter involved here is the sign pattern of the metric (18.24).) Conversely, for any choice of the parameters (18.25), the above formula defines a Kottler metric.

The local-isometry types of those Kottler metrics g which are not of constant curvature thus are, at points in general position, *classified* by the parameters (18.25) along with the sign pattern of (18.24). Since Schwarzschild metrics are those Kottler metrics for which $\mathrm{s} = 0$, we also obtain the following *universal description of all Schwarzschild metrics* g, other than those of constant curvature:

$$g = \varepsilon(\lambda - 1/3r)\,dt^2 + (\lambda - 1/3r)^{-1}\,dr^2 + r^2 h^{[\lambda]}_{jk}(x^1, x^2)\,dx^j dx^k\,, \tag{18.26}$$

in the coordinates t, r, x^1, x^2, depending on the parameters $\lambda \in \mathbf{R}$ and $\varepsilon = \pm 1$ and the surface metric (18.24) with the constant Gaussian curvature λ.

The remainder of this section is devoted to a proof of Calabi's Theorem 18.14.

Recall that for a Killing field w on a pseudo-Riemannian manifold (M, g), ∇w may be treated as a bundle morphism $TM \to TM$, skew-adjoint at every point $x \in M$. By $\operatorname{rank}[\nabla w](x)$ we then mean, as usual, the dimension of the image of $[\nabla w](x) : T_xM \to T_xM$.

The following lemma remains valid (with the same proof) if compactness of M is replaced by completeness of (M, g). To derive the existence of a flow homorphism (17.30), one then has to use, instead of Remark 2.3, the fact that every Killing field w in a complete Riemannian manifold is complete (as a vector field). For details, see e.g., Kobayashi and Nomizu (1963).

Lemma 18.12. *Let w be a Killing field on a compact Riemannian manifold (M, g), and let $x \in M$ be a point such that $w(x) = 0$ and $\operatorname{rank}[\nabla w](x) < 4$. If w is not identically zero, then $[\nabla w](x) \neq 0$ and the kernel of the flow homorphism* (17.30) *is given by*

$$\{t \in \mathbf{R} : e^{tw} = \mathrm{Id}\} = \frac{2\pi}{\mathfrak{a}(x)}\,\mathbf{Z}\,, \qquad \mathfrak{a} = |\nabla w|\,, \tag{18.27}$$

the scale-factor convention about $|\nabla w|$ being such that

$$2\,|\nabla w|^2 = w_{j,k}w^{j,k} = -\operatorname{Trace}(\nabla w \circ \nabla w)\,. \tag{18.28}$$

Proof. According to Remark 17.24, for $t \in \mathbf{R}$ we have $e^{tw} = \mathrm{Id} : M \to M$ if and only if $d[e^{tw}]_x = \mathrm{Id} : T_xM \to T_xM$. On the other hand, the assignment $\mathbf{R} \ni t \mapsto F(t) = d[e^{tw}]_x$ is a C^1 homomorphism into the group of all linear isomorphisms $T_xM \to T_xM$, and hence $F(t) = e^{tA}$ with $A = \dot F(0)$. (This is obvious from the uniqueness-of-solutions theorem for ordinary differential equations, since for such a homomorphism we clearly have $\dot F(t) = AF(t)$, $F(0) = \mathrm{Id}$.) Furthermore, $A \neq 0$, or else it would follow that $F(t) = \mathrm{Id}$ for all t and so, by Remark 17.24, $e^{tw} = \mathrm{Id}$ for all t, contrary to the hypothesis that $w \neq 0$ somewhere. Our assumption about the rank of $A = [\nabla w](x)$ means that the image $L = A(T_xM)$ is of dimension r with $r \le 3$. It now follows that $r = 2$. In fact, $r > 0$ as $A \neq 0$ and r is necessarily even. (To see this, note that, if r were odd, L would contain an eigenvector of A, as it is A-invariant, with the eigenvalue equal to zero due to skew-adjointness of A; that is in turn impossible as $\operatorname{Ker} A = L^\perp$.) We can thus identify L with a complex line in such a way that A restricted to L is the multiplication by ic for some real $c \neq 0$. Hence $T_xM = L^\perp \oplus L$, with $A(v + u) = icu$ for $v \in L^\perp$ and $u \in L$. Consequently, $e^{tA}(v + u) = v + e^{ict}u$, and so $e^{tA} = \mathrm{Id}$ if and only if $ct \in 2\pi\mathbf{Z}$. However, by (18.28) with $A = [\nabla w](x)$, $2[\mathfrak{a}(x)]^2 = -\operatorname{Trace}_{\mathbf{R}} A^2 = 2c^2$. Thus, $|c| = \mathfrak{a}(x)$, which completes the proof. ■

Corollary 18.13. *Suppose that we are given a Killing field w on a compact Riemannian manifold (M,g) of dimension 2 or 3 and points $x,y \in M$ with $w(x)=w(y)=0$. Then $|[\nabla w](x)| = |[\nabla w](y)|$.*

In fact, according to (18.27), $|[\nabla w](x)|$ then is the same for all $x \in M$ with $w(x)=0$. ■

Theorem 18.14 (Calabi, 1982)**.** *An extremal positive-definite metric h on a compact surface Σ must have a constant Gaussian curvature.*

Proof. Let us set $b = \max\kappa$, $a = \min\kappa$ and choose $x,y \in \Sigma$ with $\kappa(x) = b$, $\kappa(y) = a$. As $\kappa_{,jl} = \partial_j\partial_l\kappa$ at points where $d\kappa = 0$, we have $\nabla d\kappa \le 0$ at x and $\nabla d\kappa \ge 0$ at y and so, by (18.4),

$$c - b^2 \le 0\,, \qquad c - a^2 \ge 0\,. \tag{18.29}$$

By passing to a two-fold covering surface of Σ, if necessary, we may assume that Σ is orientable, and choose an orientation. According to Remark 18.8, formula $w = \alpha(\nabla\kappa)$ (with the nonzero parallel bivector field α introduced in Remark 18.6) then defines a Killing field w on (Σ,h) such that $w(x)=w(y)=0$ and $2\,\nabla w = 2\,\alpha(\nabla d\kappa) = (c-\kappa^2)\alpha$. Thus, $|[\nabla w](x)| = |[\nabla w](y)|$ in view of Corollary 18.13, which, since $|\alpha|$ is constant, means that $|c-b^2| = |c-a^2|$. Therefore, by (18.29), $b^2 - c = c - a^2$, i.e.,

$$b^2 + a^2 - 2c = 0\,. \tag{18.30}$$

As $\nabla\kappa = 0$ both at x and at y, (18.3) gives $3p = 3cb - b^3$ and $3p = 3ca - a^3$. Subtracting, we obtain $(b-a)(3c - b^2 - ba - a^2) = 0$. Therefore, $(b-a)^3 = (b-a)\,[2(3c - b^2 - ba - a^2) + 3(b^2 + a^2 - 2c)] = 0$ in view of (18.30), and so $\max\kappa = b = a = \min\kappa$. This completes the proof. ■

§19. Other conformally-Einstein product metrics

This section deals with the pseudo-Riemannian Einstein 4-manifolds (M,g) which are *locally conformal to* $(1+3)$*-dimensional products* in the sense that every point of M has a neighborhood on which g is conformally related to a product metric $\tilde g = e^{2f}g$ with factors of dimensions 1 and 3. The results presented here are well-known; see, e.g, see Petrov (1969, p. 345).

Recall that in §18 we discussed the four-dimensional Einstein manifolds (M,g) that are locally conformally related to products of surfaces. Somewhat surprisingly, each of those (M,g) turns out to have a dense open subset on which g is locally conformal to a $(1+3)$-dimensional product metric. (See Remark 19.4 below.) The subsequent discussion may therefore be regarded as a natural generalization of §18.

Let (M,g) be a pseudo-Riemannian manifold. By a *steady-state field* in (M,g) we mean any Killing vector field w on (M,g) such that $\langle w,w\rangle \ne 0$ everywhere in M (with $\langle\,,\rangle$ standing for g) and

$$d\xi = 0\,, \qquad \text{where} \quad \xi = \frac{w}{\langle w,w\rangle}\,, \tag{19.1}$$

that is, $w/\langle w,w\rangle$ is *closed* when regarded, with the aid of g, as a differential 1-form on M. According to Poincaré's Lemma (Corollary 11.3), this amounts to requiring that locally, i.e., in a suitable neighborhood of any given point of M, we have $\xi = dt$ or, equivalently,

$$w \,=\, \langle w,w\rangle \nabla t \tag{19.2}$$

for some C^∞ function t (where ∇t is the g-gradient of t). Any such t will be referred to as a (local) *time function* for w.

The terminology just introduced suggests that steady-state fields appear in cosmological models. (See §48.) They also appear naturally in geometry, as illustrated by this section and Lemma 21.1(c) in §20.

Example 19.1. In dimension 2, *every* Killing vector field w such that $\langle w,w\rangle \neq 0$ everywhere is a steady-state field. To see this, let us fix such a field w on a pseudo-Riemannian surface and choose, locally, a C^∞ unit vector field u orthogonal to w. Then $[u,w]=0$. In fact, from (4.4) and (17.1) we obtain $\langle [u,w],u\rangle = \langle \nabla_u w - \nabla_w u, u\rangle = -\langle \nabla_v u, u\rangle$, which is zero as $\langle u,u\rangle = \pm 1$ is constant; similarly, skew-adjointness of ∇w along with the relation $\langle u,w\rangle = 0$ gives $\langle [u,w],w\rangle = \langle \nabla_u w - \nabla_w u, w\rangle = -\langle \nabla_w w, u\rangle + \langle \nabla_w w, u\rangle = 0$. Setting $\xi = w/\langle w,w\rangle$ and computing $d\xi$ via (2.16) (with $\xi(v) = \langle \xi, v\rangle$ for all tangent vectors v), we now find that $(d\xi)(u,w) = 0$ (as $\langle \xi, w\rangle = 1$). Thus, $d\xi = 0$, as required.

Let us now return to the case of steady-state fields w in manifolds (M,g) of any dimension n. As we will see next, the metric $|\langle w,w\rangle|\, g$, conformally related to g, then admits, in a suitable neighborhood U of any given point of M, a Riemannian-product decomposition

$$|\langle w,w\rangle|\, g \,=\, \varepsilon\, dt^2 \,+\, h\,, \qquad \varepsilon \,=\, \pm 1\,, \tag{19.3}$$

whose first factor manifold is an open interval $I \subset \mathbf{R}$, while t, the natural coordinate for I (i.e., the Cartesian-product projection $U \to I$) is just any fixed local time function for w. More precisely, we have the following easy classification result.

Lemma 19.2. *Let w be a C^∞ vector field on an n-dimensional pseudo-Riemannian (M,g), $n \geq 2$. Then, the following two conditions are equivalent*:

(i) *w is a steady-state Killing field on (M,g);*

(ii) *Every point $y \in M$ has a neighborhood U which can be diffeomorphically identified with a product $I \times N$ of an open interval $I \subset \mathbf{R}$ and a manifold N of dimension $n-1$, in such a way that*

a) *g restricted to $U = I \times N$ has the warped-product form*

$$g \,=\, e^{-2f}\left[\varepsilon\, dt^2 \,+\, h\right], \qquad \varepsilon \,=\, \pm 1\,, \tag{19.4}$$

cf. Remark 16.1, *with some function $f : I \times N \to \mathbf{R}$ that is constant in the direction of I and some product metric $\varepsilon\, dt^2 + h$ with factors $\varepsilon\, dt^2$ and h that are metrics on I and, respectively, on N. Here t is the natural coordinate for the I factor, i.e., the projection $I \times N \to I$; and*

b) *w restricted to $U = I \times N$ is the "coordinate field" in the direction of I, that is, for any fixed $y \in N$, formula $I \ni t \mapsto (t,y)$ defines an integral curve of w. In other words, $d_w = \partial/\partial t$.*

Furthermore, in (b), $e^{2f} = |\langle w, w\rangle|$ *and the function* t *is a local time function for* w.

Proof. Throughout this argument, the indices a, b are assumed to range over the set $\{1, \dots, n-1\}$.

Let us suppose that (i) holds, and let t be a local time function for w. In local coordinates chosen for t and g as in Lemma 2.6, we thus have $g_{00} = \langle \nabla t, \nabla t\rangle = 1/\langle w, w\rangle$ and $g_{0a} = 0$, while $t = x^0$ and w coincides with the coordinate vector field e_0 in the direction of t. Setting $\varepsilon = \operatorname{sgn}\langle w, w\rangle$ and defining the function f by $e^{2f} = |\langle w, w\rangle|$, we now have $\partial f/\partial t = d_w f = 0$, as $d_w\langle w, w\rangle = 2\langle \nabla_w w, w\rangle = 0$ by (17.1). Also, the metric $\tilde g = e^{2f} g$ satisfies $\tilde g_{00}\,\varepsilon$ and $\tilde g_{0a} = 0$, so that $\tilde g = \varepsilon\, dt^2 + h$ with $h_{00} = h_{0a} = 0$, $h_{ab} = e^{-2f} g_{ab}$. Finally, since the coordinate vector field $w = e_0$ is a Killing field for g and $\partial f/\partial t = 0$, we have $\partial g_{ab}/\partial t = 0$ and $\partial h_{ab}/\partial t = 0$ (see Example 17.1). Thus, $\varepsilon\, dt^2 + h$ is a Riemannian-product decomposition of $\tilde g$, which proves (ii).

Conversely, let us assume (ii) and let x^j, $j = 0, 1, \dots, n-1$, be a local product-coordinate system on $U = I \times N$, consisting of the coordinate $x^0 = t$ on I and some coordinates x^a in N. In view of (19.4) with $\partial f/\partial t = 0$, we have $\partial g_{jk}/\partial t = 0$, and so (cf. Example 17.1) the coordinate vector field $w = e_0$ in the direction of t is a Killing field for g. Also, $\varepsilon\, e^{-2f} = g_{00} = \langle e_0, e_0\rangle = \langle w, w\rangle$. The 1-form $\xi = dt = dx^0$ has the components $\xi_j = \partial_j x^0 = \delta_j^0$, and so the components of the correspoding vector field $v = \nabla t$ satisfy $v^j = g^{jk}\xi_k = g^{0j}$, that is, $v^0 = g^{00} = 1/g_{00} = 1/\langle w, w\rangle$ and $v^a = 0$. Since $w = e_0$ has the components $w^0 = 1$ and $w^a = 0$, relation (19.2) follows. Hence (ii) implies (i), as required. ∎

Corollary 19.3. *Given a pseudo-Riemannian surface* (Σ, h) *and a point* $x \in \Sigma$, *the following two conditions are equivalent*:

(i) h *restricted to some neighborhood of* x *is a warped-product metric*;
(ii) *There exists a Killing field* w *in* (Σ, h) *defined on a neighborhood of* x, *with* $\langle w, w\rangle \neq 0$ *at* x.

In fact, if h has, near x, the warped-product form $h = A(t)\, dt^2 + B(t)\, dr^2$ in some coordinates $(t, r) = (x^1, x^2)$ (so that $A(t)B(t) \neq 0$), the coordinate vector field w in the direction of t is a Killing field for h (Example 17.1; note that $h_{11} = A(t)$, $h_{12} = 0$, $h_{22} = B(t)$), and $\langle w, w\rangle = A(t)$. Conversely, (ii) implies (i) in view of Lemma 19.2 along with Example 19.1. ∎

Remark 19.4. Every pseudo-Riemannian Einstein metric in dimension four that is locally conformally related to a product of surface metrics must, at all points "in general position" (as defined in the paragraph following Remark 18.10 in §18), be also locally conformal to a $(1+3)$-dimensional product metric.

To see this, let us first note that the latter conclusion is valid, more generally, for a product $g' = h_1 + h_2$ of two surface metrics, one of which (say, h_1) is a warped product. In fact, all we have to do is write, in product coordinates t, θ, y^3, y^4,

$$g' \,=\, A(t)\, d\theta^2 \,+\, B(t)\, dt^2 \,+\, h'_{\nu\rho}\, dy^r\, dy^s\,, \qquad \nu, \rho \in \{3, 4\}\,, \tag{19.5}$$

with $h'_{\nu\rho}$ (the components of h_2) depending only on y^3, y^4; dividing g' by $e^{2f} = |A(t)|$, we obtain a $(1+3)$-dimensional product metric. On the other hand, if an Einstein metric is conformal to a product of two surface metrics, then, at points in general position, these factor metrics are both extremal (see §18, as quoted above), and so they are, locally, warped products (by (18.13) or, for constant-curvature surfaces, by Example 17.19 along with Corollary 19.3).

Remark 19.5. Let us consider a pseudo-Riemannian product metric

$$\tilde{g} \,=\, \varepsilon\, dt^2 \,+\, h\,, \qquad \varepsilon \,=\, \pm 1\,, \tag{19.6}$$

on an n-dimensional Cartesian-product manifold $M = I \times N$, $n \geq 4$, with factor metrics $\varepsilon\, dt^2$ (on an interval $I \subset \mathbf{R}$), and h (on a manifold N with $\dim N = n-1$). Here t is the natural coordinate for I which, treated as a function on $M = I \times N$, is the projection $I \times N \to I$. Let us also fix a product coordinate system x^j, in M, $j = 0, 1, \ldots, n-1$, consisting of the coordinate $x^0 = t$ on I and some coordinates x^a in N, $a = 1, \ldots, n-1$. Denoting R, Ric, $\nabla\mathrm{Ric}$, s, $d\mathrm{s}$ and W the curvature tensor, the Ricci tensor and its covariant derivative, the scalar curvature and its differential and, finally, the Weyl tensor of h, and using the symbols $\tilde{R}$, $\tilde{\mathrm{Ric}}$, $\tilde{\nabla}\tilde{\mathrm{Ric}}$, $\tilde{\mathrm{s}}$, $d\tilde{\mathrm{s}}$ and $\tilde{W}$ for their analogues corresponding to $\tilde{g}$, we obviously have

$$\tilde{R} \,=\, R\,, \quad \tilde{\mathrm{Ric}} \,=\, \mathrm{Ric}\,, \quad \tilde{\nabla}\tilde{\mathrm{Ric}} \,=\, \nabla\mathrm{Ric}\,, \quad \tilde{\mathrm{s}} \,=\, \mathrm{s}\,, \quad d\tilde{\mathrm{s}} \,=\, d\mathrm{s}\,, \tag{19.7}$$

where (covariant) tensors in N are identified with their pull-backs to $M = I \times N$. In terms of components relative to product coordinates as above, this means that $\tilde{R}_{jklm}$ are given by $\tilde{R}_{abcd} = R_{abcd}$ and $\tilde{R}_{jklm} = 0$ whenever at least one of the indices is 0, and similarly for the other quantities in (19.7). The situation is not so simple for the Weyl tensor, however; for instance, using (5.8) with $g_{00} = \varepsilon$, we obtain

$$(n-2)\, \tilde{W}_{0a0b} \,=\, -\,\varepsilon\, [R_{ab} \,-\, \mathrm{s}\, h_{ab}/(n-1)]\,, \qquad \tilde{W}_{0abc} \,=\, 0\,. \tag{19.8}$$

More precisely, substituting (19.6), (19.7) into the "tilde version" of (5.6), we obtain

$$\tilde{W} \,=\, R \,-\, \frac{\mathrm{s}}{(n-1)(n-2)}\, h \circledast h \,-\, \frac{2}{n-2}\, \tilde{g} \circledast \mathrm{E}\,, \tag{19.9}$$

where $\mathrm{E} = \mathrm{Ric} - \mathrm{s}\, g/(n-1)$ is the traceless Ricci tensor of h, given by (5.5) (with n replaced by $n-1$). It follows now that, in dimensions $n \geq 4$, *the product metric* $\tilde{g}$ *in* (19.6) *is conformally flat if and only if the* $(n-1)$*-dimensional factor metric* h *is of constant curvature.* In fact, conformal flatness of $\tilde{g}$ means that $\tilde{W} = 0$ identically (Theorem 16.5). Thus, if $\tilde{g}$ is conformally flat, we have $\mathrm{E} = 0$ (by (19.8)); combined with (19.9), this gives $(n-1)(n-2)\, R = \mathrm{s}\, h \circledast h$, that is, (10.1) with (10.3) (in dimension $n-1$ rather than n), as required. Conversely, if h is of constant curvature, (10.1) and (10.3) (in dimension $n-1$), along with (19.9), imply $\tilde{W} = 0$. Finally, the $\tilde{g}$-divergence $\tilde{\mathrm{div}}\, \tilde{W}$ of $\tilde{W}$ (cf. (5.28), (5.29)) satisfies

$$[\tilde{\mathrm{div}}\, \tilde{W}]_{a0b} \,=\, 0\,, \tag{19.10}$$

as one easily sees using (5.29) and (19.7).

Lemma 19.6. *Let* $(M,\tilde g)$ *pseudo-Riemannian* 4*-manifold obtained as the Riemannian product with factors of dimensions* 1 *and* 3 *and such that, for some* C^∞ *function* $f : M \to \mathbf{R}$*, the metric* $g = e^{-2f}\tilde g$ *conformally related to* $\tilde g$ *satisfies the condition* $\operatorname{div} W = 0$*, where* W *stands for the Weyl tensor of* g*. If the Weyl tensor* $\tilde W$ *of* $\tilde g$ *is not identically zero, then* f *is constant in the direction of the one-dimensional factor of* M*.*

Proof. Suppose that g is an Einstein metric on M and $\tilde g = e^{2f}g$ is of the form (19.6). Using (16.12) (and noting that, by (16.11), $W(df,\,\cdot\,,\,\cdot\,,\,\cdot\,) = \tilde W(df,\,\cdot\,,\,\cdot\,,\,\cdot\,)$, where $\tilde W$ denotes the Weyl tensor of $\tilde g$), we obtain, from (19.10) and (19.8), $0 = [\widetilde{\operatorname{div}}\,\tilde W]_{a0b} = (n-3)\tilde W^0{}_{a0b}f_{,0}$, and so $0 = \tilde W^0{}_{a0b}f_{,0} = \varepsilon\,\tilde W_{0a0b}f_{,0}$. We now must have $f_{,0} = 0$. In fact, otherwise (19.8) would imply that h appearing in (19.6) is a 3-dimensional Einstein metric, and hence a metric of constant curvature (Remark 10.2(b)), so that $\tilde g$ would have $\tilde W = 0$ identically (Remark 19.5), contrary to our hypothesis. This completes the proof. ∎

The following result addresses the "Einstein-metric end" of the question stated in Remark 18.1. (See also Remark 19.9 below.)

Proposition 19.7. *Let* x *be a point in a pseudo-Riemannian Einstein* 4*-manifold* (M,g) *such that the Weyl tensor* W *is nonzero at* x*. Then, the following two conditions are equivalent*:

(a) *g restricted to a neighborhood of x is conformally related to a product metric with factors of dimensions* 1 *and* 3;
(b) *A neighborhood of x admits a steady-state Killing vector field.*

Proof. Assume (b) and let $\tilde g = e^{2f}g$ be as in Remark 19.5. In view of (5.30), Lemma 19.6 implies that f is constant along the I factor, i.e., we have the situation described in Lemma 19.1(ii). Therefore, assertion (i) of Lemma 19.1 holds with w defined in Lemma 19.1(ii)b). Hence (b) follows. Conversely, (b) implies (a) in view of Lemma 19.1. This completes the proof. ∎

Remark 19.8. Proposition 19.7 and its proof remain valid in a much more general situation. For instance, (b) implies (a) even without assuming that (M,g) is Einstein, or 3-dimensional, or has $W \ne 0$ at the given point, while (b) follows from (a) under the weaker hypothesis that the 4-manifold (M,g), instead of being Einstein, just satisfies the condition $\operatorname{div} W = 0$.

Remark 19.9. For any *Riemannian* 4-manifold (M,g) (Einstein or not), condition

$$\operatorname{spec} W^+ \;=\; \operatorname{spec} W^- \tag{19.11}$$

(at every point and for either local orientation) is necessary in order that (M,g) be locally conformal to a $(1+3)$-dimensional product. In fact, since (19.11) is conformally invariant (Remark 16.4(e)), it is sufficient to verify it for positive-definite $(1+3)$-dimensional product metrics. To this end, let us replace (M,g) with $(M,\tilde g)$, where $\tilde g$ is a product metric (19.6) on $M = I\times N$, as discussed in Remark 19.5, with $n = 4$, i.e., $\dim N = 3$. For a fixed point $x = (t,y)\in M$, let the local coordinates x^a in N, $a = 1,\dots,3$, used for calculations in Remark 19.5, defined on a neighborhood of y and such that the coordinate fields e_a in N are orthonormal at the point y and form, at y, eigenvectors of the traceless Ricci tensor $\mathrm{E} = \mathrm{Ric} - \mathrm{s}\,g/(n-1)$ of (N,h) with some eigenvalues μ_a. Let $\beta_a \in \Lambda^+_x M$,

$a = 1, \ldots, 3$, now denote bivectors at $x = (t, y)$ in M given by $\beta_a = e_0 \wedge e_a$ at $x = (t, y)$, where e_0 is the coordinate field in the direction of $x^0 = t$. In view of (19.6) combined with (5.13) and (2.15), the β_a are eigenvectors of the Weyl tensor $\tilde{W}$ of $\tilde{g}$, acting on bivectors, with the respective eigenvalues $\lambda_a = -\varepsilon\mu_a/2$. Relation (19.11) now follows in view of Remark 6.20.

Finally, we have to deal with the "other-metric end" of the "conformal question" of Remark 18.1. As in previous cases, it is more convoluted than the "Einstein-metric end".

Proposition 19.10. *Let $(M, \tilde{g})$ be a pseudo-Riemannian product 4-manifold with $M = I \times N$ and the product metric*

$$\tilde{g} = \varepsilon\, dt^2 + h\,, \qquad \varepsilon = \pm 1\,, \tag{19.12}$$

whose factor metrics are $\varepsilon\, dt^2$, on an interval $I \subset \mathbf{R}$, and h on a 3-manifold N. As before, t stands for the Cartesian-product projection function $M = I \times N \to I$. Given a point $x = (t, y) \in M$ at which the Weyl tensor $\tilde{W}$ of $\tilde{g}$ is nonzero, the following two conditions are equivalent:

(i) *$\tilde{g}$ restricted to some neighborhood of x is conformally related to an Einstein metric*;

(ii) *There exists a C^∞ function ϕ defined on a connected neighborhood U of y in N such that $\phi \neq 0$ everywhere in U and*

$$\nabla d\phi = -\frac{1}{2}\,\phi \cdot \mathrm{Ric}\,, \tag{19.13}$$

∇ and Ric *being the Levi-Civita connection and Ricci tensor of (N, h). More precisely, for a nowhere-zero C^∞ function ϕ defined on a neighborhood of x in M, the metric*

$$g = \tilde{g}/\phi^2 \tag{19.14}$$

is Einstein if and only if ϕ is constant in the direction of I that is,

$$\partial\phi/\partial t = 0\,, \tag{19.15}$$

and, treated as a function on an open set in N, ϕ satisfies (19.13). *In that case, we also have*

$$\phi\,\Delta\phi - 3\,h(\nabla\phi, \nabla\phi) = \kappa\,, \tag{19.16}$$

where κ is the constant Ricci curvature of the Einstein metric (19.14), *cf.* (5.3), *and Δ stands for the Laplacian of (N, h).*

Proof. Let $\phi \neq 0$ be a C^∞ function defined and near x in M and satisfying (19.15). Then (19.14) *is an Einstein metric if and only if ϕ satisfies* (19.13) *and* (19.16) *for some constant κ* (which is the Ricci curvature of (19.14)).

To see this, write condition (5.3) for the metric (19.14) using (16.14) for $n = 4$ (with switched rôles of g and $\tilde{g}$, and with φ replaced by ϕ), in product coordinates such as those in Remark 19.5. We then have $\tilde{R}_{00} = \tilde{R}_{0a} = 0$, cf. (19.7), and $\phi_{00} =$

$\phi_{0a} = 0$ as $\partial\phi/\partial t = 0$, while $g_{00} = \tilde{g}_{00}/\phi^2 = \varepsilon/\phi^2$. Thus, the result is condition (19.16) plus $\phi^2 R_{ab} + 2\phi\phi_{,ab} + [\phi\,\Delta\phi - 3\,h(\nabla\phi,\nabla\phi)]\,h_{ab} = \kappa\,h_{ab}$, $a,b = 1,2,3$, (with R_{ab} and $\phi_{,ab}$ referring to the geometry of (N,h)). Due to the form of (19.16), these two relations amount to (19.16) and $\phi^2 R_{ab} + 2\phi\phi_{,ab} = 0$, that is, (19.16) and (19.13), as required.

Suppose now that ϕ is a nowhere-zero C^∞ function on a neighborhood of x in M such that (19.14) is an Einstein metric. By Lemma 19.6, we have (19.15). The above discussion thus shows that ϕ satisfies (19.13) and (19.16) with a constant κ (the Ricci curvature of (19.14)). Conversely, let a C^∞ function $\phi \neq 0$ on a neighborhood of x in M satisfy (19.13) and (19.15). Then, as established above, (19.14) is an Einstein metric. This completes the proof. ■

Remark 19.11. The class of pseudo-Riemannian 3-manifolds (N,h) with nowhere-zero C^∞ functions ϕ satisfying (19.13) does not seem to have a usable local classification. However, one easily obtains following examples and partial classification results:

(a) Relation (19.13) holds whenever h is flat and so (see Theorem 14.2(ii)), (N,h) looks, locally, like a pseudo-Euclidean 3-space V with a constant metric, while the function ϕ on $N = V$ is affine (that is, "linear", but not necessarily homogeneous).

(b) Let h be a metric of constant curvature $K \neq 0$, so that, locally, (N,h) may be identified with a pseudosphere S_c in pseudo-Euclidean 4-space V (Theorem 14.2(i)). A function ϕ with (19.13) then can be obtained by restricting to $M = S_c$ any linear (homogeneous) function $V \to \mathbf{R}$.

(c) Those (N,h) and ϕ obtained by combining Remark 19.4 above with our classification (given in the paragraph following Remark 18.10 in §18) of pseudo-Riemannian Einstein metrics g in dimension four which are locally conformal to products $g' = h_1 + h_2$ of surface metrics with some Gaussian curvatures κ_1, κ_2. In fact, we may rescale h_1, h_2 so that $g = g'/(\kappa_1+\kappa_2)^2$ (cf. §18). Writing g' in the specific product form (19.5), we thus obtain a $(1+3)$-dimensional product metric $\tilde{g} = g'/|A(t)|$; the 3-dimensional factor metric h of $\tilde{g}$ admits, according to Proposition 19.10, a function ϕ with (19.13), namely, the function ϕ such that $g = \tilde{g}/\phi^2$, i.e.,

$$\phi = (\kappa_1 + \kappa_2)/\sqrt{|A(t)|}\,. \tag{19.17}$$

(d) A partial classification result: The examples of 3-manifolds (N,h) described in (a) – (c), all satisfy the condition

$$\#\,\mathrm{spec}\,\mathrm{Ric} \,\leq\, 2\,. \tag{19.18}$$

(Notation as in Lemma 6.15 or Remark 16.10.) Conversely, at suitably defined "points in general position", these examples represent all possible local-isometry types of pseudo-Riemannian 3-manifolds (N,h) admitting nonzero functions ϕ with (19.13) and simultaneously satisfying (19.18).

In fact, due to symmetry of the Hessian $\nabla d\phi$, equation (19.13) is always equivalent to the requirement that $2\,d^2\phi(y(t))/dt^2 + \phi\cdot\mathrm{Ric}\,(\dot{y},\dot{y})$ for every geodesic $t \mapsto y(t)$ of (N,h). It is therefore easy to verify (19.13) in cases (a) and (b) using an explicit description of geodesics as straight lines in V or, respectively, great "pseudocircles" in S_c. Note that, in cases (a), (b) the product metric (19.6) satisfies the condition $\tilde{W} = 0$ (see Remark 19.5), contrary to the assumption about $\tilde{W}$ in Proposition 19.10. In other words, (a) and (b) are completely irrelevant to our quest of constructing interesting Einstein metrics by conformal changes of $(1+3)$-dimensional product metrics.

As for (d), it follows easily from Remarks 16.10 and 19.9.

§20. Riemannian Einstein 4-manifolds and mobility

This section presents well-known facts on the Lie algebra of Killing fields in Riemannian Einstein 4-manifold. See also Petrov (1969), Chapters 4 and 5.

Any given 4-dimensional Riemannian manifold (M,g) gives rise to two interesting numbers. One of them is the dimension of the full isometry group $G = \mathrm{Isom}\,(M,g)$ of (M,g); the other is the dimension of the principal (i.e., highest-dimensional) orbits of G. The aim of this and the next section is to list all possible values of these numbers in the case where (M,g) is Einstein. (See Proposition 21.6.) However, since our discussion is local, we will replace the numbers just mentioned by their infinitesimal counterparts $\mathfrak{m}$ and $\mathfrak{o}$, which are related *but different* invariants, using *local* Killing fields rather than *global* Killing fields or isometries.

This local-global distinction may be significant. For instance, according to Theorem 24.8(i) in §24, a compact n-dimensional Riemannian manifold of negative constant curvature admits no (global) nontrivial Killing fields while, by Proposition 17.18 combined with Theorem 14.7, its sufficiently small open subsets have $\mathfrak{m}$-dimensional spaces of Killing fields with $\mathfrak{m} = n(n+1)/2$. Throughout this section we will ignore such discrepancies, keeping our focus entirely local. In other words, our classification procedure is not designed to detect any "purely global" effects.

Let (M,g) be a Riemannian 4-manifold. Its Weyl tensor W then may be treated, in the usual fashion, as an operator acting on bivectors via (5.13), which makes it a self-adjoint bundle morphism $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$. If M is oriented, W leaves invariant the subbundles $\Lambda^{\pm}M$ of $[TM]^{\wedge 2}$ (see (6.14)). The symbol

$$\#\,\mathrm{spec}\,W^{\pm} : M \to \{1,2,3\} \tag{20.1}$$

then will stand for the function that assigns to each $x \in M$ the number of distinct eigenvalues of the self-adjoint operator $W^{\pm}(x) : \Lambda^{\pm}_x M \to \Lambda^{\pm}_x M$. (Similar notations were used in Lemma 6.15 and Remark 10.11.) We often encounter the condition

$$\#\,\mathrm{spec}\,W^{+} \,\le\, 2\,, \tag{20.2}$$

which means that the self-adjoint bundle morphism $W : [TM]^{\wedge 2} \to [TM]^{\wedge 2}$, restricted to $\Lambda^{+}M$, has fewer than three distinct eigenvalues at every point. (See, for instance, Remark 16.4(e), formula (16.35), Proposition 20.1 and Lemma 20.9 below, as well as Proposition 22.4 in §22.)

Specific simple conditions imposed on the eigenvalue functions of W acting in $[TM]^{\wedge 2}$ on a given Riemannian Einstein four-manifold sometimes have the effect of "forcing" the existence of nontrivial Killing fields on a neighborhood of any point in M. One example is the requirement of *constancy* for the eigenvalue functions of W; in view of Corollary 7.2 combined with Theorem 14.7 and Proposition 17.18, it leads, locally, to a space of Killing fields of dimension 10, 8 or 6.

The following result states that condition (20.2) has a similar property (unless W^+ is parallel). The lengthy proof given here can be simplified by studying a Kähler metric conformally related to the Einstein metric in question; see Remark 22.5 in §22.

Proposition 20.1 (Derdziński, 1983)**.** *Let* (M, g) *be an oriented Riemannian Einstein* 4*-manifold satisfying* (20.2)*. Then*

(i) *Either* $W^+ = 0$ *identically, or* $W^+ \neq 0$ *everywhere in* M*.*

(ii) *In the case where* W^+ *is not identically zero, there exists a unique* C^∞ *function* $\lambda : M \to \mathbf{R} \smallsetminus \{0\}$ *and a unique* C^∞ *bivector field* $\pm\,\alpha$ *defined, up to a sign, at each point of* M*, such that* $W^+\alpha = \lambda\alpha$*,* $\langle\alpha,\alpha\rangle = 2$ *and, at each point* $x \in M$*,* $\lambda(x)$ *is a simple eigenvalue of* $W^+(x)$ *acting on* $\Lambda^+_x M$*. Furthermore,* $|W^+|^2 = \operatorname{Trace} W^2 = 3\,\lambda^2/2$ *and the vector field* $\pm\,w$ *on* M *defined up to a sign by the formula*

$$w \,=\, \lambda^{-4/3}\,\alpha(\nabla\lambda)\,, \tag{20.3}$$

has the following properties:

a) $\pm\,w$ *is a Killing field*;

b) $\pm\,w$ *is not identically zero unless* W^+ *is parallel.*

Finally, for any vector v *tangent to* M*,*

$$3\lambda\nabla_v\alpha \,=\, (\nabla\lambda)\wedge(\alpha v) \,-\, v\wedge[\alpha(\nabla\lambda)]\,. \tag{20.4}$$

Proof. Let M' be the open subset of M consisting of all points with $W^+ \neq 0$. Assuming that M' is nonempty, we will first prove (ii) with M replaced by any connected component of M'. To this end, let us choose objects α_j, λ_j, ξ_j and u_j of class C^∞, $j = 1,2,3$, satisfying (6.24), (6.12), (6.26) and (6.28) on a nonempty open subset of M'. The existence and uniqueness of λ and $\pm\,\alpha$ in (ii) now is obvious from (20.2) and (6.19) while, rearranging indices, we may also assume that

$$\alpha \,=\, \alpha_1\,, \qquad \lambda \,=\, \lambda_1 \,=\, -\,2\lambda_2 \,=\, -\,2\lambda_3\,. \tag{20.5}$$

In view of (6.28), we therefore have

$$u_1 = 0\,, \quad 2u_2 = -\,3\lambda\,\alpha_2\xi_2\,, \quad 2u_3 = 3\lambda\,\alpha_3\xi_3\,, \tag{20.6}$$

and so, using Lemmas 6.14 and 6.18(b) and (20.5), we obtain $d\lambda = -\,2d\lambda_2 = 2u_3 - 2u_1 = 3\lambda\,\alpha_3\xi_3$ and $d\lambda = -\,2d\lambda_3 = 2u_1 - 2u_2 = 3\lambda\,\alpha_2\xi_2$. Thus, by (6.12),

$$3\lambda\,\xi_2 \,=\, -\,\alpha_2(d\lambda)\,, \qquad 3\lambda\,\xi_3 \,=\, -\,\alpha_3(d\lambda)\,. \tag{20.7}$$

Now (6.26) with $j = 1$ becomes

$$3\lambda\nabla\alpha \,=\, [\alpha_2(d\lambda)]\otimes\alpha_3 \,-\, [\alpha_3(d\lambda)]\otimes\alpha_2 \tag{20.8}$$

(cf. (2.12)) which, in view of (6.34), is nothing else than (20.4).

Since the local components of $[\alpha(d\lambda)]$ are given by $[\alpha(d\lambda)]_j = \alpha^l{}_j\lambda_{,l}$, it follows that $3\lambda^{7/3}\,w_{j,k} = -4\lambda_{,k}[\alpha(d\lambda)]_j + 3\alpha^l{}_j\lambda_{,lk} + 3\alpha^l{}_{j,k}\lambda_{,l}$, with w given by (20.3). By

(20.8), the last term is skew-symmetric in j, k, and so it does not contribute to the symmetrized version

$$3\lambda^{7/3}\mathcal{L}_w g \,=\, 3\lambda[\alpha, \nabla d\lambda] \,+\, 4\, d\lambda \otimes [\alpha(d\lambda)] \,+\, 4\,[\alpha(d\lambda)] \otimes d\lambda \tag{20.9}$$

of the preceding equality, with $\mathcal{L}_w g$ as in (17.26). (Here $[\alpha, \nabla d\lambda]$ stands for the commutator of α and $\nabla d\lambda$ viewed as bundle morphisms $TU \to TU$.)

On the other hand, $\nabla\alpha_2 = \xi_1 \otimes \alpha_3 - \xi_3 \otimes \alpha$ and $\nabla\alpha_3 = \xi_2 \otimes \alpha - \xi_1 \otimes \alpha_2$ (by (6.26) and (20.5)), and so (20.7) gives $3\lambda\nabla\xi_2 = -3\, d\lambda \otimes \xi_2 - 3\lambda\, \xi_3 \otimes \xi + 3\lambda\, \xi_1 \otimes \xi_3 + \alpha_2 h$ (the last multiplication being the composite) and, similarly, $3\lambda\nabla\xi_3 = -\,3\, d\lambda \otimes \xi_3 + 3\lambda\, \xi_2 \otimes \xi - 3\lambda\, \xi_1 \otimes \xi_2 + \alpha_3 h$, where we have set

$$\xi \,=\, \frac{1}{3\lambda}\, \alpha(d\lambda)\,, \qquad h \,=\, \nabla d\lambda\,. \tag{20.10}$$

Skew-symmetrizing the last two relations, we obtain $3\lambda\,[d\xi_2 + \xi_3 \wedge \xi_1] = -\{\alpha_2, h\} - 3\lambda\, \xi_3 \wedge \xi - 3\, d\lambda \wedge \xi_2$ and $3\lambda\,[d\xi_3 + \xi_1 \wedge \xi_2] = -\,\{\alpha_3, h\} + 3\lambda\, \xi_2 \wedge \xi - 3\, d\lambda \wedge \xi_3$. (Here $\{\,,\,\}$ is the anticommutator, with $\{\alpha_3, h\} = \alpha_3 h + h\alpha_3$.)

Expressing the left-hand sides of the last two equalities via (6.27) and then using the commutator-anticommutator relation $[\alpha, h] = [\beta\gamma, h] = \beta\{\gamma, h\} - \{h, \beta\}\gamma$, valid whenever $\alpha = \beta\gamma$ (here $\beta = \alpha_2$, $\gamma = \alpha_3$), as well as using the obvious composition relations $\alpha(\xi \wedge \xi') = (\alpha\xi) \otimes \xi' - (\alpha\xi') \otimes \xi$, $(\xi \wedge \xi')\alpha = \xi' \otimes (\alpha\xi) - \xi \otimes (\alpha\xi')$, we obtain $[\alpha, h] = 4\,[d\lambda \otimes \xi + \xi \otimes d\lambda]$. Consequently, in view of (20.10) and (20.9), w is a Killing field. Furthermore, if $w = 0$, i.e., λ is locally constant in M', then $u_1 = u_2 = u_3 = 0$ by (20.6), (6.30) and (20.7), and hence $\nabla W^+ = 0$ in view of Lemma 6.18(a). This proves (ii) for M replaced by any connected component of M'.

Finally, to establish (i), suppose that $W^+(x) \neq 0$ at some fixed $x \in M$. Since every point can be joined to x by a broken geodesic, the conclusion that $W^+ \neq 0$ everywhere will follow immediately if we show that, for any geodesic $[a, b] \ni t \mapsto x(t) \in M$ with $W^+(x(a)) \neq 0$, we have $W^+(x(t)) \neq 0$ for all $t \in [a, b]$. To this end, let b' be the supremum of those $c \in [a, b]$ with $W^+(x(t)) \neq 0$ for all $t \in [a, c]$. Since $[a, b') \ni t \mapsto w(x(t))$ with w given by (20.3) (with a suitably chosen sign) is a solution to the Jacobi equation (4.51), it has a limit as $t \to b'$, and so it is bounded. Thus, by (20.3) and (6.27), setting $\mu(t) = \lambda(x(t))$ we have $\mu^{-4/3}|d\mu/dt| \leq 3\varepsilon$ with some constant $\varepsilon > 0$. Hence $|d\mu^{-1/3}/dt| \leq \varepsilon$ and so $\mu^{-1/3}$ is bounded on $[a, b')$, i.e., the limit of $\lambda(x(t))$ as $t \to b'(-)$ is positive. (The limit exists since, by (20.5), λ is a constant multiple of $|W^+|$.) Due to the supremum definition of b' we thus have $b' = b$, i.e., $W^+(x(t)) \neq 0$ for all $t \in [a, b]$. This completes the proof. ■

Remark 20.2. Assertion (i) is known to follow from condition (20.2) for Riemannian 4-manifolds under various assumptions weaker than the Einstein condition (0.1); see, e.g., Derdziński (1988) and Apostolov (1997).

Given a pseudo-Riemannian manifold (M, g) and a point $x \in M$, let us denote $\mathfrak{g}_x$ the set of the pairs $(w(x), [\nabla w](x))$ obtained using all Killing fields w defined on all possible neighborhoods of x. Note that, for any nonempty connected open subset U of M and any $x \in M$, the formula

$$F_x(w) \,=\, (w(x), [\nabla w](x)) \tag{20.11}$$

defines, according to Remark 17.6(i), an *injective* mapping

$$F_x : \mathfrak{isom}(U,g) \to \mathfrak{g}_x\,. \tag{20.12}$$

Obviously, $\mathfrak{g}_x$ is the union of the images of the operators (20.12) as U runs through all connected neighborhoods of x in M. Intersecting the domains of the Killing fields in question, we see that $\mathfrak{g}_x$ is closed under addition, and so it is a vector space, with the vector-space inclusion

$$\mathfrak{g}_x \subset T_xM \times \mathfrak{so}(T_xM)\,, \tag{20.13}$$

while (20.12) is an *injective linear operator.*

Remark 20.3. Every point x in a pseudo-Riemannian manifold (M,g) has a connected neighborhood U such that (20.12) is a *linear isomorphism.* In fact, by (20.13), $\mathfrak{g}_x$ is finite-dimensional; any fixed basis of $\mathfrak{g}_x$ consists of F_x-images of Killing fields whose domains are connected neighborhoods of x, and we then may choose our U to be the connected component, containing x, of the intersection of these domains. For such U, the injective operator (20.12) is obviously surjective as well.

By the *degree of mobility* $\mathfrak{m}(x)$ of a pseudo-Riemannian manifold (M,g) at a point $x \in M$ we mean the dimension

$$\mathfrak{m}(x) \,=\, \dim\, \mathfrak{g}_x \tag{20.14}$$

of the vector space $\mathfrak{g}_x$ appearing in (20.12). We thus have defined a function

$$\mathfrak{m} : M \to \{0,1,2,\ldots,\, n(n+1)/2\}\,, \qquad n = \dim M\,. \tag{20.15}$$

In fact, (20.13) along with the inequality $\dim\,\mathfrak{so}(T_xM) = n(n-1)/2$ (see (3.31)) show that, at every point $x \in M$,

$$0 \,\le\, \mathfrak{m}(x) \,\le\, n(n+1)/2\,, \qquad n \,=\, \dim M\,. \tag{20.16}$$

Note that $\mathfrak{m}$ is also *lower semicontinuous*, that is, given a point $x \in M$, we have

$$\mathfrak{m}(x) \,\le\, \mathfrak{m}(y) \qquad \text{for all} \quad y \ \text{ near } \ x\,. \tag{20.17}$$

To see this, choose U as in Remark 20.3. Thus, for each $y \in U$, the operator F_y defined by (20.11) is injective (Remark 17.6(i)), while for $y = x$ it is an isomorphism. Hence $\mathfrak{m}(x) = \dim\,[\mathfrak{isom}(U,g)] \le \mathfrak{m}(y)$, as required.

By a *continuation domain* in a pseudo-Riemannian manifold (M,g) we will mean any nonempty connected open subset U of M such that every Killing field w' on any nonempty connected open set $U' \subset U$ can be extended to a Killing field w on U.

Remark 20.4. Given a pseudo-Riemannian manifold (M,g),

(i) For every continuation domain U and any $x \in U$, the operator (20.12) is a linear isomorphism.

(ii) A nonempty connected open subset U of M is a union of continuation domains if and only if the function $\mathfrak{m}$ is constant on U, and then

$$\mathfrak{m} \,=\, \dim\,[\mathfrak{isom}(U,g)]\,. \tag{20.18}$$

(iii) The union of *all* continuation domains in (M,g) is a dense open subset of M.

In fact, (i) is clear since (20.12) is always injective. The 'only if' assertion of (ii) is obvious from (i) and (20.14). As for the 'if' part, fix $x \in U'$ and choose a connected neighborhood of x with the property stated in Remark 20.3, calling it U' (rather than U). In view of Remark 17.6(i), that property of U' will still hold if U' is replaced by a smaller, connected neighborhood of x. Hence we may assume that $U' \subset U$. The operators $F_y : \mathfrak{isom}(U', g) \to \mathfrak{g}_y$ defined by (20.11) are injective for all $y \in U'$ (Remark 17.6(i)) and act between spaces of the same dimension $\mathfrak{m}$ (since this is the case for $y = x$, due to our choice of U'). Therefore, all these F_y are isomorphisms, i.e., U' is a continuation domain, as required. Finally, to establish (iii), all we need to verify is that every nonempty open set $U' \subset M$ intersects some continuation domain U. To see this, let us choose $x \in U'$ at which $\mathfrak{m}$ assumes its maximum value in U' (cf. (20.16)). Hence, by (20.17), $\mathfrak{m}$ is constant on some neighborhood U'' of x contained in U'. Thus, according to (ii), there is a continuation domain U with $x \in U \subset U''$.

Let us again consider a fixed nonempty connected open set U in a pseudo-Riemannian manifold (M, g), and let $\mathfrak{g} = \mathfrak{isom}(U, g)$ denote the Lie algebra of all Killing fields on U. For any vector subspace $\mathfrak{h}$ of $\mathfrak{g}$, we define the $\mathfrak{h}$-*orbit* at any point $x \in U$ to be the vector space

(20.19) $$\mathfrak{h}[x] = \{v(x) : v \in \mathfrak{h}\} \subset T_x M\,.$$

The number

(20.20) $$\mathfrak{s}(x) = \dim \mathfrak{h}[x]$$

Then will be referred to as the $\mathfrak{h}$-*orbit dimension* at x. This defines the $\mathfrak{h}$-*orbit dimension function*

(20.21) $$\mathfrak{s} : U \to \{0, 1, 2, \ldots, n\}\,, \qquad n = \dim M\,.$$

A point $x \in U$ will be called $\mathfrak{h}$-*generic* if this function $\mathfrak{s}$ is constant in some neighborhood of x.

Remark 20.5. Since linear independence of continuous vector fields is an open condition, the functions $\mathfrak{s}$ with (20.20) for any given $\mathfrak{h}$ (including the special case $\mathfrak{o}$ given by (20.23) below) is lower semicontinuous (cf. (20.17)). Consequently, the open set U' of all $\mathfrak{h}$-generic points in U then is dense in U. This follows from the same argument as the one we used to establish Remark 20.4(iii), with $\mathfrak{m}$ is replaced by $\mathfrak{s}$.

In the case where $\mathfrak{h}$ is the whole space $\mathfrak{g} = \mathfrak{isom}(U, g)$, we will skip the prefix '$\mathfrak{g}$-' and simply speak of the *orbit dimension function* of U, which we will denote

(20.22) $$\mathfrak{o} : U \to \{0, 1, 2, \ldots, n\}\,, \qquad n = \dim M\,.$$

Thus,

(20.23) $$\mathfrak{o}(x) = \dim \mathfrak{g}[x]\,, \qquad \mathfrak{g}[x] = \{v(x) : v \in \mathfrak{isom}(U, g)\}\,.$$

For U and $\mathfrak{h}$ as above, we then have, at every point of U,

(20.24) $$0 \le \mathfrak{s} \le \mathfrak{o} \le \min(n, \mathfrak{m})\,, \qquad n = \dim M\,.$$

In fact, each $\mathfrak{h}[x]$ is contained both in $T_x M$ and in the image of $\mathfrak{g}_x$ under the projection $\mathrm{pr} : \mathfrak{g}_x \to T_x M$ given by $\mathrm{pr}\,(w(x), [\nabla w](x)) = w(x)$ (notation as in (20.11)).

Lemma 20.6. *Suppose that* U *is a nonempty connected open subset of a pseudo-Riemannian manifold* (M,g)*,* $\mathfrak{h}$ *is a vector subspace of* $\mathfrak{isom}(U,g)$*, and* $\mathfrak{s}$ *is the* $\mathfrak{h}$*-orbit dimension function with* (20.20)*,* (20.21)*. If* $\mathfrak{s}=\varepsilon$ *everywhere in* U*, with a constant* $\varepsilon\in\{0,1\}$*, then* $\dim\mathfrak{h}=\varepsilon$*.*

This is clear since, by (17.2), for a Killing field w and a C^1 function f, the product fw is *not* a Killing field unless f is constant or $w=0$ identically. ■

Applied to $\mathfrak{h}=\mathfrak{isom}(U,g)$, Lemma 20.6 yields

Corollary 20.7. *Let* U *be a nonempty connected open subset of a pseudo-Riemannian manifold* (M,g)*, and let* $\mathfrak{g}=\mathfrak{isom}(U,g)$*. If the orbit dimension function* $\mathfrak{o}$ *given by* (20.23) *satisfies* $\mathfrak{o}=\varepsilon$ *on* U *for some constant* $\varepsilon\in\{0,1\}$*, then* $\dim\mathfrak{g}=\varepsilon$*.* ■

Remark 20.8. An n-dimensional pseudo-Riemannian manifold (M,g) such that, for some vector subspace $\mathfrak{h}$ of $\mathfrak{isom}(U,g)$, the $\mathfrak{h}$-orbit dimension function $\mathfrak{s}$ satisfies $\mathfrak{s}=n$ everywhere in M, must be locally homogeneous. This is immediate from Lemma 17.20.

We also have the following result (see Derdziński, 1983, Lemma 9)

Lemma 20.9. *Let* (M,g) *be a Riemannian four-manifold such that the degree of mobility and orbit dimension functions* $\mathfrak{m}$ *and* $\mathfrak{o}$*, given by* (20.14) *and* (20.23) *for* $U=M$*, satisfy*

$$\mathfrak{m}>\mathfrak{o} \tag{20.25}$$

at every point. Then $\#\operatorname{spec}W^+\le 2$ *and* $\#\operatorname{spec}W^-\le 2$ *for any local orientation of* M*, i.e.,* (M,g) *satisfies condition* (20.2) *for both local orientations.*

Note that condition (20.25) *amounts to requiring that, for each* $x\in M$*, there exist a Killing field* w*, defined in a neighborhood of* x*, which is not identically zero and vanishes at* x*.*

Proof. By (20.24), all we need to do is assume that $\#\operatorname{spec}W^+=3$ everywhere in some orientable open submanifold U of M, for one fixed orientation, and show that we then must have $\mathfrak{m}=\mathfrak{o}$ somewhere in U. To do this, let us choose α_j, λ_j, ξ_j and u_j, $j=1,2,3$, satisfying (6.24), (6.12), (6.26) and (6.28) on a nonempty open subset of U. If the ξ_j were all identically zero, the α_j would be parallel and so, by (5.19), we would have $\lambda_j=\mathrm{s}/6$ for all p, contradicting the assumption that $\#\operatorname{spec}W^+=3$. Therefore, we have $\xi\ne 0$ on some nonempty open connected subset U' of U, where $\xi=\xi_j$ for some fixed j. Now, in view of (6.12), (6.27) and skew-symmetry of the α_j, the vector fields $u=\xi/|\xi|$ and $\alpha_j u$, $j=1,2,3$, form an orthonormal basis of T_xM at every point $x\in U'$.

On the other hand, the α_j and the ξ_j are unique up to permutations and sign changes, and so u and the $\alpha_j u$ must be invariant under the local isometries e^{tw} constituting the flow of any Killing field w in U' (Lemma 17.16). Therefore, a Killing field w cannot vanish somewhere in U' without being identically zero. In fact, if $x\in U'$ and $w(x)=0$, then, for all t, $d\,[e^{tw}]_x:T_xM\to T_xM$ must be the identity operator, as it keeps $u(x)$ and the $\alpha_j[u(x)]$ fixed; applying Remark 17.24 to the isometries $e^{tw}:U\to U$, where $U\subset U'$ is a ball of a sufficiently small radius centered at x, we see that $e^{tw}=\mathrm{Id}$ on U, and hence $w=0$ on U. Consequently, the set of zeros of w in U' is both open and closed.

Thus, for any $x \in U'$, the assignment $\mathfrak{g}_x \ni (w(x), [\nabla w](x)) \mapsto w(x) \in \mathfrak{t}_x$, with $\mathfrak{g}_x$, $\mathfrak{t}_x$ as in (20.14), is an isomorphism and so, by (20.14), $\mathfrak{m} = \mathfrak{o}$ on U'. This contradiction completes the proof. ∎

§21. Degree of mobility: Possible values

In this section we continue the discussion of §20. Most results presented here go back at least six decades; see Petrov (1969), Chapters 4 and 5 (especially pp. 136–143), and references therein.

Lemma 21.1. *Let* $\mathfrak{h}$ *be a Lie subalgebra of* $\mathfrak{g} = \mathfrak{isom}(M, g)$ *for an* n*-dimensional pseudo-Riemannian manifold* (M, g) *such that all points of* M *are* $\mathfrak{h}$*-generic, with the constant* $\mathfrak{h}$*-orbit dimension function* (20.20) *given by* $\mathfrak{s} = n - 1$*, and let* u *be a* C^∞ *unit vector field on* M *normal, at every point* x*, to the* $\mathfrak{h}$*-orbit* $\mathfrak{h}[x]$ *defined by* (20.19)*. Then*

(i) $[u, v] = 0$ *for all* $v \in \mathfrak{h}$.

(ii) $du = 0$*, that is,* u *is closed when treated, with the aid of* g*, as a differential* 1*-form on* U.

Proof. To prove (i), let us note that u satisfies the conditions

$$\langle u, u\rangle = \pm 1 \qquad \text{and} \qquad \langle u, v\rangle = 0 \tag{21.1}$$

everywhere in M, for every $v \in \mathfrak{h}$ (where $\langle\,,\rangle$ stands for g). Now, using (4.4) and (17.1), we obtain, for any $v \in \mathfrak{h}$, $\langle [u, v], u\rangle = \langle \nabla_u v - \nabla_v u, u\rangle = -\langle \nabla_v u, u\rangle = 0$, which is zero in view of (21.1). On the other hand, for any two Killing fields $v, w \in \mathfrak{h}$, formula (17.27) gives $\langle [u, v], w\rangle = -\langle [v, u], w\rangle = \langle [v, w], u\rangle$ as $\langle u, w\rangle = 0$ by (21.1). Since $[v, w] \in \mathfrak{h}$, (21.1) thus yields $\langle [u, v], w\rangle = 0$. Now assertion (i) follows, since u and all $w \in \mathfrak{h}$ together span the tangent space $T_x M$ at each point x.

To establish (ii), note that the exterior derivative du is given by (2.16) (with $\xi(v) = \langle u, v\rangle$ for all vector fields v). As an obvious consequence of (21.1), (i) and the fact that $[v, w] \in \mathfrak{h}$, we thus have $(du)(u, v) = (du)(v, w) = 0$ for all $v, w \in \mathfrak{h}$. Hence $du = 0$, as required. ∎

Lemma 21.2. *Let* U *be a continuation domain in a Riemannian* 4*-manifold* (M, g)*, and let* $\mathfrak{h}$ *be a Lie subalgebra of* $\mathfrak{g} = \mathfrak{isom}(U, g)$ *such that all points of* U *are both* $\mathfrak{g}$*-generic and* $\mathfrak{h}$*-generic, with the constant values* $\mathfrak{o}$ *of the orbit dimension and* $\mathfrak{s}$ *of the* $\mathfrak{h}$*-orbit dimension. Let us also assume that*

$$\dim \mathfrak{g} = 4\,, \qquad \dim \mathfrak{h} = 3\,, \qquad \mathfrak{o} = 3\,, \qquad \mathfrak{s} = 2\,, \tag{21.2}$$

and let $w \in \mathfrak{g}$ *be a Killing field on* U *which commutes with* $\mathfrak{h}$ *and, along with* $\mathfrak{h}$*, spans* $\mathfrak{g}$*, i.e.,* $\mathfrak{g} = \mathfrak{h} + \mathbf{R}w$*. Then*

(a) *At every point* $x \in U$*,* w *is orthogonal to the* $\mathfrak{h}$*-orbit* $\mathfrak{h}[x] \subset T_x M$.

(b) w *is a steady-state field, as defined in* §19*. In other words,* $\xi = w/\langle w, w\rangle$ *is closed when treated, with the aid of* g*, as a differential* 1*-form on* U.

Proof. In view of (20.19), (20.20) and (21.2), for any given $x \in U$ we may choose $v \in \mathfrak{h} \smallsetminus \{0\}$ with $v(x) = 0$. Then $A = [\nabla v](x)$ is a skew-adjoint operator

$T_xM \to T_xM$ and $A \neq 0$ in view of Remark 17.6(i). Setting $w_0 = w(x)$, we now have $Aw_0 = 0$; in fact, in view of (4.4) with $v(x) = 0$, Aw_0 is the value at x of

$$\nabla_w v \;=\; \nabla_w v \,-\, \nabla_v w \;=\; [w, v]\,, \tag{21.3}$$

while $[w, v] = 0$ since w commutes with $\mathfrak{h}$. On the other hand, A leaves the plane $P = \mathfrak{h}[x] \subset T_xM$ invariant, as one sees replacing w in (21.3) by an arbitrary element of $\mathfrak{h}$ and using the fact that $\mathfrak{h}$ is closed under the Lie-bracket operation. Hence the plane $P^\perp \subset T_xM$ is A-invariant as well. Writing $w_0 = w_1 + w_2$ with $w_1 \in P$, $w_2 \in P^\perp$, we now have $Aw_1 = Aw_2 = 0$. Moreover, $w(x)$ along with the plane $P = \mathfrak{h}[x]$ span the three-dimensional $\mathfrak{g}$-orbit $\mathfrak{g}[x]$ (as $\mathfrak{g} = \mathfrak{h} + \mathbf{R}w$), so that $w(x) = w_0 \notin P$ and, consequently, $w_2 \neq 0$. Using the obvious fact that a nonzero skew-adjoint operator in a Euclidean plane must be injective, we now conclude that $A(P^\perp) = \{0\}$, and so A restricted to P is nonzero (as $A \neq 0$). This in turn implies injectivity of $A : P \to P$. Consequently, $w_1 = 0$, i.e., $w(x) = w_2 \in P^\perp$. This gives (a).

To prove (b), let us choose, locally in U, a C^∞ unit vector field u normal at each point x of its domain to the $\mathfrak{g}$-orbit $\mathfrak{g}[x] \subset T_xM$ defined in (20.23). For $\xi = w/\langle w, w\rangle$, the exterior derivative $d\xi$ is given by (2.16) (with $\xi(v) = \langle \xi, v\rangle$ for all vector fields v). We thus have $(d\xi)(u, w) = (d\xi)(u, v) = (d\xi)(v, w) = (d\xi)(v, v') = 0$ for all $v, v' \in \mathfrak{g}$, which is an obvious consequence of the relations $\langle \xi, u\rangle = 0$ (due to our choice of u), $\langle \xi, v\rangle = \langle \xi, v'\rangle = 0$ (from (a)), $\langle \xi, w\rangle = 1$, $[u, v] = [u, w] = 0$ (from Lemma 21.1(i)), $[v, w] = 0$ (from our hypothesis), and $\langle \xi, [v, v']\rangle = 0$ (from $[v, v'] \in \mathfrak{h}$ along with (a)). Hence $d\xi = 0$. This completes the proof. ∎

Lemma 21.3. *Let $\mathfrak{h}$ be a three-dimensional vector space of C^∞ vector fields on a surface Σ, and let there exist a pseudo-Riemannian metric h on Σ such that every $v \in \mathfrak{h}$ is a Killing field for (Σ, h). Then*

(a) *The space $\mathfrak{h}$ determines such a metric h uniquely up to a constant factor.*
(b) *The Gaussian curvature of (Σ, h) is constant.*
(c) $\mathfrak{h} = \mathfrak{isom}(\Sigma, h)$.
(d) $T_x\Sigma = \{v(x) : v \in \mathfrak{h}\}$ *for every $x \in \Sigma$, i.e., (Σ, h) is infinitesimally homogeneous, as defined in §17.*

Proof. Fix $x \in \Sigma$. Since $\dim \mathfrak{so}(T_x\Sigma) = 1$ by(3.31), the restriction to $\mathfrak{h}$ of the injective operator $F_x : \mathfrak{isom}(\Sigma, h) \to T_x\Sigma \times \mathfrak{so}(T_x\Sigma)$, given by (20.12), must be a linear isomorphism. This implies (c), (d) as well as the existence, for any given $x \in \Sigma$, of a vector field $v \in \mathfrak{h}$ with $v(x) = 0$ and $[\nabla v](x) \neq 0$. Let us fix such a vector field v. In any fixed coordinate system x^j at x, $j = 1, 2$, we then have $v^j(x) = 0$ while, by (17.2), the 2×2 matrix $\mathfrak{B} = [v_{j,k}(x)]$ is nonzero and skew-symmetric, so that it must have the form $\mathfrak{B} = \lambda\mathfrak{C}$ for some real $\lambda \neq 0$ and the matrix $\mathfrak{C} = [c_{jk}]$ with $c_{12} = -c_{21} = 1$ and $c_{11} = c_{22} = 0$. On the other hand, $v_{j,k} = g_{js}v^s{}_{,k}$, which amounts to the matrix-product relation $\mathfrak{B} = \lambda\mathfrak{C} = \mathfrak{G}\mathfrak{A}$ for the 2×2 matrices $\mathfrak{A} = [v^j{}_{,k}(x)] = [(\partial_k v^j)(x)]$ (see (4.12)) and $\mathfrak{G} = [g_{jk}(x)]$. Since $\lambda\mathfrak{C}$ is invertible, so is $\mathfrak{A}$, and we have $\mathfrak{G} = \lambda\mathfrak{C}\mathfrak{A}^{-1}$. This yields (a). Finally, (b) is immediate from (d) in view of Remark 20.8, which completes the proof. ∎

Lemma 21.4. *Using the ranges of indices given by*

$$j, k \,\in\, \{1, 2\}\,, \qquad a, b \,\in\, \{3, 4\}\,, \tag{21.4}$$

let us suppose that (M,g) *is a pseudo-Riemannian* 4*-manifold forming the domain of a coordinate system* $x^1,\ldots,x^4$ *such that the component functions of* g *satisfy*

$$g_{ja} = 0\,, \tag{21.5}$$

and let (M,g) *admit a three-dimensional vector space* $\mathfrak{h}$ *of Killing fields such that every* $v\in\mathfrak{h}$ *satisfies the component relations*

$$v^a = 0\,, \tag{21.6}$$

i.e., is tangent, at each point, to the span of the first two coordinate directions. Then

- (i) *There exist functions* h_{jk} *of the variables* x^1,x^2 *and a function* ϕ *of* x^3,x^4*, such that* $g_{jk} = \phi(x^3,x^4)\,h_{jk}(x^1,x^2)$.
- (ii) *The components* g_{ab} *are functions of* x^3,x^4 *alone, that is,* $\partial_j\, g_{ab}=0$.
- (iii) *The metric* $\tilde g$ *conformally related to* g*, given by* $\tilde g = e^{2f}g$ *with* f *such that* $\varphi(x^3,x^4)=e^{-2f}$ *is, locally, the Riemannian product of two pseudo-Riemannian surface metric, such that* x^1,x^2 *and, respectively,* x^3,x^4 *are coordinates for the factor surfaces.*
- (iv) *The first factor metric of the Riemannian product in* (iii)*, with the components* h_{jk}*, has a constant Gaussian curvature.*

Proof. Let us fix any $v\in\mathfrak{h}$. By (21.6) and (21.5), we have $v_a = 0$. Hence, combining (4.20) with (21.6) and (17.2), we obtain $0 = v_{a,b}+v_{b,a} = -\,2\Gamma_{abj}v^j$ and $0 = v_{a,j}+v_{j,a} = \partial_a v_j - 2\Gamma_{ajk}v^k = g_{jk}\partial_a v^k + [\partial_a g_{jk} - 2\Gamma_{ajk}]\,v^k$. Combined with (4.9) and (21.5), these two relations give

$$v^j\,\partial_j g_{ab} = 0\,, \qquad g_{jk}\,\partial_a v^k = 0\,. \tag{21.7}$$

As $\det[g_{jk}]\neq 0$, the last equality implies that the components v^j of any $v\in\mathfrak{h}$ satisfy $\partial_a v^j = 0$, that is, are functions of x^1,x^2. Combined with (21.6), this allows us to treat $\mathfrak{h}$ as a three-dimensional vector space of vector fields on a surface $\Sigma\subset M$ obtained by arbitrarily fixing the values of x^3 and x^4. According to Example 17.3(a), all $v\in\mathfrak{h}$ then are Killing fields on (Σ,h), where h is the metric on Σ whose components in the coordinates x^1,x^2 are g_{jk} (with x^3,x^4 fixed). The first equality in (21.7), combined with Lemma 21.3(d) for any such Σ (that is, any fixed x^3,x^4) now implies assertion (ii), while (i) and (iv) are immediate from Lemma 21.3(a), (b) (where h_{jk} is defined to be g_{jk} with fixed x^3,x^4). Finally, (iii) is obvious from (i) and (ii), which completes the proof. ■

Lemma 21.5. *Suppose that* $\mathfrak{z}$ *is a one-dimensional ideal in a three-dimensional Lie algebra* $\mathfrak{g}$ *such that the quotient Lie algebra* $\mathfrak{q} = \mathfrak{g}/\mathfrak{z}$ *has a basis* u_1,u_2,u_3 *with*

$$[u_1,u_2] = \delta u_3\,,\quad [u_2,u_3] = u_1\,,\quad [u_3,u_1] = u_2 \tag{21.8}$$

for some $\delta\in\{-1,0,1\}$*, while* $[\mathfrak{z},\mathfrak{g}]=\{0\}$*, in the sense that* $[v,w]=0$ *whenever* $v\in\mathfrak{g}$ *and* $w\in\mathfrak{z}$*.*

(i) *If* $\delta=\pm1$*, then* $\mathfrak{g}$ *has a basis* e_0,e_1,e_2,e_3 *with*

$$e_0\in\mathfrak{z}\,,\qquad [e_0,e_1] = [e_0,e_2] = [e_0,e_3] = 0 \tag{21.9}$$

and

$$[e_1,e_2] = \delta e_3\,,\quad [e_2,e_3] = e_1\,,\quad [e_3,e_1] = e_2\,. \tag{21.10}$$

(ii) *If* $\delta = 0$*, then* $\mathfrak{g}$ *has a basis* e_0,e_1,e_2,e_3 *which satisfies condition* (21.9) *and either* (21.10)*, or*

$$[e_1,e_2] = e_0\,,\quad [e_2,e_3] = e_1\,,\quad [e_3,e_1] = e_2\,. \tag{21.11}$$

In either case, $\mathfrak{g}$ *contains a unique three-dimensional ideal* $\mathfrak{h}$*, spanned by* e_1*,* e_2*,* e_3 *in case* (21.10)*, and by* e_1*,* e_2*,* e_0 *when we have* (21.9) *and* (21.11)*.*

Proof. Denoting $\mathrm{pr}:\mathfrak{g}\to\mathfrak{q}$ the quotient projection, we may choose a basis e_0,e_1,e_2,e_3 with $e_0\neq 0$ satisfying (21.9) and $\mathrm{pr}\,e_j = u_j$, $j=1,2,3$. Hence, by (21.8), $[e_1,e_2]=\delta e_3 + a_3e_0$, $[e_2,e_3]=e_1+a_1e_0$, $[e_3,e_1]=e_2+a_2e_0$ for some real numbers a_1,a_2,a_3. Let us set $\delta_1=\delta$, $\delta_2=\delta_3=1$. Replacing each e_j, $j=1,2,3$, by $e_j+\delta_ja_je_0$ if $\delta_j\neq 0$ (and leaving it unchanged when $\delta_j=0$), we now obtain (21.9) and (21.10) in the cases where either $\delta\neq 0$, or $\delta=a_3=0$. Finally, when $\delta=0$ and $a_3\neq 0$, we can obtain (21.9) and (21.11) if, in addition, we replace e_0 by a_3e_0. This completes the proof. ■

The following result may be viewed as a crude classification theorem for Lie algebras of all germs of Killing fields in Riemannian Einstein of dimension four. (See also Remark 21.7 below.) The notion of a $\mathfrak{h}$-generic point appearing in the statement was defined immediately before Remark 20.5.

Proposition 21.6. *Given a Riemannian Einstein* 4*-manifold* (M,g)*, let* U *be a continuation domain in* (M,g) *such that all points of* U *are* $\mathfrak{g}$*-generic, with* $\mathfrak{g}=\mathfrak{isom}(U,g)$*; according to* Remarks 20.4(iii) *and* 20.5*, the union of such* U *is an open dense subset of* M*. The functions* $\mathfrak{s}$ *and* $\mathfrak{o}$ *given by* (20.20) *and* (20.23) *thus are both constant on* U*. The constant value of the pair* $(\mathfrak{m},\mathfrak{o})$ *on* U *then must be one of*

$$(10,4)\,,\quad (8,4)\,,\quad (6,4)\,,\quad (4,3)\,,\quad (3,3)\,,\quad (2,2)\,,\quad (1,1)\,,\quad (0,0)\,. \tag{21.12}$$

More precisely, one of the following three cases occurs.

(i) $\mathfrak{o}=4$ *and* (U,g) *is locally symmetric, i.e., locally isometric to one of*
  a) S^4, $\mathbf{R}^4$, H^4*, with* $\mathfrak{m}=10$*; or,*
  b) $\mathbf{C}\mathrm{P}^2$, $(\mathbf{C}\mathrm{P}^2)^*$*, with* $\mathfrak{m}=8$*; or,*
  c) $S^2\times S^2$*,* $H^2\times H^2$ *with* $\mathfrak{m}=6$*,*

  each carrying a constant multiple of its standard Einstein metric obtained as in Example 10.3*,* 10.5 *or* 10.6*.*

(ii) $(\mathfrak{m},\mathfrak{o})=(4,3)$*. Then the Lie algebra* $\mathfrak{g}=\mathfrak{isom}(U,g)$ *contains a unique* 3*-dimensional ideal* $\mathfrak{h}$*. Let* U' *now be any subset of* U *consisting of* $\mathfrak{h}$*-generic points*; *by* Remark 20.5*, the union of such* U' *is an open dense subset of* U*. If we denote* $\mathfrak{s}$ *the constant value of the* $\mathfrak{h}$*-orbit dimension function on* U'*, only two subcases are possible*:
  a) $\mathfrak{s}=3$; *see* Remark 21.7 *below, or,*
  b) $\mathfrak{s}=2$ *and* g *restricted to* U' *is a Kottler metric, cf.* Remark 18.11*, while* $\mathfrak{h}$ *has a basis* e_1,e_2,e_3 *satisfying* (21.10) *for some* $\delta\in\{-1,0,1\}$*.*

(iii) $\mathfrak{m}=\mathfrak{o}\leq 3$; *see* Remark 21.7*.*

Proof. If $\mathfrak{o} = 4$, (i) is obvious from Remark 20.8 combined with Jensen's theorem (Corollary 7.3), Theorem 14.7 and Proposition 17.18. On the other hand, Corollary 20.7 along with (20.18) implies that $\mathfrak{m} = \mathfrak{o}$ whenever $\mathfrak{o} \le 1$. Since, by (20.24), we always have $\mathfrak{o} \le 4$ and $\mathfrak{o} \le \mathfrak{m}$, our assertion thus has already been established whenever $\mathfrak{o} = \mathfrak{m}$, and the only remaining case is

$$\mathfrak{o} \in \{2, 3\}\,, \qquad \text{with} \quad \mathfrak{m} > \mathfrak{o}\,. \tag{21.13}$$

Assume (21.13). By Lemma 20.9 and Proposition 20.1, a nonempty open subset of U admits a Killing field $\pm w$, which is defined *explicitly*, up to a sign, by formula (20.3), and hence is invariant under the local isometries e^{tv} constituting the flow of any Killing field v (Lemma 17.16). Note that $w \ne 0$ for at least one choice of the orientation; otherwise, Proposition 20.1(ii)b) would give $\nabla W^+ = 0$ for both orientations, i.e., $\nabla W = 0$, and so (M, g) would be locally symmetric, with $\mathfrak{o} = 4$ (Example 17.19), contradicting (21.13).

The Riemannian metric g now gives rise to a quotient metric g' on the 3-manifold N obtained, locally, as the quotient of U modulo w (so that the points of N are suitable short segments of integral curves of w), and the action of the flow e^{tv} of any Killing field v in U obviously descends to the quotient N. The resulting "quotient flow" in N leaves invariant the quotient metric g' on N (defined in as in the paragraph preceding Example 10.6), and hence, by Lemma 17.16, it constitutes the flow of a Killing field in N. This gives rise to a linear operator Φ from $\mathfrak{g} = \mathfrak{isom}(U, g)$ into $\mathfrak{isom}(N, g')$, the kernel of which is clearly spanned by w.

If we now had $\mathfrak{o} = 2$, applying Lemma 20.6 to the subspace $\Phi(\mathfrak{g})$ of $\mathfrak{isom}(N, g')$ (with the $\Phi(\mathfrak{g})$-orbit dimension function equal to 1), we would conclude that $\dim[\Phi(\mathfrak{g})] = 1$ and hence (cf. (20.18)) $\mathfrak{m} = \dim \mathfrak{g} = 2 = \mathfrak{o}$, which contradicts (21.13). Hence $\mathfrak{o} \ne 2$ and so, by (21.13), $\mathfrak{o} = 3$.

Let us now choose, locally in U, a C^∞ unit vector field u normal, everywhere in its domain, to the $\mathfrak{g}$-orbits given by (20.23). (As we just saw, they are all 3-dimensional). By Lemma 21.1(ii) and Poincaré's Lemma (Corollary 11.3), u is, locally, the gradient of some C^∞ function f. Any fixed nonempty level set $f^{-1}(c)$ of f thus is a 3-dimensional submanifold of U and, since $u = \nabla f$ is, at each point, both normal to $f^{-1}(c)$ and orthogonal to every Killing field $v \in \mathfrak{g}$, it follows that every $v \in \mathfrak{g}$ is tangent to $f^{-1}(c)$ at all points of $f^{-1}(c)$. Restricting each $v \in \mathfrak{g}$ to $f^{-1}(c)$, we thus obtain a Killing field on $f^{-1}(c)$ with the submanifold metric (Example 17.3). However, this restriction procedure is injective (Lemma 17.7), and so it gives rise to a vector space $\mathfrak{g}'$ of Killing fields on the 3-manifold $f^{-1}(c)$ with $\dim \mathfrak{g}' = \dim \mathfrak{g} = \mathfrak{m}$. The operation of forming the quotient modulo w described above now can be applied to $f^{-1}(c)$ with its metric and the space $\mathfrak{g}'$ of Killing fields, resulting in some quotient surface $\varSigma$ with a quotient metric h and a linear operator Φ, defined as before, from $\mathfrak{g}'$ into $\mathfrak{isom}(\varSigma, h)$, the kernel of which is again spanned by w. Since $\mathfrak{m} \ge 4$ (by (21.13) with $\mathfrak{o} = 3$), the image $\Phi(\mathfrak{g}') \subset \mathfrak{isom}(\varSigma, h)$ is of dimension $\mathfrak{m} - 1 \ge 3$ while, by (17.6) with $n = 2$, $\dim[\mathfrak{isom}(\varSigma, h)] \le 3$. Hence $\mathfrak{m} - 1 = 3$ and so $(\mathfrak{m}, \mathfrak{o}) = (4, 3)$. Thus, $\Phi(\mathfrak{g}') = \mathfrak{isom}(\varSigma, h)$ is 3-dimensional, and so the Gaussian curvature κ of the Riemannian surface $(\varSigma, h)$ is constant. (In fact, $(\varSigma, h)$ is infinitesimally homogeneous, as defined in §17, since in the case of $(\varSigma, h)$ the injective linear operator (17.5) must, for dimensional reasons, be surjective as well; that in turn implies constancy of κ, either by Lemma 17.4 and Remark 10.1, or as a consequence of Lemma 17.20.)

Since $\Phi(\mathfrak{g}')$ is Lie-algebra isomorphic to the quotient $\mathfrak{g}/\mathfrak{z}$ with $\mathfrak{g} = \mathfrak{isom}(U, g)$ and $\mathfrak{z} = \mathbf{R}w$, Remark 17.10 shows that the pair $\mathfrak{g}$, $\mathfrak{z}$ satisfies the hypotheses of Lemma 21.5. The conclusion of Lemma 21.5 now states that $\mathfrak{g}$ represents one of four possible Lie-algebra isomorphism types and that it contains a unique 3-dimensional ideal $\mathfrak{h}$. For this ideal $\mathfrak{h}$, the $\mathfrak{h}$-orbit dimension function $\mathfrak{s}$, with $0 \le \mathfrak{s} \le 3$, is constant on a set U' chosen as in (ii) and, in view of Lemma 20.6, its value cannot be 0 or 1. Hence $\mathfrak{s} = 3$ or $\mathfrak{s} = 2$.

Since our assertion is true by default when $\mathfrak{s} = 3$, we may now assume that $\mathfrak{s} = 2$ on U'. We can choose u and f as above, defined on a neighborhood in U' of any given point $x \in U'$ and such that $f(x) = 0$. In view of Lemma 21.2(b) combined with the Poincaré Lemma (Corollary 11.3), we can also find a function t defined near x with $t(x) = 0$ and $\nabla t = w/\langle w, w\rangle$. Applying Lemma 2.6 to the submanifold $t^{-1}(0)$ (rather than M) and the function f restricted to it, we find coordinates x^2, x^3, x^4 in $t^{-1}(0)$ such that, near x in $t^{-1}(0)$, we have $x^2 = f$, $e_2 = u$ and $g(e_2, e_3) = g(e_2, e_4) = 0$. Applying Lemma 2.6 once again, this time to M itself and the function t, we can find local coordinates x^1, x^2, x^3, x^4 near x in M such that in on the submanifold $t^{-1}(0)$ the "new" coordinate functions x^2, x^3, x^4 are the same as the old ones, and $g_{12} = g_{13} = g_{14} = 0$ everywhere in the coordinate domain. Also, the coordinate vector field e_1 now coincides with w, which also satisfies (2.36). Since $w = e_1$ is a Killing field, according to Example 17.1 we have $\partial_1 g_{jk} = 0$ for all j, k. Thus, since relations $g_{23} = g_{24} = 0$ hold wherever $x^1 = 0$, they must hold everywhere in the coordinate domain. Furthermore, the coordinate vector field e_2 now coincides with u, also on the whole coordinate domain. To see this, we may use the component characterization (2.1) of $cvf2$, so that all we need to show is the equalities $u^2 = 1$ and $u^1 = u^3 = u^4 = 0$. However, due to our choice of the coordinates x^2, x^3, x^4 in $t^{-1}(0)$, we already have these equalities at points with $x^1 = 0$ (with $u^1 = 0$ there since u is tangent to the submanifold $t^{-1}(0)$, given by $x^1 = 0$). On the other hand, since u commutes with the Killing field $w = e_1$ (see Lemma 21.1(i)), its components are, by (2.5), locally constant in the direction of x^2. We have thus shown that $u = e_2$. The span of the coordinate vector fields e_3, e_4 at any point y near x now must coincide with the $\mathfrak{h}$-orbit $\mathfrak{h}[y] \subset T_yM$, since both are 2-dimensional and orthogonal to u and w: The former, since (as we have seen) $g_{ja} = 0$ for $j = 1, 2$ and $a = 3, 4$, and the latter in view of our choice of u and Lemma 21.2(a). Consequently, Lemma 21.4 may be applied to $\mathfrak{h}$, giving assertion (ii)b) (the statement on the structure of $\mathfrak{h}$ being immediate from Lemma 21.3(c), along with Lemma 21.3(b) and Remark 17.10). This completes the proof. ■

Remark 21.7. The meaning of Proposition 20.14 can be summarized as follows. Given a continuation domain U in a Riemannian 4-manifold (M, g), and a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g} = \mathfrak{isom}(U, g)$, one says that $\mathfrak{h}$ *acts locally freely on* U if the $\mathfrak{h}$-orbit dimension function $\mathfrak{s}$ (see (20.19), (20.20)) is constant on U (i.e., all points of U are $\mathfrak{h}$-generic) and, in addition, $\mathfrak{s} = \dim \mathfrak{h}$. Each of the (sub)cases in Proposition 20.14 contains a conclusion stating that either

(a) The metric is explicitly known, or
(b) Some Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ acts locally freely on the open set in question. (In this case we also have $\mathfrak{h} = \mathfrak{g}$ unless $\dim \mathfrak{h} = 3$ and $\dim \mathfrak{g} = 4$.)

The phrase 'explicitly known' refers to classes of metrics for which a complete local classification was provided; namely, by Theorem 14.7 for locally symmetric Einstein metrics (case (i) of Proposition 20.14), and by Remark 18.11 for Kottler metrics (case (ii)b) of Proposition 20.14).

Of course, Proposition 20.14 does not include a classification for case (b) above. It is clear, however, that case (b) leads to large families of metrics, parametrized by solutions to some system of partial differential equations in $4-\mathfrak{s}$ real variables. (Again, $\mathfrak{s} = \dim \mathfrak{h}$.) When $\mathfrak{s} = 3$, these become ordinary differential equations.

§22. Einstein metrics conformal to Kähler metrics

This section is devoted to those oriented Riemannian Einstein 4-manifolds which are locally conformally Kähler in a manner compatible with the orientation (as defined below). The reasons why we take a look at them are threefold. First, this class includes all those (suitably oriented) Einstein manifolds which have a pseudogroup of local isometries whose dimension is greater than that of its orbits. (See Remark 22.6 below; the pseudogroup of isometries is there treated infinitesimally, that is, replaced with Killing fields.) In other words, the manifolds in question include a familiar and geometrically natural category. Second, they are characterized (among Riemannian Einstein 4-manifolds) by a pointwise algebraic condition imposed on the curvature tensor (see Proposition 22.4). Finally, we already encountered a special case of this situation, as explained next.

Specifically, in §18 we discussed Einstein metrics on 4-manifolds that are obtained by applying a conformal change (16.5) to a product of surface metrics. Such a construction turned out to be possible when the factor metrics were both extremal (Lemma 18.4).

Since oriented *Riemannian* surfaces constitute Kähler manifolds (Remark 18.7), so do their products. More precisely, the Riemannian product of two orientable Riemannian surfaces can be made into a Kähler manifold in a manner compatible with either orientation, using the Kähler forms α^+ and α^- defined as in (16.34). Thus, at least in the Riemannian case, we can now generalize the above idea by replacing product-of-surfaces metrics with (Riemannian) Kähler metrics. More precisely, we will say that an oriented pseudo-Riemannian manifold $(M, \tilde{g})$ (of any dimension) is *locally conformally Kähler* (as an *oriented* manifold) if every point of M has a neighborhood U with a C^∞ function $f : U \to \mathbf{R}$ and a C^∞ bivector field α defined on U, such that the triple (U, g, α), with the metric $g = e^{-2f}\tilde{g}$, is a Kähler manifold whose canonical orientation (§9) coincides with the original orientation. For positive-definite metrics in dimension four, the orientation condition amounts to requiring α to be a section of $\Lambda^+ U$ (cf. Corollary 9.4).

Our main interest here lies clarifying what being locally conformally Kähler means for oriented Riemannian Einstein manifolds of dimension four. (In the terminology of Remark 18.1, this is the "Einstein-metric end" of the question.) The answer turns out to be (at least in the case where $W^+ \neq 0$) the familiar eigenvalue condition (20.2), which follows the general pattern of simplicity at the Einstein-metric end, mentioned in Remark 18.1.

However, for completeness, we begin with a discussion of the "Kähler-metric end" of the problem. The presentation in this section follows Derdziński (1983).

Proposition 22.1. *Let (M, g, α) be a Riemannian Kähler manifold of real dimension four and let U be a nonempty connected open set in M with $W \neq 0$*

everywhere in U. *Then*

(a) (U,g) *locally conformally Einstein if and only if it satisfies the condition*

$$2\nabla d\mathrm{s} \,+\, \mathrm{s}\cdot\mathrm{Ric} \,=\, \phi\, g\,, \qquad \text{i.e.,} \qquad 2\,\mathrm{s}_{,jk} \,+\, \mathrm{s}\, R_{jk} \,=\, \phi\, g_{jk}\,, \tag{22.1}$$

for some function $\phi : U \to \mathbf{R}$, *where* s *is the scalar curvature of* g.

(b) *An Einstein metric on* U *conformally related to* g, *if it exists, must, up to a constant factor, be given by*

$$\tilde g \,=\, g/\mathrm{s}^2\,. \tag{22.2}$$

Proof. Condition $W^+ \neq 0$ on U, for the canonical orientation, amounts to $\mathrm{s} \neq 0$ on U (cf. Corollary 9.9). Assertion (b) now follows from Lemma 16.8 combined with Lemma 5.2. As a consequence of (b), (U,g) locally conformally Einstein if and only if (22.2) is an Einstein metric, that is, its Ricci tensor $\widetilde{\mathrm{Ric}}$ equals a function times g. Therefore, (a) is immediate from (16.13) with $n = 4$ and $\varphi = \mathrm{s}$. This completes the proof. ∎

Remark 22.2. Relation (22.1) implies, by contraction, $4\phi \,=\, \mathrm{s}^2 + 2\,\Delta\mathrm{s}$. Another consequence of (22.1) is that

$$\mathrm{s}^3 \,+\, 6\,\mathrm{s}\,\Delta\mathrm{s} \,-\, 12\, g(\nabla\mathrm{s},\nabla\mathrm{s}) \,=\, q \tag{22.3}$$

for some real constant q. Although this can be verified directly (using the contracted Ricci-Weitzenböck formula (4.39) and the Bianchi identity (5.2)), it also follows from the fact that, according to Schur's Theorem 5.1, the scalar curvature of any Einstein metric is constant. In fact, in the open set U where $\mathrm{s} \neq 0$, q is nothing else than the scalar curvature of the Einstein metric (22.2) (as one sees contracting (16.14) with $n = 4$ and $\varphi = \mathrm{s}$). Also, the left-hand side of (22.3) equals zero in the interior U' of the set of points in M at which $\mathrm{s} \neq 0$. Since the union of the sets U and U' is obviously dense in M (while one of them may be empty) and, as we just saw, the function q defined by (22.3) satisfies $dq = 0$ everywhere in $U \cup U'$, we have $dq = 0$ identically on M, i.e., q is constant.

Remark 22.3. If a Riemannian Kähler manifold (M,g,α) of real dimension 4 satisfies (22.1), then

(i) The vector field $\nabla\mathrm{s}$ holomorphic, i.e., g is an extremal Kähler metric in the sense of Calabi (1982); see also Remark 18.8.

(ii) $\alpha(\nabla\mathrm{s})$ is a Killing vector field on (M,g).

In fact, (i) amounts to the claim that $\nabla d\mathrm{s}$ commutes with α, which in turn is clear from (22.1), since so do g (i.e., Id) and Ric (by (9.6)). Now (ii) is an obvious consequence of Lemma 17.11.

We now proceed to study the "Einstein-metric end" of the question mentioned above.

Proposition 22.4. *Let* $(M,\tilde g)$ *be an oriented Riemannian Einstein* 4*-manifold whose self-dual Weyl tensor* $\tilde W^+$ *is not identically zero. Then, the following two conditions are equivalent*:

(i) $(M,\tilde g)$ *is locally conformally Kähler*;

(ii) *Condition* (20.2), *that is,* $\#\,\mathrm{spec}\,\tilde W^+ \,\leq\, 2$, *is satisfied at every point of the oriented manifold* $(M,\tilde g)$.

Furthermore, if (i) *or* (ii) *is satisfied, then* $\tilde W^+ \neq 0$ *everywhere and a Kähler metric* g *conformally related to* $\tilde g$ *is, locally, unique up to a constant factor and, up to a factor, must be given by*

$$g = [24\,\tilde g(\tilde W^+, \tilde W^+)]^{1/3}\tilde g\,, \tag{22.4}$$

with notations analogous to (5.32)*, while a skew-adjoint* C^∞ *bundle morphism* $\pm\alpha : TM \to TM$ *serving as the multiplication by* i *for a Kähler manifold* (M, g, α) *is, locally, unique up to a sign and, at each point* $x \in M$*, the bivector corresponding to* $\pm\alpha(x)$ *via* $\tilde g$ *is an eigenvector of length* $\sqrt 2$ *for* $\tilde W^+(x)$ *associated with the unique simple eigenvalue* $\lambda(x)$*. The function* $\lambda : M \to \mathbf{R}$ *thus defined is of class* C^∞ *and*

$$\tilde g(\tilde W^+, \tilde W^+) = 3\,\lambda^2/2\,, \tag{22.5}$$

Finally, the scalar curvature s *of* g *is given by*

$$\mathrm{s} = (6\,\lambda)^{-1}\,, \tag{22.6}$$

so that relation (22.4) *is equivalent to*

$$\tilde g = g/\mathrm{s}^2\,,$$

where s *is the scalar curvature of* g*.*

Proof. Condition (20.2) is conformally invariant (Remark 16.4(e)) and holds for Kähler manifolds of real dimension 4 (by Corollary 9.9(a)). Therefore, (i) implies (ii). Conversely, let us assume (ii). Applying Proposition 20.1 to a metric which is now denoted $\tilde g$ (rather than g), let us choose, in a neighborhood U of any given point of M, a function λ and bivector field α described in assertion (ii) of Proposition 20.1. Treating α as a skew-adjoint bundle morphism $TU \to TU$ (with the aid of $\tilde g$), we now see that, according to (16.9), relation (20.4) states that α is ∇-parallel, where ∇ stands for the Levi-Civita connection of the metric $g' = (6\lambda)^{2/3}\tilde g$. We also have (22.5) (see Proposition 20.1) and so g' coincides with g given by (22.4), i.e., α is parallel in (M, g). Since $\alpha^2 = -1$ by the '(c) implies (a)' assertion in Lemma 9.3, this shows that (M, g, α) is a Kähler manifold, and hence proves (i).

Let us now consider any nonempty connected open set U in M with a C^∞ function f and a section α of $\Lambda^+ U$ such that, for $g = e^{-2f}\tilde g$, (M, g, α) is a Kähler manifold. With λ as above, (16.19) implies that the function $e^{2f}\lambda$ provides the unique simple eigenvalue of self-dual Weyl tensor W^+ of g at any point of U. Thus, by Corollary 9.9(a), $e^{2f}\lambda = \mathrm{s}/6$ (where s is the scalar curvature of g). Since the metric $g' = g = (6\lambda)^{2/3}\tilde g$ given by (22.4) is obtained in this way for f characterized by $e^{-2f} = (6\lambda)^{2/3}$, relations (22.6) and (22.2) follow.

On the other hand, the remaining uniqueness assertion now can be established as follows. For any f as above, since $\tilde g = e^{2f} g$ is an Einstein metric conformal to g, the uniqueness assertion of Proposition 22.1(b) shows that the function $e^{2f}\mathrm{s}^2 = (6e^{3f}\lambda)^2$ is constant. Thus, e^{2f} is, up to a constant factor, uniquely determined by λ, as required. ■

Remark 22.5. Propositions 22.1 and 22.4 give rise to a proof of Proposition 20.1 which is much more concise than the argument in §20. Specifically, the part of that proof consisting of the three paragraphs following formula (20.8) is devoted just to showing that w defined by (20.3) has the properties a), b) in the assertion of Proposition 20.1. This can also be established as follows. Under the assumptions of Proposition 20.1, let the symbols g and λ originally appearing in Proposition 20.1 be replaced with $\tilde{g}$ and λ, just as it is done in Proposition 22.4. The Kähler metric g given by (22.4) (wherever $\tilde{W}^+ \neq 0$) now is locally conformally Einstein, so that, by Proposition 22.1, it satisfies (22.1). Consequently, according to Remark 22.3(ii), $\alpha(\nabla \mathrm{s})$ is a Killing field in (M, g). However, in view of (22.2), the $\tilde{g}$-gradient of any C^1 function f is related to its g-gradient by $\tilde{\nabla} f = \mathrm{s}^2 \nabla f$, and so, by (22.6), $\alpha(\nabla \mathrm{s}) = \alpha(\tilde{\nabla}\mathrm{s}/\mathrm{s}^2) = \alpha[\tilde{\nabla}(\mathrm{s}^{-1})] = 6\alpha(\tilde{\nabla}\lambda)$. Hence w is a Killing field in (M, g), and so $d_w \mathrm{s} = 0$ (Lemma 17.4), which, according to Example 17.3(b) with (22.2), w is a Killing field for the original Einstein metric $\tilde{g}$ as well. Finally, if w is identically zero, so is $\alpha(\nabla \mathrm{s})$, so that s is constant. Therefore, in view of Corollary 9.9(b), W^+ is g-parallel, and hence $\tilde{W}^+$ is $\tilde{g}$-parallel, as $\tilde{g}$ now is a constant multiple of g (by (22.2)).

Remark 22.6. Suppose that $(M, \tilde{g})$ is an orientable Riemannian Einstein 4-manifold with the property that every point $x \in M$ has a connected neighborhood U with a Killing field w defined on U such that $w(x) = 0$ and $w \neq 0$ somewhere in U. Then, suitably oriented, $(M, \tilde{g})$ is locally conformally Kähler. In fact, by Lemma 20.9, we have $\#\operatorname{spec}\tilde{W}^+ \leq 2$ for either choice of an orientation and so our assertion follows from Proposition 22.4. (Note that, if $\tilde{W}^+$ and $\tilde{W}^-$ are both identically zero, then, by Theorem 16.5, $\tilde{g}$ is conformally flat, and hence still locally conformally Kähler.)

The remainder of this section is devoted to an alternative presentation of the above discussion, namely, in the context of an important conformal invariant known as the Bach tensor. Since the concepts discussed here will not be used elsewhere in the text, we omit computational details.

By the *Bach tensor* of a pseudo-Riemannian manifold (M, g) we mean the symmetric twice-covariant tensor field Bac with the local components B_{jk} given by

$$B_{jk} = W_{pjks,}{}^{ps} + \frac{1}{2} R^{ps} W_{pjks}\,. \tag{22.7}$$

(See Bach, 1921.) In dimension four, the Bach tensor of a metric $\tilde{g} = e^{2f} g$ conformally related to g then has the components

$$\tilde{B}_{jk} = e^{-2f} B_{jk}\,. \tag{22.8}$$

To prove (22.8), one can use a straightforward but tedious direct computation based on (16.7), (16.11), (16.12) and (16.17).

Proposition 22.7. *Vanishing of the Bach tensor* Bac *of a pseudo-Riemannian metric* g *on a* 4*-manifold* M *is a necessary condition in order that* g *be locally conformally related to an Einstein metric.*

In fact, according to (16.8), vanishing of Bac is a conformally invariant condition, while, by Lemma 5.2 and (5.25), $\mathrm{Bac} = 0$ identically whenever (M, g) is Einstein. ■

On the other hand, for a Kähler manifold of real dimension 4, condition $\mathrm{Bac} = 0$ is equivalent to (22.1) (see, e.g., Lemma 5 in Derdziński, 1983). This in a way explains "the real meaning" of the 'only if' part in Proposition 22.1(a).

§23. Potentials for Kähler-Einstein metrics

A Kähler metric g on a complex manifold is locally described by a single real-valued C^∞ function ϕ, called a *potential* for g. A sufficient condition for such a metric to be Einstein is provided by a single nonlinear second-order partial differential equation imposed on ϕ, namely, the *Monge-Ampère equation* (see (23.29) below). In the Ricci-flat case, the equation takes the much simpler form (23.30).

The construction just outlined can be used to produce a large variety of examples of Einstein metrics. Its detailed description is the subject of this section.

Our presentation follows standard sources such as Weil (1958) and Wells (1979).

Let M be a manifold. Besides ordinary "real" tangent and cotangent vectors (or vector fields) in M, it is sometimes convenient to use *complex,* or *complexified,* (co)tangent vectors at any point $x \in M$, that is, elements of the complexified (co)tangent space $[T_xM]^{\mathbf{C}}$ or $[T_x^*M]^{\mathbf{C}}$. Complexified tangent (or, cotangent) vectors at x thus should be thought of as formal combinations $v + iw$ (or, $\xi + i\eta$), where v, w (or, ξ, η) are ordinary ("real") tangent or, respectively, cotangent vectors at x. Note that complexified *cotangent* vectors $\xi + i\eta$ just described may also be identified with arbitrary real-linear functions $\xi + i\eta : T_xM \to \mathbf{C}$. An obvious example of a complexified cotangent vector field is the differential $\xi + i\eta = df$ of any complex-valued C^1 function f, with $\xi = d\,[\mathrm{Re}\, f]$, $\eta = d\,[\mathrm{Im}\, f]$.

The complexification $V^{\mathbf{C}} = V + iV$ of any real vector space V carries the anti-automorphism $u \mapsto \overline{u}$ of *complex conjugation*, with $\overline{v + iw} = v - iw$ for $v, w \in V$. Applied to complexified (co)tangent vectors at $x \in M$ as above this gives, for instance, $d\overline{f} = \overline{df}$ for complex-valued C^1 functions f, where $\overline{f}$ is the valuewise complex conjugate of f.

Let us also recall that a ρ times contravariant and σ times covariant tensor at a point x in any manifold M is a $(\rho + \sigma)$-linear real-valued function of σ tangent and ρ cotangent vectors. Every such tensor B now can be extended to a *complex-valued* function of σ complexified tangent and ρ complexified cotangent vectors, the extension being uniquely characterized by the requirement of complex-multilinearity. In this way, ordinary "real" tensors form a special case of a complex (or complexified) tensors just described. Among the complex tensors B, the real ones are characterized by $\overline{B} = B$, where

$$\overline{B}(v_{j_1}, \ldots, v_{j_\sigma}, \xi^{k_1} \ldots \xi^{k_\rho}) \,=\, \overline{B(\overline{v_{j_1}}, \ldots, \overline{v_{j_\sigma}}, \overline{\xi^{k_1}} \ldots \overline{\xi^{k_\rho}})}\,.$$

Remark 23.1. All natural multilinear operations involving tensor fields (of appropriate regularity) will from now on, without further comments, be also applied to complexified tensor fields, the extension being made unique by the requirement of complex (multi)linearity. This includes the natural pairing $\xi(v)$, directional derivative $d_v f$, Lie bracket $[v, w]$, exterior product $\xi \wedge \eta$, exterior derivatives $d\xi$ and $d\alpha$, as well as the inner product $g(v, w)$ and covariant derivative $\nabla_v w$, applied to C^1-differentiable complex 1-forms ξ, η, vector fields v, w, functions f, 2-forms α. In the last two examples, we use a given pseudo-Riemannian metric g on the manifold in question and its Levi-Civita connection ∇. Note that $g(v, w)$ then is *complex bilinear and symmetric* in v, w, rather than sesquilinear and Hermitian; the latter will be the case if we use the expression $g(v, \overline{w})$. Due to uniqueness of this extension, all algebraic relations valid in the real case still hold: For instance, $d_v f = (df)(v)$ for any f and v as above.

For the remainder of this section, we adopt the following conventions about ranges of indices:

$$\begin{aligned} &p, q, r, s \in \{1, \dots, m\}, \quad \bar{p}, \bar{q}, \bar{r}, \bar{s} \in \{\bar{1}, \dots, \bar{m}\}, \\ &j, k, l \in \{1, \dots, m, \bar{1}, \dots, \bar{m}\}, \end{aligned} \tag{23.1}$$

where $m \geq 1$ is a fixed integer, and the sets $\{1, \dots, m\}$ and $\{\bar{1}, \dots, \bar{m}\}$ are regarded as disjoint. The disjoint index sets $\{1, \dots, m\}$ and $\{\bar{1}, \dots, \bar{m}\}$ are *not* to be treated as unrelated; in other words, given $p \in \{1, \dots, m\}$, the symbol $\bar{p}$ is "tied" to p, that is, stands for the same numeral in $\{\bar{1}, \dots, \bar{m}\}$. In particular, m and $\bar{m}$ represent the same numeral (which, in the subsequent discussion, will be the complex dimension of the complex manifold in question).

Let M now be a *complex manifold* of complex dimension m, that is, a real $2m$-dimensional manifold along with a maximal atlas of $\mathbf{C}^m$-valued coordinate systems z^p, $p = 1, \dots, m$, such that the transition mappings between them are all complex-analytic. Such *complex-analytic coordinates* z^p lead to the real coordinates $x^1, y^1, \dots, x^m, y^m$, with $x^p = \operatorname{Re} z^p$, $y^p = \operatorname{Im} z^p$. However, it is often convenient to use instead of x^p, y^p the coordinates $z^1, \dots, z^m, z^{\bar{1}}, \dots, z^{\bar{m}}$, where

$$z^{\bar{p}} = \overline{z^p} \tag{23.2}$$

is the valuewise complex conjugate of the coordinate function z^p. The coordinate system $z^p, z^{\bar{p}}$ just described is valued in the real $2m$-dimensional vector subspace of $\mathbf{C}^m$, consisting of all $(z^1, \dots, z^m, z^{\bar{1}}, \dots, z^{\bar{m}})$ with (23.2) for all $p = 1, \dots, m$. Given such complex coordinates z^p, let us provisionally denote $e_1, h_1, \dots, e_m, h_m$ the coordinate vector fields in the directions of the corresponding real coordinates $x^1, y^1, \dots, x^m, y^m$ (cf. (2.1)). Formulae

$$\mathbf{e}_p = \frac{1}{2}(e_p - i h_p), \qquad \mathbf{e}_{\bar{p}} = \frac{1}{2}(e_p + i h_p), \tag{23.3}$$

$$dz^p = dx^p + i\, dy^p, \qquad dz^{\bar{p}} = dx^p - i\, dy^p, \tag{23.4}$$

now describe bases $\mathbf{e}_1, \dots, \mathbf{e}_m, \mathbf{e}_{\bar{1}}, \dots, \mathbf{e}_{\bar{m}}$ and $dz^1, \dots, dz^m, dz^{\bar{1}}, \dots, dz^{\bar{m}}$ of the complex spaces $[T_x M]^{\mathbf{C}}$ and, respectively, of $[T_x^* M]^{\mathbf{C}}$ at any point x of the coordinate domain. Obviously,

$$\overline{dz^p} = dz^{\bar{p}}, \quad \overline{dz^{\bar{p}}} = dz^p, \quad \overline{\mathbf{e}_p} = \mathbf{e}_{\bar{p}}, \quad \overline{\mathbf{e}_{\bar{p}}} = \mathbf{e}_p, \tag{23.5}$$

and we have the duality relations (cf. (2.3))

$$(dz^k)(\mathbf{e}_j) = \delta_j^k, \tag{23.6}$$

i.e., with (23.1), $(dz^p)(\mathbf{e}_q) = \delta_q^p$, $(dz^{\bar{p}})(\mathbf{e}_{\bar{q}}) = \delta_{\bar{q}}^{\bar{p}}$, $(dz^{\bar{p}})(\mathbf{e}_q) = (dz^p)(\mathbf{e}_{\bar{q}}) = 0$.

Let us now define the *Cauchy-Riemann partial-derivative operators* relative to the coordinates z^p to be the directional derivatives $\partial/\partial z^p$ and $\partial/\partial z^{\bar p}$ in the direction of $\mathbf{e}_p$ and, respectively, $\mathbf{e}_{\bar p}$. We have, by (23.3),

$$\frac{\partial}{\partial z^p} = \frac{1}{2}\left[\frac{\partial}{\partial x^p} - i\,\frac{\partial}{\partial y^p}\right], \qquad \frac{\partial}{\partial z^{\bar p}} = \frac{1}{2}\left[\frac{\partial}{\partial x^p} + i\,\frac{\partial}{\partial y^p}\right]. \tag{23.7}$$

Remark 23.2. Conditions $\partial f/\partial z^{\bar p} = 0$ for $p = 1, \ldots, m$, imposed on a complex-valued C^1 function f defined in the coordinate domain are the familiar *Cauchy-Riemann equations*; their solutions are precisely those functions f which are *holomorphic* (i.e., complex-analytic). Similarly, equations $\partial f/\partial z^p = 0$ characterize the *antiholomorphic* functions, that is, conjugates of holomorphic functions. It will also be useful later to note that a complex-valued C^∞ functions f satisfies $\partial_p\partial_{\bar q} f = 0$ for all indices $p, \bar q$ with (23.1) if and only if, locally, f is the sum of a holomorphic and an antiholomorphic function. In fact, the 'if' part is now obvious. Conversely. let $\partial_p\partial_{\bar q} f = 0$. Then $\partial_p f = 0$ is holomorphic for each p, and so we can find a holomorphic function φ with $\partial_p f = \partial_p \varphi$ for all p. Thus, $f - \varphi$ is antiholomorphic, as required.

Any ρ times contravariant and σ times covariant complexified tensor (field) B on our complex manifold of M now can be described via its components relative to a complex coordinate system z^p, which arise by the same formal operations involving the dz^j and $\mathbf{e}_j$ as in the case of an ordinary real coordinate system. Specifically, these components are

$$B_{j_1\ldots j_\sigma}{}^{k_1\ldots k_\rho} = B(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_\sigma}, dz^{k_1} \ldots dz^{k_\rho})\,. \tag{23.8}$$

Thus, for instance, an ordinary (real) tangent vector v can be expanded as

$$v = v^j\mathbf{e}_j = v^p\mathbf{e}_p + v^{\bar p}\mathbf{e}_{\bar p}\,, \tag{23.9}$$

with the (complex) components $v^p = (dz^p)(v)$, $v^{\bar p} = (dz^{\bar p})(v)$ related by $v^{\bar p} = \overline{v^p}$. The corresponding expansion in the real coordinates $x^1, y^1, \ldots, x^m, y^m$ is $v = (\mathrm{Re}\, v^p)\, e_p + (\mathrm{Im}\, v^p)\, h_p$.

Let us denote $J : TM \to TM$ the real vector-bundle morphism of multiplication by i in our complex manifold M. We then have

$$J\mathbf{e}_p = i\,\mathbf{e}_p\,, \qquad J\mathbf{e}_{\bar p} = -\,i\,\mathbf{e}_{\bar p}\,. \tag{23.10}$$

This is immediate from the relations $Je_p = h_p$, $Jh_p = -\,e_p$. In fact, just as in the real case, multiplying a tangent vector $v \in T_xM$ by a scalar λ can be realized by choosing a differentiable curve $t \mapsto x(t) \in M$ with $x(0) = x$, $\dot x(0) = v$ and then replacing it with $t \mapsto x(\lambda t) \in M$. For e_p at x, we may use the curve characterized by the components $z^q(t) = z^q$ (for $q \neq p$) and $z^p(t) = z^p + t$, with $z^q = z^q(x)$ for all q.

More generally, let us consider any finite-dimensional complex vector space V which we chose to treat as a real vector space endowed with the fixed operator $J : V \to V$ of complex multiplication by i, satisfying the condition $J^2 = -\,\mathrm{Id}$. (Cf. Remark 3.9.) Also, let $h : V \times V \to \mathbf{R}$ be a real-bilinear function. We will say that h is a *Hermitian tensor* (or, an *anti-Hermitian form*) in V if it is symmetric (or, respectively, skew-symmetric) and, for all $v, w \in V$, we have $h(Jv, Jw) = h(v, w)$. Both the Hermitian tensors in V and, separately, the anti-Hermitian forms in V, form real vector spaces. These two spaces are canonically isomorphic, under the assignment

$$b \mapsto \beta \tag{23.11}$$

that sends each Hermitian tensor b to the anti-Hermitian form β with

$$\beta(v, w) \,=\, b(Jv, w)\,. \tag{23.12}$$

In the case of alomost complex manifolds M (see the the beginning of §9) we will speak of Hermitian tensor *fields* and anti-Hermitian (differential) forms on M, that is, tensor fields or forms whose value at each point x is a Hermitian tensor (or, an anti-Hermitian form) in T_xM. Our primary examples of such M will be almost Hermitian manifolds and complex manifolds.

Example 23.3. Let (M, g, α) be an almost Hermitian pseudo-Riemannian manifold (§9). Thus, $J = \alpha$ turns each tangent space $V = T_xM$ into a complex vector space (§9), and then

(a) g is a Hermitian tensor field on M.

(b) α, treated (with the aid of g) as a twice-covariant tensor field on M, is an anti-Hermitian form on M.

(c) At every point x, $\alpha(x)$ is the image of $g(x)$ under the isomorphism (23.11) (see (2.19)).

(d) If we use $g(x)$ to identify real-bilinear functions on T_xM with real-linear operators $T_xM \to T_xM$ (see (2.12)), then Hermitian tensors (or, anti-Hermitian forms) in T_xM correspond precisely to those self-adjoint (or, skew-adjoint) operators in T_xM that commute with J, i.e., are *complex-linear*. Note that the operators corresponding in this way to $g(x)$ and $\alpha(x)$ are Id and J, that is, the complex multiplications by 1 and i.

(e) If, in addition, (M, g, α) is a (pseudo-Riemannian) Kähler manifold (see §9), then the Ricci tensor Ric is a Hermitian tensor field on M and, for any point x and any vectors $v, w \in T_xM$, the "curvature operator" $R(v, w)$ is an anti-Hermitian form in T_xM. This is immediate from (c) combined with Proposition 9.6.

Let (M, g, α) now be any pseudo-Riemannian Kähler manifold. We define the *Ricci form* of (M, g, α) to be the differential 2-form on M whose value at each point $x \in M$ is the image of $\mathrm{Ric}(x)$ under the isomorphism (23.11) for T_xM. Thus,

$$\rho(v, w) \,=\, \mathrm{Ric}\,(\alpha v, w) \tag{23.13}$$

for vectors v, w tangent to M at any point.

On the other hand, let M now be a complex manifold. By a *Hermitian metric* on M we then mean any pseudo-Riemannian metric g on M which at the same time is a Hermitian tensor field on the complex manifold M. In other words, for each $x \in M$, the operator $J : T_xM \to T_xM$ of multiplication by i is assumed to be be g-isometric. (Since $J^2 = -\,\mathrm{Id}$, this is the same as requiring the real vector-bundle morphism $J : TM \to TM$ to be skew-adjoint relative to g, cf. Remark 3.18.)

Remark 23.4. A triple (M, g, α) obtained from a Hermitian metric g on a complex manifold M by declaring α to be the bivector field corresponding to J via g is called a *Hermitian manifold*; this clearly is a special case of an almost Hermitian manifold as defined in §10. On the other hand, let us defined a *Kähler metric on a complex manifold* M to be any Hermitian pseudo-Riemannian metric g on M such that the bivector field α just described is parallel relative to the Levi-Civita connection ∇ of g. In other words, Kähler metrics on the given complex manifold M are precisely those Hermitian metrics g for which the Hermitian manifold (M, g, α) defined above is a Kähler manifold in the sense of §9.

Remark 23.5. The adverb 'almost' as in an "almost Hermitian manifold" is essential; not every almost Hermitian manifold is a Hermitian manifold, that is, comes from a *complex* manifold, as described above. On the other hand, it is well-known (see, e.g., Kobayashi and Nomizu, 1963) that a Kähler manifold defined as in §9 is automatically "complex", that is, the complex-bundle structure of its tangent bundle TM is induced by a complex-manifold structure in M.

Let b and β now be arbitrary real, twice-covariant tensors at a point x in a complex manifold M, and let z^p be a fixed complex coordinate system whose domain contains x. Then b is a Hermitian tensor in T_xM if and only if

$$b_{pq} = b_{\bar p\bar q} = 0\,, \qquad b_{p\bar q} = b_{\bar q p} = \overline{b_{\bar p q}}\,, \tag{23.14}$$

while β is an anti-Hermitian form in T_xM if and only if

$$\beta_{pq} = \beta_{\bar p\bar q} = 0\,, \qquad \beta_{p\bar q} = -\,\beta_{\bar q p} = -\,\overline{\beta_{\bar p q}}\,, \tag{23.15}$$

for all indices with (23.1). If b and β satisfy (23.14) and (23.15), then β corresponds to b under the isomorphism (23.11) if and only if

$$\beta_{p\bar q} = i\,b_{p\bar q} \tag{23.16}$$

(indices as in (23.1)) or, equivalently,

$$\beta = b_{p\bar q}\,dz^p \wedge dz^{\bar q}\,. \tag{23.17}$$

In fact, the components of b are given by the usual formula

$$b_{kl} = b(\mathbf{e}_k,\, \mathbf{e}_l) \tag{23.18}$$

(and analogously for β, with indices as in (23.1)), and so the symmetry and Hermitian symmetry conditions $b_{kl} = b_{lk}$ and $b(\mathbf{e}_k, \mathbf{e}_l) = b(J\mathbf{e}_k, J\mathbf{e}_l)$ combined with (23.10) and reality of b, amount to (23.14). The other two assertion can be verified similarly. As for (23.17), note that in general, for any 2-form β we have $2\beta = \beta_{jk}\,dx^j \wedge dx^k$ in any real coordinates x^j (cf. (2.15)) and so, according to Remark 23.1, the same will hold with the dx^j replaced by the dz^j. Using the ranges of indices given by (23.1), we can now rewrite (23.16) in the form (23.17).

Lemma 23.6. *Let b be a Hermitian tensor field of class C^∞ on a complex manifold M, and let β be the the anti-Hermitian differential 2-form on M which corresponds to b under the isomorphism* (23.11)*. Then the following three conditions are equivalent*:

(a) *β is closed as a differential form, i.e., $d\beta = 0$.*

(b) *In any complex coordinates z^p, the components of b satisfy*

$$\partial_p b_{q\bar r} \;=\; \partial_q b_{p\bar r}\,. \tag{23.19}$$

with indices as in (23.1)*.*

(c) *Every point of M has a neighborhood U on which there exists a* potential *for b, that is, a C^∞ function $\phi : U \to \mathbf{R}$ with*

$$b_{p\bar q} \;=\; \partial_p\partial_{\bar q}\,\phi \tag{23.20}$$

in any complex coordinates z^p, for all indices p, $\bar q$ with (23.1)*.*

Proof. (a) and (b) are equivalent in view of (23.15) and (23.16), along with the component description of $d\beta$ in the paragraph following formula (4.22) in §4. (One could also verify this by applying d to (23.17).) Also, (b) follows from (c) since all partial derivatives commute. Conversely, let us assume (b). In view of the Poincaré Lemma (Remark 11.5), we can find, locally in M, a C^∞ 1-form ϑ with $\alpha = d\vartheta$. Writing $\vartheta = \vartheta_p\,dx^p + \vartheta_{\bar p}\,dy^{\bar p}$, we easily see that relation $\alpha = d\vartheta$ amounts, locally, to the existence of functions ψ, χ with $\vartheta_p = \partial_p\psi$, $\vartheta_{\bar p} = \partial_p\chi$ (cf. Corollary 11.3), and setting $\phi = \psi - \chi$ we obtain (23.20). This function ϕ need not be real-valued. However, reality of b (i.e., the second relation in (23.14)) shows that $\partial_p\partial_{\bar q}\,(\phi - \overline{\phi}) = 0$, so that, according to Remark 23.2 we have, locally, $\operatorname{Im}\phi = \theta + \chi$, where θ is holomorphic and χ is antiholomorphic. Replacing ϕ with $\phi - i(\theta + \chi)$ we now get a new potential for b (cf. Remark 23.2), which this time is real-valued. This completes the proof. ∎

Before proceeding further, let us observe that the full analogy between real and complex coordinate formulae on a complex manifold M (cf. Remark 23.1) extends to the Christoffel symbols Γ_{jk}^l of any metric g on M, and their modified versions (4.6). For instance, we still have relation (4.9). This is of particular interest for *Hermitian* metrics g, for which we have relations (23.14) with $b = g$. Since analogous relations then hold for the reciprocal metric components (with superscripts), we have, for instance, by (4.9),

$$\Gamma_{pqr} \;=\; \Gamma_{\bar p\bar q\bar r} \;=\; 0\,, \tag{23.21}$$

and so $\Gamma_{pq}^{\bar r} = 0$ (as $\Gamma_{pq}^{\bar r} = \Gamma_{pqs}g^{s\bar r}$), etc.

Lemma 23.7. *Let g be a fixed Hermitian metric on a complex manifold M, and let $\alpha : TM \to TM$ be the morphism of multiplication by i. In other words, α corresponds to g under the isomorphism* (23.11)*, cf.* Example 23.3(c)*. Then the following four conditions are equivalent*:

(i) *g is a Kähler metric on the complex manifold M, as defined in* Remark 23.4.

(ii) *α is closed as a differential 2-form, i.e., $d\alpha = 0$.*

(iii) *Locally, g has a potential, as defined in* Lemma 23.6(c).
(iv) *In any complex coordinates, the only Christoffel symbols Γ_{jkl} of g that are not automatically zero are those of the form $\Gamma_{pq\bar{r}}$ and $\Gamma_{\bar{p}\bar{q}r}$, with indices as in* (23.1).

Proof. Equivalence between (ii) and (iii) is immediate from Lemma 23.6. Furthermore, (i) obviously implies (ii) (see the paragraph following formula (4.22) in §4). Also, (ii) implies (iv), as one sees using (23.14) and (23.19) for $b = g$ and (4.9), as well as (23.21). Finally, let us assume (iv). Combining the usual coordinate formula for $\alpha_{jk,l}$ with (23.15) for $\beta = \alpha$, we now obtain $\nabla\alpha = 0$. This completes the proof. ■

Relation (23.20) for a Hermitian tensor field b on a complex manifold, which admits a potential, and the potential function itself, is often written in terms of the anti-Hermitian differential 2-form β corresponding to b under (23.11), and it then reads

$$\beta = i\,\partial\overline{\partial}\phi\,. \tag{23.22}$$

Here ∂ and $\overline{\partial}$ are two operators, one taking 1-forms to 2-forms, the other sending functions to 1-forms, and they are given by $\overline{\partial}\phi = (\partial_{\bar{p}}\phi)\,dz^{\bar{p}}$, $\partial\xi = (\partial_p\xi_q)\,dz^p \wedge dz^q + (\partial_p\xi_{\bar{q}})\,dz^p \wedge dz^{\bar{q}}$, whenever $\xi = \xi_p\,dz^p + \xi_{\bar{p}})\,dz^{\bar{p}}$. (These operators are actually independent of the complex coordinates used, which will not really matter in our discussion).

Let b again be a Hermitian tensor field of class C^∞ on a complex manifold M. The determinant $\det b = \det[b_{p\bar{q}}]$ then is a C^∞ function (real-valued by (23.14)), defined on the coordinate domain and, of course, depending on the choice of the complex coordinate system z^p used. However, if b is assumed nondegenerate (i.e., $\det b \neq 0$ everywhere), a Hermitian tensor field h for which the natural logarithm $\log|\det b|$ of the absolute value of this determinant is a potential is defined globally and coordinate-independent; in fact, when complex coordinates are changed, $\det b$ becomes multiplied by $F\overline{F}$, where F is the (holomorphic) Jacobian determinant of the coordinate transition, and so $\log|\det b|$ is replaced by $\log|\det b| + \log F + \overline{\log F}$ (with some complex-analytic local branch of log). The coordinate-independence of h now is obvious (cf. Remark 23.2). Denoting γ the anti-Hermitian differential 2-form on M which corresponds to b under (23.11), we will now write

$$\gamma = i\,\partial\overline{\partial}\log|\det b|\,. \tag{23.23}$$

Proposition 23.8. *Let g be a Kähler metric on a complex manifold M, as defined in* Remark 23.4. *Then*

$$\rho = -\,i\,\partial\overline{\partial}\log|\det g|\,, \tag{23.24}$$

i.e., relation (23.23) *holds when $b = g$ and $\gamma = -\rho$, where ρ denotes the Ricci form of the Kähler manifold (M, g, α), defined by* (23.13).

Proof. Using assertion (iv) of Lemma 23.7 and (4.25), we see that the curvature components all vanish, except maybe those of the form

$$R_{p\bar{q}r}{}^{s} = \partial_{\bar{q}}\Gamma^{s}_{pr}\,,\ R_{\bar{p}q\bar{r}}{}^{\bar{s}} = \partial_{q}\Gamma^{\bar{s}}_{\bar{p}\bar{r}}\,,\ R_{\bar{q}pr}{}^{s} = -\,R_{p\bar{q}r}{}^{s}\,,\ R_{q\bar{p}\bar{r}}{}^{\bar{s}} = -\,R_{\bar{p}q\bar{r}}{}^{\bar{s}}\,. \tag{23.25}$$

In view of (4.36) and Example 23.3(e), the components of the Ricci tensor Ric now are $R_{pq} = R_{\bar p\bar q} = 0$, $R_{p\bar q} = R_{\bar q p} = R_{\bar q rp}{}^r = -\,\partial_{\bar q}\Gamma^r_{rp}$. Thus, by (4.11), $2R_{p\bar q} = -\,\partial_p\partial_{\bar q}\log|\det[g_{jk}]| = -\,\partial_p\partial_{\bar q}\log|\det[g_{r\bar s}]| = -\,\partial_p\partial_{\bar q}\log|\det g|$, since $\det[g_{jk}] = (\det[g_{r\bar s}])^2$ by (23.14) with $b = g$. This completes the proof. ■

Formula (23.24) can be used to produce examples of Kähler-Einstein metrics on complex manifolds. More precisely, assuming that our discussion is local, and a complex coordinate system has been chosen z^p, all we need to do to create a Kähler metric g, is to provide a potential for g, which is just any real-valued C^∞ function ϕ. The metric g then has the components functions

$$g_{pq} = g_{\bar p\bar q} = 0\,, \qquad g_{p\bar q} = g_{\bar q p} = \partial_p\partial_{\bar q}\phi \tag{23.26}$$

(indices as in (23.1); see (23.14) and Lemma 23.7(iii). The requirement that this metric be Einstein, i.e., satisfy (5.3) with some constant $\kappa \in \mathbf{R}$, is nothing else than

$$\rho = \kappa\alpha\,, \tag{23.27}$$

to be satisfied by the Ricci form ρ and the Kähler form α. (In fact, ρ corresponds to Ric, and α to g, under the isomorphism (23.11); see Example 23.3(c),(e).) As in (5.4), we then have

$$\kappa = \mathrm{s}/n\,, \quad n = 2m = \dim_{\mathbf{R}}M\,, \tag{23.28}$$

where s is the scalar curvature of (M,g).

In view of (23.24) and (23.26) with $\alpha_{p\bar q} = i\,g_{p\bar q}$ (cf. (23.16)), the Einstein condition (23.27) will automatically follow if we choose ϕ such that $\log|\det[g_{p\bar q}]| = -\,\kappa\phi$, i.e., if ϕ is a solution to the *Monge-Ampère equation*

$$|\det[\partial_p\partial_{\bar q}\phi]| = e^{-\kappa\,\phi}\,, \tag{23.29}$$

with a given constant κ. For Ricci-flat metrics, the equation becomes

$$|\det[\partial_p\partial_{\bar q}\phi]| = 1\,. \tag{23.30}$$

PART II: SOME TOPOLOGICAL OBSTRUCTIONS

This part describes various obstructions to the existence of, or nonexistence results for, Riemannian Einstein metrics on compact 4-manifolds. We present the well-known theorems of Bochner (1946), Berger (1965), Thorpe (1969[a]), Myers (1935) and Lichnerowicz (1963); see Corollary 28.2, Theorem 26.1, formula (26.5), Theorem 28.7 and Theorem 31.3. Also, in §27, we describe various arguments due to Sambusetti (1998), which provide generalizations of the Berger and Thorpe inequalities based on a powerful result of Besson, Courtois and Gallot (1995).

Four of the following nine sections contain exposition of necessary background material; these are §24, §25, §29, and §30. Finally, in §32 we present a brief argument showing that the U(2)-invariant Riemannian Einstein metric on the compact complex surface $M = \mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}\mathrm{P}^2}$, discovered by Page (1978), must be globally conformal to a Kähler metric; the latter metric was independently found by Calabi (1982).

§24. The Ricci curvature and Bochner's theorems

By a *volume density* in an n-dimensional real vector space V we mean any nonzero n-linear skew-symmetric function $\pm\nu : V \times \ldots \times V \to \mathbf{R}$, defined only up to a sign. For instance, any nondegenerate symmetric bilinear form $g = \langle\, ,\rangle$ in a real vector space V with $\dim V = n$, $1 \le n < \infty$, gives rise to a naturally distinguished volume density $\pm\nu$, defined up to a sign by the requirement that

$$\nu(e_1, \ldots, e_n) = \pm 1 \tag{24.1}$$

for some (or any) g-orthonormal basis e_j of V. Thus,

$$\nu = \pm e^1 \wedge \ldots \wedge e^n , \tag{24.2}$$

e^j being the dual of any g-orthonormal basis e_j. (Note that $\pm\nu$ is well-defined according to by (3.11).) For an arbitrary basis e_j of V, we can use the component matrix $[g_{jk}] = [g(e_j, e_k)]$ of g, to write

$$\nu(e_1, \ldots, e_n) = \pm\sqrt{|\det[g_{jk}]|} \, . \tag{24.3}$$

In fact, both sides obey the same transformation rule under a change of basis, and coincide when the e_j are orthonormal. Applied to a pseudo-Riemannian metric g on an n-dimensional manifold M, this construction gives the *Riemannian volume density* vol_g of g, which assigns to each $x \in M$ the volume density $\pm\nu_x$ of g_x in the tangent space T_xM. Note that vol_g is "almost" a section $\pm\nu$ of $[T^*M]^{\wedge n}$ (except that it is defined at each point only up to a sign). Formula (24.3) shows that ν is of class C^∞, since so is its essential component $\pm\nu_{1\ldots n} = \pm\nu(e_1, \ldots, e_n)$ in any local coordinate system x^j for M; ignoring the ambiguity of sign, we may express this as

$$\nu_{1\ldots n} = \sqrt{|\det[g_{jk}]|} \, . \tag{24.4}$$

Remark 24.1. If, in the above discussion, the vector space V and the pseudo-Riemannian manifold (M, g) are *oriented*, the the ambiguity of sign can be removed.

All we need to do is require the bases $e_1, \ldots, e_n$ in (24.1) – (24.4) to be *positive-oriented*, and replace all the $\pm$ signs with pluses. In this way, ν becomes just a nonzero n-linear skew-symmetric function (in the case of V), or a differential n-form on M, i.e., a section of $[T^*M]^{\wedge n}$. The resulting n-form ν on M is called the *volume form* or *volume element* of the oriented pseudo-Riemannian manifold (M, g). (See also (3.34).) Note that the sign of ν changes when the orientation of M is reversed.

Remark 24.2. The Riemannian volume density vol_g gives rise to a Borel measure on the underlying manifold M, which can be briefly described as follows. (For more details, see, e.g., Sulanke and Wintgen, 1972.) To integrate functions f of suitable regularity (e.g., continuous ones) over reasonably simple sets $\Omega \subset M$, let us first assume that Ω is contained in the domain of a local coordinate system $x^1, \ldots, x^n$, $n = \dim M$; then, $\int_\Omega f \,\mathrm{vol}_g$ is by definition the integral in $\mathbf{R}^n$ of $f\sqrt{|\det[g_{jk}]|}$ treated as a function of the $x^1, \ldots, x^n$, over the set in $\mathbf{R}^n$ which is the coordinate image of Ω. (Independence of the coordinate system used is clear from the change-of-variables formula for multiple integrals.) For more general sets Ω, one can first decompose Ω into a countable union $\bigcup_s \Omega_s$ of disjoint sets, each contained in a coordinate domain, and then define $\int_\Omega f \,\mathrm{vol}_g$ by additivity. (To see that the result is the same for another decomposition $\bigcup_{s'} \Omega'_{s'}$ of Ω, consider the decomposition $\Omega = \bigcup_{s,s'}(\Omega_s \cap \Omega'_{s'})$ and use countable additivity of the Lebesgue integral as a function of the integration domain.)

The following clasical result is known as the *divergence theorem* or the *integration-by-parts* formula.

Theorem 24.3 (Gauss)**.** *Let (M, g) be a pseudo-Riemannian manifold, and let w be a C^1 vector field on M which vanishes outside a compact set. Then*

$$\int_M (\operatorname{div} w)\ \mathrm{vol}_g \ = \ 0\,. \tag{24.5}$$

Proof. Let us assume that w has a *small support*, that is, vanishes outside a compact set contained in the domain of a local coordinate system $x^1, \ldots, x^n$. Then, by (4.42), $\operatorname{div} w = w^j{}_{,j}$ and so, in view of (4.12) and (4.11), $(\operatorname{div} w)\sqrt{|\det[g_{kl}]|} = \partial_j v^j$ with $v^j = w^j\sqrt{|\det[g_{kl}]|}$. Relation (24.5) then is an obvious consequence of the definition of $\int_\Omega f \,\mathrm{vol}_g$ for sets Ω contained in a coordinate domain (see Remark 24.2), combined with iterated integration. The general case now follows since, due to an obvious argument involving a finite partition of unity, every C^1 vector field vanishing outside a compact set is a finite sum of C^1 vector fields with small supports. This completes the proof. ■

As an immediate consequence of the divergence theorem, we obtain the following result known as *Bochner's Lemma*:

Corollary 24.4 (Bochner, 1946)**.** *Let $f : M \to \mathbf{R}$ be a C^2 function on a compact Riemannian manifold (M, g) such that $\Delta f \geq 0$. Then f is constant.*

In fact, using (4.43) and (24.5), we obtain

$$\int_M \Delta f \ \mathrm{vol}_g \ = \ 0\,, \tag{24.6}$$

and so $\Delta f \geq 0$ implies that $\Delta f = 0$. We also have, in general,

$$f\Delta f \;=\; \operatorname{div}(f\nabla f) \,-\, \langle \nabla f, \nabla f\rangle\,, \tag{24.7}$$

or, in local coordinates, $ff_{,j}{}^{j} = (ff_{,j})^{j} - f^{,j}f_{,j}$, as one sees using differentiation by parts and (4.44). Integrating (24.7) we thus find that, in our case, $0 = \int_M f\,\Delta f \,\operatorname{vol}_g = -\int_M |\nabla f|^2 \operatorname{vol}_g$, and so f is constant, as required. ■

Remark 24.5. The divergence formula (24.5) remains valid in a much more general situation, such as when the manifold M, instead of carrying a fixed metric g, is just endowed with a nowhere-zero continuous volume density $\pm\nu$ (that is, a continuous "section" $\pm\nu$ of $[T^*M]^{\wedge n}$, $n = \dim M$, defined at each point only up to a sign). Such $\pm\nu$ then gives rise to a *divergence operator* div assigning to each C^1 vector field w on M the function $\operatorname{div} w$ given by the local-coordinate expression $\operatorname{div} w = \partial_j w^j + w^j\,\partial_j \log|\nu_{1\dots n}|$, with $\pm\nu_{1\dots n} = \pm\nu(e_1,\dots,e_n)$ in any given local coordinate system x^j. (This is independent of the coordinate system used, just as it was in the special case (24.4), due to the transformation rule for $\pm\nu_{1\dots n}$.) On the other hand, $\pm\nu$ leads to a Borel measure on M, which in turn allows us to form integrals $\int_\Omega f\,\nu$, exactly as in the case where $\pm\nu = \operatorname{vol}_g$. For instance, for sets Ω contained in a coordinate domain $\int_\Omega f\nu$ if defined to be the integral in $\mathbf{R}^n$ of $f\nu_{1\dots n}$, as a function of the $x^1,\dots,x^n$, over the coordinate image of Ω. The divergence theorem

$$\int_M (\operatorname{div} w)\,\nu \;=\; 0\,. \tag{24.8}$$

for compactly supported C^1 vector fields w now follows, as before, from the local-coordinate relation $(\operatorname{div} w)\nu_{1\dots n} = \partial_j v^j$ with $v^j = w^j\nu_{1\dots n}$.

Remark 24.6. In an n-dimensional manifold M which is *oriented* one can, besides integrals $\int_\Omega f\,\nu$ of functions f relative to a fixed nowhere-zero continuous volume density $\pm\nu$ (Remark 24.5), also form so-called *oriented integrals* $\int_\Omega \omega$ of continuous differential n-forms ω over "reasonably simple" sets $\Omega \subset M$. To this end, one first assumes that Ω is contained in a connected open set U which is the domain of a local coordinate system $x^1,\dots,x^n$, and sets $\int_\Omega \omega$ equal ± 1 times the integral in $\mathbf{R}^n$ of $\omega_{1\dots n} = \omega(e_1,\dots,e_n)$ treated as a function of the $x^1,\dots,x^n$, over the coordinate image of Ω. The sign factor ± 1 equals 1 (or, -1) depending on whether the coordinates $x^1,\dots,x^n$ are (or, are not) compatible with the given orientation of M. For the remaining details (independence of the coordinates used, integration over more general sets Ω), see Remark 24.2. To describe how both integrations are related, let us assume that we are given $\pm\nu$ as in Remark 24.5 on an *oriented* n-manifold M, a set Ω and a function f as before. Then, we may "choose a sign" for ν by requiring that $\nu(e_1,\dots,e_n) > 0$ for some (any) positive-oriented basis $e_1,\dots,e_n$ of T_xM at any $x \in M$. In this way, ν is a differential n-form on M, and the "nonoriented" integral of f over Ω relative to $\pm\nu$ coincides with the oriented integral, over Ω, of the differential n-form $\omega = f\nu$.

Let (M,g) be a Riemannian manifold. We denote $\mathrm{P}(TM)$ the *projectivized tangent bundle* of M, that is, the set of all pairs (x,L) formed by a point $x \in M$ and a 1-dimensional vector subspace L of the tangent space T_xM. The Ricci tensor Ric of (M,g) gives rise to the *Ricci curvature function* of (M,g), which is a function $\mathrm{P}(TM) \to \mathbf{R}$ sending each (x, L) to $\mathrm{Ric}\,(v,v)/g(v,v)$, where $v \in T_x M$ is any nonzero vector in L.

Remark 24.7. The Ricci curvature function of a Riemannian manifold (M, g) uniquely determines its Ricci tensor Ric (due to symmetry of Ric), and is constant if and only if (M, g) is Einstein; in fact, κ appearing in (5.3) then is the constant value of the Ricci curvature. Also, positivity/nonnegativity of the Ricci curvature corresponds to positive definiteness/semidefiniteness of Ric, and similarly for its negativity and nonpositivity. The conditions just named will from now on be written as $\mathrm{Ric} > 0$, $\mathrm{Ric} \geq 0$, $\mathrm{Ric} < 0$, etc.

The sign of the constant Ricci curvature κ in (5.3) leads to specific global consequences for compact Einstein manifolds, that are valid in general for manifolds whose Ricci curvature has a fixed sign. For instance, let (M, g) be a compact pseudo-Riemannian manifold. From the contracted Ricci-Weitzenböck formula (4.39) we obtain

$$R_{jk} w^j w^k \,=\, w^j w^k{}_{,jk} \,-\, w^j w^k{}_{,kj} \tag{24.9}$$

for C^2 vector fields w on M. Integrating this by parts (i.e., using Theorem 24.3), we obtain the following relation due to Bochner (1946):

$$\int_M \mathrm{Ric}\,(w,w)\,\mathrm{vol}_g \,=\, \int_M (\mathrm{div}\, w)^2\,\mathrm{vol}_g \,-\, \int_M \mathrm{Trace}\,(\nabla w \circ \nabla w)\,\mathrm{vol}_g\,, \tag{24.10}$$

valid for all C^2 vector fields w on M.

The following classical result of Bochner is an obvious consequence of (24.10).

Theorem 24.8 (Bochner, 1946)**.** *Let* Ric *denote the Ricci curvature function of a compact Riemannian manifold* (M, g)*, with the same notational conventions as in* Remark 24.7.

(i) *If* $\mathrm{Ric} < 0$ (*or,* $\mathrm{Ric} > 0$)*, then* (M, g) *admits no nontrivial Killing fields* (*or, respectively, harmonic* 1*-forms*)*.*

(ii) *If* $\mathrm{Ric} \leq 0$ (*or,* $\mathrm{Ric} \geq 0$)*, then every Killing field* (*or, respectively, harmonic* 1*-form*) *on* (M, g) *is parallel.*

In fact, for a Killing field w, ∇w is skew-adjoint at every point (see §17), and so $\mathrm{div}\, w = 0$, while harmonic 1-forms on compact Riemannian manifolds may be identified with those vector fields w for which ∇w is self-adjoint at every point and $\mathrm{div}\, w = 0$. ∎

Later in §36 we will need the *Stokes formula*, which is, basically, an alternative version of Gauss's divergence formula (24.5). It states that, given a differential $(n-1)$-form γ of class C^1 on an n-dimensional manifold M, such that $\gamma = 0$ outside a compact set, we have

$$\int_M d\gamma \,=\, 0\,, \tag{24.11}$$

the integral being taken in the sense of oriented integration of n-forms. To prove it, we proceed as in the proof of Theorem 24.3, first using a finite partition of unity to reduce the problem to the case where γ has a small support, and then observing that the assertion in that case is completely straigtforward, since $d\gamma$ has the essential component $(d\gamma)_{1\ldots n} = \partial_1\gamma_{2\ldots n} - \partial_2\gamma_{13\ldots n} + \ldots - (-1)^n\partial_n\gamma_{1\ldots(n-1)}$.

Finally, let us note that Stokes's formula (24.11) can also be derived directly from (24.5) if one chooses a Riemannian metric g on M and observes that $d\gamma = [\mathrm{div}\,(*\gamma)]\,\mathrm{vol}$, where vol is the volume form of the oriented Riemannian manifold (M, g) and $*$ is the Hodge star $*$ acting on $(n-1)$-forms (and sending them to vectors).

§25. Curvature and characteristic numbers

The existence of a Riemannian Einstein metric on a compact 4-manifold M imposes topological restrictions (see §26) on the Euler characteristic $\chi = \chi(M)$ and signature $\tau = \tau(M)$ of M. Formulae (25.1) and (25.6) below give the standard Chern-Weil integral expressions for χ and τ in terms of any Riemannian metric on M. The reader not familiar with the Chern-Weil theory may consider treating (25.1) and (25.6) as *definitions* of χ and τ (which then appear to be just real numbers, even though in fact they always are integers). The independence of χ and τ of the metric used can easily be seen by connecting any two Riemannian metrics g, g' on M with a C^2 curve of metrics, such as $[0,1] \ni t \mapsto g^{(t)} = (1-t)g + tg'$. Applying d/dt to the expressions (25.1) and (25.6), one then easily verifies that they are constant in t (along *any* C^2 curve of metrics) in view of the divergence theorem (Theorem 24.3).

Specifically, the Euler characteristic $\chi(M)$ of a compact 4-manifold M is given by

$$8\,\pi^2\,\chi(M) \;=\; \|R\|^2 \;-\; \|\mathrm{E}\|^2\,. \tag{25.1}$$

Here $\|\;\|$ is the L^2 norm relative to g, while R and E are the curvature tensor and, respectively, the traceless Ricci tensor of any Riemannian metric g on M; the coefficient conventions for the integrands are (5.32) and $|A|^2 = A_{jk}A^{jk}$ for curvature-like 4-tensors R and symmetric 2-tensors A. Thus, for instance, for R, E, the scalar curvature function s and the Weyl conformal tensor W of a Riemannian manifold of any dimension $n \geq 3$, we have

$$|R|^2 \;=\; |W|^2 \;+\; \frac{1}{n-2}\,|\mathrm{E}|^2 \;+\; \frac{\mathrm{s}^2}{2n(n-1)}\,, \qquad |\mathrm{Ric}|^2 \;=\; |\mathrm{E}|^2 \;+\; \frac{\mathrm{s}^2}{n}\,, \tag{25.2}$$

as one easily sees using (5.9), the relation

$$4\,\langle g \circledast A, g \circledast B\rangle \;=\; (n-2)\langle A, B\rangle \;+\; (\mathrm{Trace}\,A)(\mathrm{Trace}\,B)\,, \tag{25.3}$$

for symmetric 2-tensors A, B (immediate from (5.7)), and the fact that the three terms in (5.9) are mutually orthogonal (which in turn is clear from (25.3) and (5.25)). In view of (25.2), with $n = 4$, we can rewrite (25.1) as

$$192\,\pi^2\,\chi(M) \;=\; 24\,\|W\|^2 \;-\; 12\,\|\mathrm{E}\|^2 \;+\; \|\mathrm{s}\|^2\,. \tag{25.4}$$

If, in addition, the compact 4-manifold M is oriented and $W^\pm$ are the $\Lambda^\pm M$ components of W, we have

$$\|W\|^2 \;=\; \|W^+\|^2 \;+\; \|W^-\|^2\,, \tag{25.5}$$

and the signature $\tau = \tau(M)$ of M is given by

$$12\,\pi^2\,\tau(M) \;=\; \|W^+\|^2 \;-\; \|W^-\|^2\,. \tag{25.6}$$

We also set

$$\tau(M) \;=\; 0 \qquad \text{if} \quad M \quad \text{is nonorientable}\,. \tag{25.7}$$

Finally, for any compact oriented Riemannian 4-manifold (M, g), (25.4) and (25.6) imply the important relation

$$96\,\pi^2\,[2\chi(M) \,+\, 3\tau(M)] \;=\; 48\,\|W^+\|^2 \;-\; 12\,\|\mathrm{E}\|^2 \;+\; \|\mathrm{s}\|^2\,, \tag{25.8}$$

which will be used later in §26 and §34.

The remainder of this section devoted to a remark that will not be needed until §34.

Specifically, the left-hand side of (25.8) acquires an additional interpretation in the case of compact Kähler manifolds (M, g, α) (see §10) of real dimension four. The almost complex structure α then is integrable, that is, comes from a complex-manifold structure in M (Remark 23.5). According to formula (36.2) in §36, the (real) first Chern class $c_1(M)$ of M is represented in the de Rham cohomology (§36) by the closed 2-form $\rho/2\pi$, where ρ is the Ricci form given by (23.13). Since Ric, treated as a bundle morphism $TM \to TM$, is complex-linear by (9.6), we may choose, at any given point x, a complex-orthonormal basis u, v of T_xM which diagonalizes Ric so that $\mathrm{Ric}\,u = \lambda u$, $\mathrm{Ric}\,v = \lambda v$ for some $\lambda, \mu \in \mathbf{R}$. Now we have $\mathrm{s} = 2(\lambda + \mu)$ and $\rho = \lambda u \wedge (\alpha u) + \mu v \wedge (\alpha v)$, and so $\rho \wedge \rho$ equals the volume form $u \wedge (iu) \wedge v \wedge (iv)$ (Remark 24.1) times $2\lambda\mu = \mathrm{s}^2/4 - |\mathrm{Ric}|^2/2 = \mathrm{s}^2/8 - |\mathrm{E}|^2/2$. As $c_1^2(M) = \int_M \rho \wedge \rho$ and $|W^+|^2 = \mathrm{s}^2/24$ (see (10.10)), (25.8) yields

$$c_1^2(M) \;=\; 2\,\chi(M) \,+\, 3\,\tau(M)\,. \tag{25.9}$$

§26. The Berger and Thorpe inequalities

The simplest restrictions that the existence of a Riemannian Einstein metric on a compact 4-manifold M imposes on its topology are inequalities involving its Euler characteristic $\chi = \chi(M)$ and signature $\tau = \tau(M)$, due to Berger (1965) and Thorpe (1969[a]); see also Hitchin (1974). Further generalizations of these inequalities are immediate from a result of Besson, Courtois and Gallot (1995), as observed by Sambusetti (1998); see §27.

Theorem 26.1 (Berger, 1965)**.** *Every compact* 4*-manifold* M *carrying an Einstein metric* g *satisfies the inequality*

$$\chi(M) \geq 0\,, \tag{26.1}$$

and then

$$\chi(M) \;=\; 0 \qquad \text{if and only if} \quad g \quad \text{is flat.} \tag{26.2}$$

This is immediate from (25.1) with $\mathrm{E} = 0$. ∎

Following Sambusetti (1998), let us now set, for a compact 4-manifold M,

$$[M] = 2\chi(M) - 3|\tau(M)|\,. \tag{26.3}$$

Applying (25.8) and (26.3) to a fixed Einstein ($\mathrm{E} = 0$) metric g on a compact orientable 4-manifold M with an orientation chosen so that $\tau(M) \le 0$, we find that

$$96\,\pi^2\,[M] = 48\,\|W^+\|^2 + \|\mathrm{s}\|^2 \ge \|\mathrm{s}\|^2 \ge 0\,. \tag{26.4}$$

By (26.3), this leads to the *Thorpe inequality* (Thorpe, 1969)

$$|\tau(M)| \le \frac{2}{3}\,\chi(M) \tag{26.5}$$

valid for any compact 4-manifold M admitting an Einstein metric.

Note that, even though we used orientability of M, relation (26.5) remains valid for nonorientable manifolds as well (with (25.7)), in view of Berger's Theorem 26.1. (This can also be established by passing to a two-fold orientable covering of M.)

The equality case in (26.5) is in turn settled by a theorem of Hitchin; see §33.

Remark 26.2. Integral formulae analogous to (25.1) and (25.6) remain valid for (indefinite) pseudo-Riemannian metrics on compact 4-manifolds M, which in turn leads to estimates resembling (26.5) and (26.1). For the Lorentzian sign pattern $-+++$, these provide no further information as we then have $\tau(M) = \chi(M) = 0$ solely due to the existence of a Lorentz metric (whether Einstein or not). However, for neutral Einstein metrics (with the sign pattern $--++$), some interesting estimates hold. See Law, 1991, and references therein.

§27. Degrees of mappings into hyperbolic manifolds

A generalization of the Thorpe inequality (26.5) involves the concept of the *volume entropy* $\mathrm{Ent}_h \in [0,\infty)$ for a compact n-dimensional Riemannian manifold (N,h). It is given by $\mathrm{Ent}_h = \lim_{r\to\infty} \dfrac{1}{r}\log V(\tilde{x},r)$, with $V(\tilde{x},r)$ denoting the volume of the ball of radius r centered at any fixed point $\tilde{x}$ in the Riemannian universal covering of (N,h). We also set

$$\{N\}_h = \mathrm{Vol}_h \cdot [\mathrm{Ent}_h]^n\,, \qquad \langle M\rangle_g = \mathrm{Vol}_g \cdot |\min \mathrm{Ric}_g|^{n/2} \tag{27.1}$$

for compact Riemannian n-manifolds (M,g) and (N,h), where $\mathrm{Vol}_h \in (0,\infty)$ and Ric_g denote the volume of (N,h) and the Ricci curvature function of (M,g) (described in §24). For instance, Einstein n-manifolds (M,g) satisfy

$$n^{n/2}\langle M\rangle_g = \mathrm{Vol}_g \cdot |\mathrm{s}_g|^{n/2}\,,$$

as the Ricci curvature then is constant and equal to $1/n$ times the scalar curvature s_g (see (0.1)). When $n = 4$ (and $\|\mathrm{s}_g\|$ stands for the L^2 norm of the scalar curvature function s_g of the Einstein 4-manifold (M,g)), this becomes

$$16\langle M\rangle_g = \|\mathrm{s}_g\|^2\,. \tag{27.2}$$

On the other hand, if (N,h) is a real or complex hyperbolic space of real dimension n (Examples 10.4, 10.6), we have, respectively (see Remark 27.9 below),

$$[\mathrm{Ent}_h]^2 = \frac{n-1}{n}\,|\mathrm{s}_h| \qquad \text{or} \qquad [\mathrm{Ent}_h]^2 = \frac{n}{n+2}\,|\mathrm{s}_h|\,. \tag{27.3}$$

We will need the following powerful result.

Theorem 27.1 (Besson, Courtois and Gallot, 1995). *Let M and N be compact orientable manifolds with* $\dim M = \dim N \geq 3$ *and let h be a negatively curved locally symmetric Riemannian metric on N. The inequality*

$$\{M\}_g \,\geq\, |\deg f| \cdot \{N\}_h \tag{27.4}$$

then is satisfied by the degree $\deg f$ *of every continuous mapping* $f : M \to N$; *in addition,* equality *in* (27.4) *with* $\deg f = d \neq 0$ *then implies that f is homotopic to a $|d|$-fold covering which is homothetic, i.e., isometric up to a constant factor.*

For a proof, see the paper of Besson, Courtois and Gallot. ■

Formula (27.4) leads to a generalization of the Berger and Thorpe estimates given in §26, as shown by Sambusetti (1998). Namely, we have

Corollary 27.2 (Sambusetti, 1998). *Let M and N be compact orientable* 4*-manifolds and let h be a locally symmetric Riemannian metric of negative curvature on N. If M admits a Riemannian Einstein metric, then the degree* $\deg f$ *of every continuous mapping* $f : M \to N$ *satisfies the estimate*

$$\frac{1}{108\,\pi^2}\,\{N\}_h \cdot\, |\deg f| \,\leq\, \chi(M) \,-\, \frac{3}{2}\,|\tau(M)|\,. \tag{27.5}$$

Furthermore, inequality (27.5) *is strict except when either* $\deg f = 0$, *or* $\deg f = d \neq 0$, *N admits a real hyperbolic metric, and f is homotopic to a $|d|$-fold covering homothety.*

Proof. For any compact orientable Riemannian manifold (M, g), Bishop's comparison theorem (see Besson ***et al.***, 1995) and the first formula in (27.3) give

$$(n-1)^{n/2}\langle M\rangle_g \,\geq\, \{M\}_g\,, \qquad n = \dim M\,, \tag{27.6}$$

with $\langle M\rangle_g$ as in (27.1). If g now is an Einstein metric and $n = 4$, combining (26.4) and (27.2) with (27.6) and (27.4), we obtain

$$864\pi^2\,[M] \,\geq\, 9\,\|\mathrm{s}_g\|^2 \,=\, 144\langle M\rangle_g \,\geq\, 16\{M\}_g \,\geq\, 16\,|\deg f| \cdot \{N\}_h\,, \tag{27.7}$$

which, by (26.3), implies (27.5). Finally, the equality-case assertion now is clear from the equality-case statement for (27.4). ■

Remark 27.3. Following Sambusetti (1998), we may rewrite the inequality (27.5) in two equivalent ways, each of which becomes a convenient source of further conclusions. Specifically, let M and N be compact oriented 4-manifolds such that M admits an Einstein metric, and let q be a real parameter. Furthermore, let us assume that N admits a Riemannian metric h on N such that either $q = 1$ and h is a real hyperbolic metric, or $q = 32/81$ and h is complex hyperbolic. (In view of Theorem 14.7, these are the only possible types of (N, h) satisfying the hypotheses of Corollary 27.2.) The estimate (27.5) on the degree $\deg f$ of any continuous mapping $f : M \to N$ then takes the form

$$q\,|\deg f| \cdot \chi(N) \,\leq\, \chi(M) \,-\, \frac{3}{2}\,|\tau(M)|\,, \tag{27.8}$$

as well as

$$|\deg f| \cdot [N] \,\le\, [M]\,, \tag{27.9}$$

with $[M], [N]$ as in (26.3). These inequalities are immediate from (27.5) and the relations

(a) $\{N\}_h = 108\,\pi^2\,\chi(N)$ if (N, h) has constant curvature $K < 0$.
(b) $\{N\}_h = 128\,\pi^2\,\chi(N)/3$ if (N, h) is complex hyperbolic.

To verify (a) and (b), note that in both cases (N, h) is Einstein and satisfies $W^- = 0$ for a suitable orientation (see (10.11)). Thus, (27.2), (27.3) and (26.5) yield $\{N\}_h = 9\langle N\rangle_h = 54\pi^2[N]$ (case (a)), or $9\{N\}_h = 64\langle N\rangle_h = 384\pi^2[N]$ (case (b)), while in (a) $\tau(N) = 0$ (by (25.6) with $W = 0$) and, in (b), $|\tau(N)| = \chi(N)/3$. (The last relation is clear from (25.4) – (25.6) along with $\mathrm{E} = 0$, $W^- = 0$, and (10.10); it also follows directly from the Hirzebruch signature formula.)

Corollary 27.4 (Besson, Courtois and Gallot, 1995)**.** *Let N be a compact orientable 4-manifold admitting a locally symmetric Riemannian metric h of negative curvature. Then, up to diffeomorphisms and constant factors, h is the unique Riemannian Einstein metric on N.*

In fact, the assertion follows from the equality-case conclusion in Corollary 27.2, applied to inequality (27.5) rewritten as (27.9), with $M = N$, for any given Einstein metric g on M, and for the identity mapping f. ■

Corollary 27.5 (Sambusetti, 1998)**.** *Given compact orientable 4-manifolds M and N such that N admits a real hyperbolic metric, the connected sums*

$$kN \,=\, N \,\#\, \ldots \,\#\, N\,, \qquad\qquad N \,\#\, M$$

carry no Einstein metrics, provided that $k \ge 2$ or, respectively, $\chi(M) - \frac{3}{2}\,|\tau(M)| \le 2$ and M is not a topological sphere.

This in turn follows from (27.8) applied to the obvious collapsing mappings f into N with $\deg f = k$ (or, respectively, $\deg f = 1$); the argument also relies in part on the equality case of (27.4) and Freedman's solution of the 4-dimensional Poincaré conjecture. For details, see Sambusetti (1998). ■

The connected-sum examples just mentioned lead in turn to the following conclusion.

Corollary 27.6 (Sambusetti, 1998)**.** *Every pair (χ, τ) that can be realized as the Euler characteristic and signature of a compact 4-manifold, can also be realized by a compact 4-manifold admitting no Einstein metric.* ■

Remark 27.7. The above results can be extended to the case of nonorientable manifolds, using Epstein's notion of *absolute degree* and (25.7). See Sambusetti (1998).

Remark 27.8. Denoting u_n the volume of a unit ball in $\mathbf{R}^n$, and letting $V(r)$ be the volume of the radius r ball with a fixed center x in a given n-dimensional Riemannian manifold (M, g), we have, for $r \ge 0$ close to zero,

$$V(r) \,=\, u_n r^n \,-\, \frac{\mathrm{s}(x)}{6(n+2)}\, u_n r^{n+2} \,+\, \varphi(r) r^{n+3} \tag{27.10}$$

with some continuous function φ of the variable r, where $\mathrm{s}(x)$ stands for the scalar curvature of (M,g) at x. To see this, use geodesic normal coordinates x^j at x; then, at x, the volume integrand $\sqrt{\det[g_{jk}]}$ has the second order Taylor expansion $1 + a_{jk}x^j x^k$ with $4a_{jk} = g^{ls}\partial_j\partial_k g_{ls}$, and so $\mathrm{s}(x) = -6g^{jk}a_{jk}$ (note that the cyclic sum of the $\partial_j\partial_k g_{ls}$ over j,k,l is zero). At the same time,

$$\int_{|x|\le r} \langle Tx, x\rangle \, dx \;=\; \frac{u_n}{n+2}\, r^{n+2}\ \mathrm{Trace}\, T$$

for any linear operator T in an n-dimensional Euclidean space. Integration now yields the required coefficient of r^2 in (27.10).

Remark 27.9. Using Lorentzian pseudosphere models (Examples 10.4, 10.6), we get

$$V(r) \;=\; nu_n c^n F_{n-1}(r/c)$$

in the pseudosphere (hyperboloid) given by $\langle x,x\rangle = -\,c^2$ in an $(n+1)$-dimensional real pseudo-Euclidean space of the sign pattern $-+\ldots+$, with

$$F_n(\rho) \;=\; f(\rho) \;+\; \frac{1}{2^{n-1}} \sum_{j=0}^{[(n-1)/2]} (-1)^j \binom{n}{j} \frac{\varPsi((n-2j)\rho)}{n-2j}\,,$$

where, for odd n, $f(\rho) = 0$ and $\varPsi(t) = \cosh t - 1$, while, for even n, $f(\rho) = 2^{-n}\binom{n}{n/2}\rho$ and $\varPsi(t) = \sinh t$. At the same time, the metric h in question has the sectional curvature $K = -1/c^2$, so that and the scalar curvature $\mathrm{s}_h = n(n-1)K$. Similarly,

$$V(r) \;=\; u_n\, c^n\, \sinh^n \frac{r}{c}$$

for a complex hyperbolic space of (even) real dimension n, obtained as the Riemannian quotient (under the obvious S^1 action) of the pseudosphere $\langle x,x\rangle = -c^2$ in a complex pseudo-unitary space of the complex dimension $\frac{1}{2}n+1$ and sign pattern $-+\ldots+$. Induction on n shows that $\sinh^n\rho$ has the Taylor expansion of order $n+2$ given by $\rho^n + \frac{n}{6}\rho^{n+2}$. Thus, in view of (27.10) (see also (10.9))

$$\mathrm{s}_h = -\,\frac{n(n+2)}{c^2}\,, \qquad \mathrm{Ent}_h \;=\; \frac{n}{c}$$

in the latter case.

§28. Positive Ricci curvature and Myers's theorem

Bochner's Theorem 24.8 can be rephrased as a statement about isometry groups and Betti numbers. First, since the Killing fields on a compact Riemannian manifold (M,g) form the Lie algebra of the compact Lie group of all isometries of (M,g), we obtain

Corollary 28.1 (Bochner, 1946)**.** *Every compact Riemannian manifold* (M,g) *with* $\mathrm{Ric} \le 0$ *and an Euler characteristic* $\chi(M) \ne 0$ *admits no nontrivial Killing field, i.e., its group of isometries is finite.*

In fact, a nontrivial Killing field would be parallel and hence nonzero everywhere, thus implying $\chi(M) = 0$. ■

Since harmonic forms represent the real cohomology of M via Hodge theory (see Wells, 1979), we similarly have

Corollary 28.2 (Bochner, 1946)**.** *The first Betti number of any compact Riemannian manifold with* Ric >0 *is zero.* ■

Corollary 28.3 (Bochner, 1946)**.** *Let* b_1 *denote the first Betti number of any given compact* n*-dimensional Riemannian manifold* (M,g) *with* Ric ≥ 0.

(i) *If* $b_1 > 0$, *then* $\chi(M) = 0$.
(ii) *If* $b_1 \ge n-1$, *then* (M,g) *is flat.*

Proof of Corollary 28.3. Assertion (i) follows from Theorem 24.8 just like in Corollary 28.1 above. To prove (ii), select $n-1$ linearly independent harmonic 1-forms ξ_j, $j = 1,\dots,n-1$. Since they are now parallel, we can find, locally, a parallel 1-form ξ which is unit and pointwise orthogonal to the preceding ξ_j. Treating all the ξ_j as vector fields, we obtain $R = 0$ from (4.23). ■

Remark 28.4. In particular, any compact Ricci-flat Riemannian n-manifold with $b_1 \ge n-1$ is flat.

Let *nabla* be a connection in the tangent bundle TM of a manifold M. For any geodesic $[a,b] \ni t \mapsto x(t) \in M$ of ∇, we define its *Jacobi operator* $\mathcal{J}$ to be the linear operator that assigns to every C^2 tangent vector field w along the geodesic the continuous vector field $\mathcal{J}w$ along it, given by

$$\mathcal{J}w \,=\, \nabla_{\dot x}\nabla_{\dot x}w \,-\, R(w,\dot x)\dot x\,. \tag{28.1}$$

In the case where ∇ is the Levi-Civita connection of a pseudo-Riemannian manifold (M,g), we will call $\mathcal{J}$ the *Jacobi operator of* (M,g).

Suppose now that $F:\Omega\to M$ is a C^k mapping of a rectangle $\Omega = [a,b]\times[c,d]$ the given manifold M endowed with a connection *nabla* in TM. We will use the generic symbols t and s for the variables with $a \le t \le b$ and $c \le s \le d$. Thus, in view of (11.1) and (4.3), F_{st} has the component functions

$$F_{st}^j \,=\, \frac{\partial^2 F^j}{\partial t\,\partial s} \,+\, (\varGamma_{kl}^j \circ F)\frac{\partial F^k}{\partial t}\,\frac{\partial F^l}{\partial s}\,. \tag{28.2}$$

If F is fixed, we will simply write $x(t,s)$ and x_t, x_s instead of $F(t,s)$, $\partial F/\partial t$ and $\partial F/\partial s$. If $k \ge 1$, the partial derivatives x_t, x_s are C^{k-1} sections of the tangent bundle TM along the mapping $\Omega \ni (t,s) \mapsto x(t,s)$ (see §11). Consequently,

$$x_{ts} \,=\, x_{st} \qquad \text{if} \quad \nabla \quad \text{is torsionfree} \tag{28.3}$$

and $k \ge 2$, as one easily sees using (4.3) and (28.2) along with its analogue for F_{ts}. From (28.3) and (11.2) we now obtain (for $k \ge 3$)

$$x_{tts} \,=\, x_{stt} \,-\, R(x_s,x_t)x_t \qquad \text{if} \quad \nabla \quad \text{is torsionfree.} \tag{28.4}$$

Of particular interest for us are those C^3 mappings $F:[a,b]\times[c,d]\to M$ for which the curve $[a,b]\ni t\mapsto x(t,c) = F(t,c)$ is a geodesic, i.e.,

$$x_{tt}(\,\cdot\,,c) \,=\, 0\,. \tag{28.5}$$

Remark 28.5. A mapping $F:\Omega\to M$ as above is often termed a *variation* of curves in M, and then it is to be regarded as a one-parameter family of curves (a "curve of curves") in M. We then think of $s \in [c,d]$ as the *variation parameter*, labeling the individual curves $t \mapsto x(t,s)$ of the family, while $t \in [a,b]$ then is the parameter along each curve. If these curves all are geodesics of (M,g), one calls F a *variation of geodesics* in (M,g); clearly, this is the case if and only if $x_{tt} = 0$ everywhere in Ω. Thus, for any C^3 variation of geodesics, (28.4) shows that the vector field $w = x_s(\,\cdot\,,c)$ along the curve $t \mapsto x(t,c)$ is a Jacobi field, that is, satisfies the Jacobi equation (4.51) (which, by (28.1), means nothing else than $\mathcal{J}w = 0$). The fact that any Killing field w restricted to a geodesic $t \mapsto x(t)$ in a pseudo-Riemannian Einstein manifolds (M,g) must satisfy the Jacobi equation (see Remark 17.5) now may be explained by applying this principle to the variation of geodesics defined by $(t,s) \mapsto e^{sw}(x(t))$ with s near 0, where e^{sw} denotes the flow of w (cf. Lemma 17.16).

By the *length* and *action* functionals for a given Riemannian manifold (M,g) one means the real-valued functions L and A, which associate with every C^1 curve $\gamma : [a,b] \to M$, defined on any closed interval $[a,b]$, the numbers

$$(28.6) \qquad \mathsf{L}[\gamma] \,=\, \int_a^b |\dot\gamma(t)|\,dt\,, \qquad \mathsf{A}[\gamma] \,=\, \frac{1}{2}\int_a^b |\dot\gamma(t)|^2\,dt\,,$$

with $|v|^2 = g(v,v)$ for tangent vectors v. Note that we then have the Schwarz inequality

$$(28.7) \qquad (\mathsf{L}[\gamma])^2 \,\le\, 2\,(b-a)\,\mathsf{A}[\gamma]\,,$$

which becomes an equality for constant-speed curves, i.e., when $|\dot\gamma|$ is constant.

Lemma 28.6. *Given a Riemannian manifold (M,g), let us denote $(\,,\,)$ the L^2 inner product of vector fields along any fixed curve in M parametrized by $t \in [a,b]$, so that $(w,u) = \int_a^b \langle w,u\rangle\,dt$, and let $\gamma : [a,b] \to M$ be a geodesic of (M,g) with a constant-speed parameter t. For any C^3 tangent vector field w along γ such that $w(a) = 0$ and $w(b) = 0$, we then have*

$$(28.8) \qquad (w,\,\mathcal{J}w) \,\le\, 0\,.$$

Proof. We may choose a C^3 variation $\Omega \ni (t,s) \mapsto x(t,s)$ of curves in M, with $\Omega = [a,b]\times[c,d]$, in such a way that $\gamma(t) = x(t,c)$, $w(t) = x_s(t,c)$ and $x(a,s) = \gamma(a)$, $x(b,s) = \gamma(b)$ for all t and s. (For instance, $x(t,s) = \exp_{\gamma(t)} sw(t)$, where $\exp$ is the geodesic exponential mapping of (M,g).) Let us set

$$\mathsf{L}(s) \,=\, \mathsf{L}[\gamma^{(s)}]\,, \qquad \mathsf{A}(s) \,=\, \mathsf{A}[\gamma^{(s)}]\,,$$

where $\gamma^{(s)} : [a,b] \to M$ is given by $\gamma^{(s)}(t) = x(t,s)$. Thus, $2\,\mathsf{A}(s) = (x_t, x_t)$. The derivative $\mathsf{A}'(s) = d\mathsf{A}(s)/ds$, now is given by

$$(28.9) \qquad \mathsf{A}'(s) \,=\, -\,(x_{tt}, x_s)\,,$$

as one sees using (28.3) and differentiation by parts, and noting that $x_s(a,\,\cdot\,) = 0$, $x_s(b,\,\cdot\,) = 0$. Thus, since $\gamma^{(c)} = \gamma$ is a constant-speed minimizing geodesic, we have $2\,(b-a)\,\mathsf{A}(c) = [\mathsf{L}(c)]^2 \le [\mathsf{L}(s)]^2 \le 2\,(b-a)\,\mathsf{A}(s)$ for all $s \in [c,d]$ (in view of (28.7)), while (28.5) and (28.9) give $\mathsf{A}'(c) = 0$. These two relations clearly imply $\mathsf{A}''(c) \ge 0$. However, differentiating (28.9), we obtain $\mathsf{A}''(s) = -\,(x_{tts}, x_s) - (x_{tt}, x_{ss})$, and so (28.5), (28.4) and (28.1) yield $\mathsf{A}''(c) = -\,(w, \mathcal{J}w)$. As $\mathsf{A}''(c) \ge 0$, this proves our assertion. ∎

Theorem 28.7 (Myers, 1935)**.** *Let* (M,g) *be a complete Riemannian manifold of dimension* $n \ge 2$*, with*

$$\text{Ric} \;\ge\; (n-1)\,\delta \;>\; 0 \tag{28.10}$$

for some $\delta \in \mathbf{R}$*. Then*

(i) *The diameter of* (M,g) *satisfies the estimate*

$$\text{diam}\,(M,g) \;\le\; \pi/\sqrt{\delta}\,. \tag{28.11}$$

(ii) M *is compact and its fundamental group* $\pi_1 M$ *is finite.*

Proof. We will establish (i), by proving that the length estimate

$$(\mathsf{L}[\gamma])^2 \;\le\; \pi^2/\delta \tag{28.12}$$

holds for every minimizing geodesic $\gamma : [a,b] \to M$. To this end, let us fix parallel orthonormal vector fields $[a,b] \ni t \mapsto e_\lambda(t)$, $\lambda = 1,2,\ldots,n-1$, tangent to M along γ and orthogonal to $\dot\gamma$. The function $f(t) = \sin[(b-a)^{-1}\pi(t-a)]$ then satisfies $f(a) = f(b) = 0$ and $(b-a)^2\ddot f = -\pi^2 f$. For $w = w_\lambda = f e_\lambda$, formula (28.1) becomes $\mathcal{J} w_\lambda = -(b-a)^{-2}\pi^2 f e_\lambda - f\,R(e_\lambda,\dot\gamma)\dot\gamma$; thus, (28.8) applied to $w = w_\lambda$, $\lambda = 1,2,\ldots,n-1$, yields $0 \le -\sum_\lambda (w_\lambda, \mathcal{J} w_\lambda) = \int_a^b f^2\varphi\,dt$, with

$$\varphi \;=\; (n-1)(b-a)^{-2}\pi^2 \;-\; \text{Ric}\,(\dot\gamma,\dot\gamma)\,. \tag{28.13}$$

Since $f \ne 0$ in (a,b), this shows that $\varphi \ge 0$ somewhere in $[a,b]$. On the other hand, by (28.10), we have $\varphi \le (n-1)(b-a)^{-2}\delta[\pi^2\delta^{-1} - (b-a)^{-2}|\dot\gamma|^2]$ and, as $\sup\varphi \ge 0$, we obtain $(\mathsf{L}[\gamma])^2 = (b-a)^{-2}|\dot\gamma|^2 \le \pi^2\delta^{-1}$. This yields (28.12) and hence proves assertion (i). As for (ii), note that finiteness of $\text{diam}\,(M,g)$ shows that M must be compact. Applying this last conclusion to the Riemannian universal covering of (M,g), we see that $\pi_1 M$ is finite, as required. ■

Since the Einstein condition (0.1), in the case where $\mathrm{s} > 0$, clearly implies (28.10) with δ given by $n(n-1)\,\delta = \mathrm{s}$, Theorem 28.7 yields

Corollary 28.8. *Any complete Riemannian Einstein manifold of dimension* n*,* $n \ge 2$*, whose scalar curvature* s *is positive, is automatically compact, has a finite fundamental group, and its diameter does not exceed* $\pi\sqrt{n(n-1)}\,/\sqrt{\mathrm{s}}\,$. ■

We conclude this section with a classical result establishing a relation between Jacobi fields and the exponential mapping (4.16).

Proposition 28.9. *Let* ∇ *be a torsionfree connection in the tangent bundle* TM *of a manifold* M*, and let a vector* $v \in T_xM$ *lie in the domain* $U_x \subset T_xM$ *of the exponential mapping* $\exp_x : U_x \to M$*, cf.* (4.16)*. Then, for any* $u \in T_xM$*, the image of* u *under the differential of* $\exp_x$ *at* v *is given by*

$$d(\exp_x)_v u \;=\; w(1)\,, \tag{28.14}$$

where $[0,1] \ni t \mapsto w(t)$ *is the Jacobi field along the geodesic* $[0,1] \ni t \mapsto \exp_x tv$*, uniquely determined by the initial conditions* $w(0) = 0$ *and* $[\nabla_{\dot x} w](0) = u$*.*

Proof. Define $\Phi : \mathbf{R}^2 \to T_xM$ by $\Phi(t,s) = t(v+su)$. Since U_x is open and contains tv for all $t \in [0,1]$, we can find $r > 0$ such that the image under Φ of the rectangle $\Omega = [0,1] \times [0,r]$ lies entirely in U_x. The composite $F = \exp_x \circ \Phi : \Omega \to M$ clearly is a variation of geodesics, so that, according to Remark 28.5, formula $w(t) = F_s(t,0)$ defines a Jacobi field w along the geodesic $[0,1] \ni t \mapsto F(t,0) = \exp_x tv$. On the other hand, as $\partial[\Phi(t,s)]/\partial s = tu$, we obtain (28.14), since the left-hand side of (28.14) is nothing else than $d[\exp_x \Phi(t,s)]/ds$ at $t = 1$ and $s = 0$. Finally, $w(0) = 0$ since $\Phi(0,s) = 0$ and $F(0,s) = x$ for all $s \in [0,r]$, while $[\nabla_{\dot{x}} w](t) = F_{st}(t,0)$ equals u when $t = 0$, as one easily sees using the component formula (28.2) in normal coordinates at x (in which F appears identical to Φ), along with (4.18). This completes the proof. ■

§29. G-structures and G-connections

Let V be a vector space over the field $\mathbf{K}$ of real or complex numbers, with $\dim V = q$, $1 \le q < \infty$, and let $\mathcal{B} = \mathcal{B}(V)$ stand for the set of all bases of V. The matrix group $\mathrm{GL}(q,\mathbf{K})$ then acts on $\mathcal{B}$ freely and transitively by matrix multiplication from the right applied to bases treated as single-row matrices (with vector entries). Restricted to any fixed subgroup G of $\mathrm{GL}(q,\mathbf{K})$, this becomes a free action of G on $\mathcal{B}$. Any orbit $\mathbf{S}$ of the action of G just defined then is called a *G-structure* in V. Note that any given basis $e_1,\dots,e_q$ of V belongs to a unique G-structure in V, which may be called *the G-structure determined by* the basis e_a, $a = 1,\dots,q$.

Most geometric structures of any interest in vector spaces V of a given dimension q can equivalently be described as G-structures for suitable matrix groups $G \subset \mathrm{GL}(q,\mathbf{K})$. For instance, a positive definite real or complex inner product in V is uniquely characterized by the set of all of its orthonormal bases, which is an $\mathrm{O}(q)$ or $\mathrm{U}(q)$-structure. The vector space structure of V alone, without any additional distinguished geometry, corresponds to the set of all bases, i.e., a $\mathrm{GL}(q,\mathbf{K})$-structure; one fixed basis of V is an $\{\mathrm{Id}\}$-structure; an orientation of V (when $\mathbf{K} = \mathbf{R}$) is a $\mathrm{GL}^+(q,\mathbf{R})$-structure. Similarly, when $\mathbf{K} = \mathbf{R}$, a Euclidean inner product in V coupled with an orientation of V constitutes an $\mathrm{SO}(q)$-structure, while a volume density $\pm\nu$ in V (see Remark 24.1) is a $\mathrm{SL}^{\pm}(q,\mathbf{R})$-structure, $\mathrm{SL}^{\pm}(q,\mathbf{R})$ being the group of all real $q \times q$ matrices of determinant ± 1. (The corresponding bases $e_1,\dots,e_q$ then are characterized by $\nu(e_1,\dots,e_q) = \pm 1$.)

Let $\mathcal{E}$ now be a real or complex vector bundle of fibre dimension q over a manifold M and let G be a Lie subgroup of the matrix group $\mathrm{GL}(q,\mathbf{K})$, with $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. By a *G-structure* in $\mathcal{E}$ we then mean a mapping assigning to each $x \in M$ a G-structure $\mathbf{S}_x$ in the fibre $\mathcal{E}_x$ with the property that for any point $x \in M$ there exists a local C^∞ trivialization e_a of $\mathcal{E}$ defined on a neighborhood U of x and such that, for each $y \in U$, the G-structure $\mathbf{S}_y$ in $\mathcal{E}_y$ is determined by (i.e., contains) the basis $e_a(y)$, $a = 1,\dots,q$. The group G is referred to as the *structure group* of the given G-structure $\mathbf{S}$, while local C^∞ trivializations e_a with the property just stated then are said to *be compatible with* (or, *belong to*) the G-structure. According to the preceding paragraph, an $\mathrm{O}(n)$ or $\mathrm{U}(n)$-structure may be identified with a Riemannian (or, respectively, Hermitian) fibre metric $\langle\,,\rangle$ in $\mathcal{E}$; for local C^∞ trivializations, compatibility with this structure amounts to being orthonormal.

Suppose now that we are given a G-structure $\mathbf{S}$ in a real or complex vector bundle $\mathcal{E}$ of some fibre dimension q over a manifold M. A connection ∇ in $\mathcal{E}$ is said to be *compatible* with the G-structure $\mathbf{S}$ (or, briefly, to be a *G-connection*) if the set of bases forming $\mathbf{S}$ is invariant under the ∇-parallel transports along all C^1 curves in M. As an example, every connection in $\mathcal{E}$ is a $\mathrm{GL}(q,\mathbf{K})$-connection; at the other extreme, an $\{\mathrm{Id}\}$-connection is one that makes the given system of global trivializing sections parallel. (Thus, by (4.52), an $\{\mathrm{Id}\}$-connection is necessarily flat.)

Lemma 29.1. *Let G be a Lie subgroup of a Lie group H, and let $I \ni t \mapsto \Psi(t) \in H$ be a C^1 curve defined on an interval $I \subset \mathbf{R}$ and such that $\Psi(t_0) \in G$ for some fixed parameter $t_0 \in I$. Furthermore, let $\Phi : I \to T_1H$ be the curve of vectors tangent to H at the unit element 1, given by $\Phi = \dot\Psi\Psi^{-1}$, in the sense that $\Phi(t)$ is the image of $\dot\Psi(t) \in T_{\Psi(t)}H$ under the differential of the right multiplication by $[\Psi(t)]^{-1}$ in H. Then, in order that $\Psi(t) \in G$ for all $t \in I$, it is necessary and sufficient that $\Phi(t) \in \mathfrak{g}$ for all t, where $\mathfrak{g}$ is the Lie algebra of G, identified with the tangent space $T_1G \subset T_1H$.*

Proof. Necessity is obvious, as the right multiplication by $[\Psi(t)]^{-1}$ then sends G into G. Sufficiency: Solve $\dot\Psi(t) = \Phi(t)\Psi(t)$ with $\Psi(t_0) = \Psi_0 \in G$ as an initial value problem with the unknown curve $\Psi(t)$, and note that its solutions in G and H must coincide, due to uniqueness of solutions in G. ■

Lemma 29.2. *Suppose that we are given a manifold M, a vector bundle $\mathcal{E}$ over M, of some fibre dimension q over a scalar field $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, and a fixed G-structure $\mathbf{S}$ in $\mathcal{E}$. For any connection ∇ in $\mathcal{E}$, the following conditions are equivalent*:

(i) *∇ is a G-connection, i.e., is compatible with $\mathbf{S}$*;

(ii) *For some, or every, family of local C^∞ trivializations e_a compatible with the G-structure $\mathbf{S}$ and local coordinate systems x^j, whose coordinate-and-trivialization domains U cover M, all the matrices with matrix indices a, b obtained by fixing j and $x \in U$ in the connection components $\Gamma_{ja}^b(x)$ defined by* (4.48) *are elements of the matrix Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(q,\mathbf{K})$ corresponding to the matrix group $G \subset \mathrm{GL}(n,\mathbf{K})$.*

Proof. Let e_a be a fixed local trivialization of $\mathcal{E}$ over an open set $U \subset M$, compatible with $\mathbf{S}$, and let $I \ni t \mapsto \psi_a(t) \in \mathcal{E}_{x(t)}$, $a = 1,\dots,q$, be a ∇-parallel field of bases of the fibres $\mathcal{E}_{x(t)}$ along a C^1 curve $I \ni t \mapsto x(t) \in U$ such that, for some $t_0 \in I$, the basis $\psi_a(t_0)$ of $\mathcal{E}_{x(t_0)}$ is compatible with $\mathbf{S}_{x(t_0)}$. We thus have $d\psi_a^b/dt = -\Gamma_{jc}^b(x(t))\dot x^j(t)\psi_a^c(t)$, $\psi_a^b(t)$ being the components of $\psi_a(t)$ characterized by $\psi_a(t) = \psi_a^b(t)e_a(x(t))$. In other words, the matrix-valued function $t \mapsto \Psi(t) = [\psi_a^b(t)]$ satisfies the condition $\dot\Psi\Psi^{-1} = \Phi$, $\Phi(t)$ being the matrix-valued function with the components $\phi_a^b(t) = -\Gamma_{ja}^b(x(t))\dot x^j(t)$. By Lemma 29.1, the basis $\psi_a(t)$ of the fibre at $x(t)$ is compatible with $\mathbf{S}_{x(t)}$ for all t if and only if $\Phi(t) \in \mathfrak{g}$ for all t. This completes the proof. ■

Corollary 29.3. *For M, $\mathcal{E}$, q and $\mathbf{K}$ as in* Lemma 29.2 *and a Lie subgroup G of $\mathrm{GL}(q,\mathbf{K})$ with the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(q,\mathbf{K})$, let ∇ be a G-connection for some fixed G-structure $\mathbf{S}$ in $\mathcal{E}$, and let an open set $U \subset M$ be the domain of both a local trivialization e_a for $\mathcal{E}$ compatible with $\mathbf{S}$, and a local coordinate system x^j in M. If $R_{jka}{}^b$ are the component functions* (4.53) *of the curvature tensor R^∇ of*

∇ *corresponding to the* e_a *and* x^j, *then the matrix* $R_{jka}{}^b(x)$ *with the indices* a, b, *obtained by fixing* j, k *and* $x \in U$, *is an element of* $\mathfrak{g}$.

In fact, this is obvious from Lemma 29.2 and (4.53) (note that the last two terms in (4.53) represent a matrix commutator). ■

Remark 29.4. The statements about "matrices with the indices a, b" in Lemma 29.2 and Corollary 29.3 have an invariant (i.e., trivialization-independent) meaning, which can be exhibited as follows. A fixed G-structure $\mathbf{S}$ in a vector bundle $\mathcal{E}$ with q, M, $\mathbf{K}$, G and $\mathfrak{g}$ as above gives rise to the *associated bundle* $\mathcal{G}$ *of Lie groups* and the *associated bundle* ϑ *of Lie algebras*, with the fibres $\mathcal{G}_x$ and ϑ_x over any $x \in M$ such that $\mathcal{G}_x$ consists of all linear isomorphisms $\mathcal{E}_x \to \mathcal{E}_x$ which, acting on bases of $\mathcal{E}_x$, leave the set $\mathbf{S}_x$ invariant, while $\vartheta_x \subset \mathfrak{gl}(\mathcal{E}_x)$ is the Lie algebra of $\mathcal{G}_x$ (i.e., its tangent space at the identity). The $\mathcal{G}_x$ and ϑ_x are all (non-canonically) isomorphic to G and, respectively, $\mathfrak{g}$; in fact, from the definition of a G-structure it is clear that, in any local trivialization e_a for $\mathcal{E}$ compatible with $\mathbf{S}$, elements F of $\mathcal{G}_x$ and ϕ of ϑ_x are precisely those operators $\mathcal{E}_x \to \mathcal{E}_x$ whose component matrix $[F_a^b]$ or $[\phi_a^b]$ is an element of G or, respectively, $\mathfrak{g}$. Corollary 29.3 thus states that the curvature R^∇ of any G-connection ∇ sends any pair of vectors $v, w \in T_xM$, $x \in M$, to an element $R^\nabla(v, w)$ of ϑ_x (and not just any operator in $\mathfrak{gl}(\mathcal{E}_x) = \operatorname{Hom}(\mathcal{E}_x, \mathcal{E}_x)$). Similarly, Lemma 29.2 implies that the difference $\tilde{\nabla} - \nabla$ (which, for any two connections ∇ and $\tilde{\nabla}$, is a "tensor"; see beginning of §16), in the case of two G-connections must be a section of the subbundle $\operatorname{Hom}(TM, \vartheta)$. This last assertion is an obvious consequence of (4.49).

§30. Spin$_c$-structures and spinor bundles

Throughout this section, G will denote the 7-dimensional Lie subgroup

$$G = \{(\mathfrak{A}, \mathfrak{B}, z) \in \mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(1) : \det \mathfrak{A} = \det \mathfrak{B} = z\} \tag{30.1}$$

of the matrix group $\mathrm{U}(2) \times \mathrm{U}(2) \times \mathrm{U}(1)$. Note that G is also isomorphic to a subgroup of $\mathrm{U}(2) \times \mathrm{U}(2)$, via the projection

$$(\mathfrak{A}, \mathfrak{B}, z) \mapsto (\mathfrak{A}, \mathfrak{B}). \tag{30.2}$$

Remark 30.1. The group G given by (30.1) is connected. In fact, let H denote the subgroup $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \{1\}$ of G (that is, the set of all $(\mathfrak{A}, \mathfrak{B}, z) \in G$ with $\det \mathfrak{A} = \det \mathfrak{B} = z = 1$). Thus, H is connected, since so is $\mathrm{SU}(2)$. (To verify the latter assertion, fix a matrix $\mathfrak{A} \in \mathrm{SU}(2)$ and choose an orthonormal basis $\mathbf{u}$, $\mathbf{v}$ of $\mathbf{C}^2$ such that $\mathbf{u}$ is an eigenvector of $\mathfrak{A}$; then, $\mathfrak{A}\mathbf{u} = e^{ip}\mathbf{u}$, $\mathfrak{A}\mathbf{v} = e^{-ip}\mathbf{v}$ for some real p, and relations $\mathfrak{A}_t\mathbf{u} = e^{itp}\mathbf{u}$, $\mathfrak{A}_t\mathbf{v} = e^{-itp}\mathbf{v}$ define a curve $[0,1] \ni t \mapsto \mathfrak{A}_t$ connecting Id to $\mathfrak{A}$ in $\mathrm{SU}(2)$.) Any $(\mathfrak{A}, \mathfrak{B}, e^{i\theta}) \in G$ now can be joined to $(e^{-i\theta/2}\mathfrak{A}, e^{-i\theta/2}\mathfrak{B}, 1) \in H$ by the curve

$$[0,1] \ni t \mapsto (e^{-it\theta/2}\mathfrak{A}, e^{-it\theta/2}\mathfrak{B}, e^{i(1-t)\theta}) \in G,$$

and hence connectivity of G follows from that of H.

For G defined by (30.1), by a G-*bundle* over a manifold M we mean a triple $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ formed by two complex plane bundles $\mathcal{S}^\pm$ and a complex line bundle

$\mathcal{K}$ over M which all carry fixed Hermitian fibre metrics (denoted $\langle\, ,\rangle$) and whose fibres over each $x \in M$ are endowed with skew-symmetric bilinear multiplications $\mathcal{S}_x^\pm \times \mathcal{S}_x^\pm \to \mathcal{K}_x$, both denoted $(\psi, \phi) \mapsto \psi\phi$, which depend C^∞-differentiably on x, and are *normalized* in the sense that

$$(30.3) \qquad |\psi\phi| = |\psi|\cdot|\phi| \qquad \text{whenever} \quad \psi\,,\,\phi \quad \text{are orthogonal.}$$

In terms of $\mathcal{S}^\pm$ alone, the above conditions simply state that there is a fixed isomorphic identification $[\mathcal{S}^+]^{\wedge 2} = [\mathcal{S}^-]^{\wedge 2}$ between the highest exterior powers of $\mathcal{S}^+$ and $\mathcal{S}^-$, with the further identifications

$$(30.4) \qquad \mathcal{K} = [\mathcal{S}^+]^{\wedge 2} = [\mathcal{S}^-]^{\wedge 2}\,,$$

where $\psi\phi$ equals $\psi \wedge \phi$ times a positive scaling function. On the other hand, the normalization condition (30.3) means that this identification is norm-preserving, which can always by achieved by a suitable choice of the scaling functions.

Remark 30.2. To verify the normalizing conditions (30.3), it suffices to verify that $|\psi^\pm\phi^\pm\| = 1$ for *some* orthonormal basis ψ^+, ϕ^+ of $\mathcal{S}_x^+$ and *some* orthonormal basis ψ^-, ϕ^- of $\mathcal{S}_x^-$, since the transition determinant between two orthonormal bases is of modulus 1.

The direct sum $\mathcal{S}^+ \oplus \mathcal{S}^- \oplus \mathcal{K}$ then carries a naturally distinguished G-structure (§29), with G as in (30.1), which associates with each $x \in M$ the G-orbit of bases of the fibre at x, consisting of all $(\psi^+, \phi^+, \psi^-, \phi^-, \rho)$ satisfying the conditions

$$(30.5) \qquad \psi^\pm, \phi^\pm \in \mathcal{S}_x^\pm\,, \quad |\psi^\pm| = |\phi^\pm| = 1\,, \quad \langle\psi^\pm, \phi^\pm\rangle = 0\,, \quad \psi^+\phi^+ = \psi^-\phi^-\,,$$

and $\rho = \psi^\pm\phi^\pm \in \mathcal{K}_x$.

For any $x \in M$, let us now define the *determinant mapping*

$$(30.6) \qquad \det : \operatorname{Hom}(\mathcal{S}_x^+, \mathcal{S}_x^-) \to \mathbf{C}$$

by requiring that any $F \in \operatorname{Hom}(\mathcal{S}_x^+, \mathcal{S}_x^-)$ act on the fibre of (30.4) over x via multiplication by $\det F$. (Equivalently, $\det F$ is the matrix determinant of F computed using bases $(\psi^\pm, \phi^\pm)$ of $\mathcal{S}_x^\pm$ that satisfy (30.5).) Furthermore, we will call $F \in \operatorname{Hom}(\mathcal{S}_x^+, \mathcal{S}_x^-)$ a *homothety* if it is the composite of a norm-preserving isomorphism with the multiplication by a real scalar, i.e., if

$$(30.7) \qquad |F\phi| = |F| \cdot |\phi|$$

for some real $|F| \geq 0$ and all $\phi \in \mathcal{S}_x^+$ (where $|\,|$ also stands for the norm in $\mathcal{S}_x^\pm$ corresponding to the Hermitian inner product $\langle\, ,\rangle$). The set

$$(30.8) \qquad \mathcal{V}_x = \{F \in \operatorname{Hom}(\mathcal{S}_x^+, \mathcal{S}_x^-) : \; F \text{ is a homothety and } \det F \in [0, \infty)\}\,,$$

with $\det$ as in (30.6), then is a 4-dimensional real vector space with a Euclidean inner product (also denoted $\langle\, ,\rangle$), whose norm $|\,|$, characterized by $\langle F, F\rangle = |F|^2$, coincides with that defined by (30.7), and satisfies the relations

$$(30.9) \qquad |F|^2 = \det F = \frac{1}{2}\operatorname{Trace} F^*F\,, \qquad F^*F = |F|^2\,\mathrm{Id}$$

for all $F \in \mathcal{V}_x$. In fact, one easily sees that any fixed pair (ψ^+, ϕ^+), (ψ^-, ϕ^-) of bases with (30.5) identifies each of $\mathcal{S}_x^+$ and $\mathcal{S}_x^-$ with $\mathbf{C}^2$ in such a way that $\mathcal{V}_x$ corresponds to the set of all matrix operators

$$F_{ab} = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \tag{30.10}$$

with $a, b \in \mathbf{C}$. As operators $\mathcal{S}_x^+ \to \mathcal{S}_x^-$, the F_{ab} then are characterized by

$$F_{ab}\psi^+ = a\psi^- + b\phi^-\,, \qquad F_{ab}\phi^+ = -\overline{b}\psi^- + \overline{a}\phi^-\,. \tag{30.11}$$

Relations (30.9) for F given by (30.10) are immediate, and so is (30.7) with ϕ replaced by a vector in $\mathbf{C}^2$. (Equality (30.7) then is nothing else than to multiplicativity of the quaternion norm.)

Remark 30.3. Each of the spaces $\mathcal{V}_x$ just described is, in addition, naturally oriented. In fact, let $\mathcal{B}$ be the set of all bases $(\psi^+, \phi^+, \psi^-, \phi^-)$ of $\mathcal{S}_x^+ \oplus \mathcal{S}_x^-$ satisfying (30.5). Assigning to $(\psi^+, \phi^+, \psi^-, \phi^-) \in \mathcal{B}$ the orthonormal basis

$$F_{10},\, F_{i0},\, F_{01}\, F_{0i} \tag{30.12}$$

of $\mathcal{V}$, with F_{ab} as in (30.11), we now obtain a natural continuous mapping $\varPhi$ of $\mathcal{B}$ into the set of all bases of $\mathcal{V}$. On the other hand, $\mathcal{B}$ is connected, since it is an orbit of the matrix group G projected into $\mathrm{U}(2) \times \mathrm{U}(2)$ via (30.2) (which itself is connected according to Remark 30.1). The bases (30.12), forming a connected set (the image of $\varPhi$), thus all represent the same distinguished orientation of $\mathcal{V}_x$.

Consequently, the $\mathcal{V}_x$ are the fibres of a real vector bundle $\mathcal{V}$ carrying a natural SO(4)-structure (§29), i.e., a Riemannian fibre metric along with an orientation.

For a *G-bundle* $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ over a manifold M as defined above, let ∇^+ and ∇^- be any pair of U(2)-connections (see §29) in $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively, that both induce, via (30.4), the same U(1)-connection A in $\mathcal{K}$. (This simply means that the direct sum $\nabla^+ \oplus \nabla^- \oplus A$ is a G-connection in $\mathcal{S}^+ \oplus \mathcal{S}^- \oplus \mathcal{K}$.) Furthermore, let ∇ be the SO(4)-connection that ∇^+, ∇^- and A naturally induce in $\mathcal{V}$. The assignment

$$(\nabla^+,\, \nabla^-) \mapsto (A\,, \nabla)\,, \tag{30.13}$$

thus defined is *bijective.* This fact (justified by a direct argument given in the next paragraph) really amounts to the statement that the Lie-group homomorphism $G \to \mathrm{U}(1) \times \mathrm{SO}(4)$, underlying the construction leading from $\mathcal{S}^+$ and $\mathcal{S}^-$ to $\mathcal{K}$ and $\mathcal{V}$, induces an *isomorphism* of Lie algebras, and so (30.13) is bijective since the local connection components constitute 1-forms valued in the appropriate matrix Lie algebra (Lemma 29.2).

To see directly why (30.13) must be bijective, let us fix local trivializations of $\mathcal{S}^\pm$ and $\mathcal{K}$, consisting of C^∞ sections $\psi^+, \phi^+, \psi^-, \phi^-, \rho$ of the appropriate bundles, all defined on an open subset U of M and satisfying (30.5) with $\rho(x) = \psi^\pm(x)\phi^\pm(x) \in \mathcal{K}_x$ at any $x \in U$. Given $\nabla^\pm$ as above, let us fix $x \in U$ and $v \in T_xM$. Then, let us apply d_v to the inner products $\langle \psi^\pm, \psi^\pm \rangle$, $\langle \phi^\pm, \phi^\pm \rangle$, $\langle \psi^\pm, \phi^\pm \rangle$, and A_v to $\rho = \psi^\pm\phi^\pm$ (A being the connection induced in $\mathcal{K}$ by ∇^+ and ∇^-). Since $\langle \rho, \rho \rangle = 1$, the Leibniz rule gives $\nabla_v^\pm \psi^\pm = is^\pm\psi^\pm + z^\pm\phi^\pm$, $\nabla_v^\pm \phi^\pm = i(r -$

$s^\pm)\phi^\pm - \overline{z^\pm}\phi^\pm$ for some $r, s^\pm \in \mathbf{R}$ and $z^\pm \in \mathbf{C}$ (all depending on x and v), with $A_v\rho = ir\rho$. Next, for any fixed $(a,b) \in \mathbf{C}^2$, let F_{ab} be the local section of $\mathcal{V}$ given by (30.11). Using the Leibniz rule $(\nabla_v F)\psi = \nabla_v^-(F\psi) - F(\nabla_v^+\psi)$ for local C^1 sections ψ of $\mathcal{S}^+$ and F of $\mathcal{V}$, with ∇ standing for the connection in $\mathcal{V}$ induced by ∇^+ and ∇^-, we easily verify that $\nabla_v F_{ab} = F_{cd}$ with $(c,d) \in \mathbf{C}^2$ given by $c = ia(s^+ - s^-) - \overline{b}z^+ - b\overline{z^-}$, $d = ib(r - s^+ - s^-) + az^- + \overline{a}z^+$, so that the column vector $[\operatorname{Re} c,\ \operatorname{Im} c,\ \operatorname{Re} d,\ \operatorname{Im} d]^{\mathrm{T}}$ equals

$$\begin{bmatrix} 0 & s^+ - s^- & -\operatorname{Re}(z^+ + z^-) & -\operatorname{Im}(z^+ + z^-) \\ s^- - s^+ & 0 & \operatorname{Im}(z^- - z^+) & \operatorname{Re}(z^+ - z^-) \\ \operatorname{Re}(z^+ + z^-) & \operatorname{Im}(z^+ - z^-) & 0 & s^+ + s^- - r \\ \operatorname{Im}(z^+ + z^-) & \operatorname{Re}(z^- - z^+) & r - s^+ - s^- & 0 \end{bmatrix} \begin{bmatrix} \operatorname{Re} a \\ \operatorname{Im} a \\ \operatorname{Re} b \\ \operatorname{Im} b \end{bmatrix}. \tag{30.14}$$

Denoting $\mathfrak{so}(4)$ the Lie algebra of all skew-symmetric real 4×4 matrices, we now easily see that the assignment

$$\mathbf{R}^3 \times \mathbf{C}^2 \ni (s^+, s^-, r, z^+, z^-) \mapsto (r, \mathfrak{M}) \in \mathbf{R} \times \mathfrak{so}(4),$$

where $\mathfrak{M}$ is the 4×4 matrix in (30.14), is bijective. This proves bijectivity of (30.13), as required: In fact, the columns of $\mathfrak{M}$ as above provide coefficients in the expansion of the ∇_v-derivatives of the local orthonormal trivializing sections given by (30.12) as combinations of the sections (30.12).

Let us now suppose that (M,g) is an oriented Riemannian 4-manifold and $\mathcal{K}$ is a complex line bundle over M with a fixed U(1)-structure (a Hermitian fibre metric $\langle\,,\rangle$). By a spin$_\mathrm{c}$-*structure for* (M,g) *associated with* $(\mathcal{K}, \langle\,,\rangle)$ we mean a G-bundle of the form $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ over M, along with a fixed, norm- and orientation-preserving, real-isomorphic identification

$$(\mathcal{V}, \langle\,,\rangle) = (TM, g). \tag{30.15}$$

(In particular, the bundles $\mathcal{V}$ and TM then are assumed isomorphic.) Every vector $v \in T_xM$ thus constitutes an operator $\mathcal{S}_x^+ \to \mathcal{S}_x^-$, called the *Clifford multiplication* by v and denoted $\phi \mapsto v\phi$. Hence, by (30.7) and (30.9) with $F = v$, we have $|v\phi| = |v| \cdot |\phi|$ and $v^*(v\phi) = g(v,v)\phi$ for all $x \in M$, $v \in T_xM$, and $\phi \in \mathcal{S}_x^+$. Let us denote $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ the vector bundle over M with a Hermitian fibre metric $\langle\,,\rangle$, obtained as the orthogonal direct sum of $\mathcal{S}^+$ and $\mathcal{S}^-$. The Clifford multiplication by a vector $v \in T_xM$ then is usually extended to a *skew-Hermitian* operator

$$v : \mathcal{S}_x \to \mathcal{S}_x, \qquad v(\mathcal{S}_x^\pm) \subset \mathcal{S}_x^\mp, \tag{30.16}$$

by declaring its action on $\mathcal{S}_x^-$ to be the negative adjoint $-v^* : \mathcal{S}_x^- \to \mathcal{S}_x^+$ of $v : \mathcal{S}_x^+ \to \mathcal{S}_x^-$. The preceding equalities then become the *Clifford-algebra relations*

$$|v\phi| = |v| \cdot |\phi|, \qquad v(v\phi) = -g(v,v)\,\phi, \tag{30.17}$$

valid for all $x \in M$, $v, w \in T_xM$ and $\phi \in \mathcal{S}_x$. Consequently (see Remark 3.12), for any such x, ϕ and $v, w \in T_xM$,

$$v(w\phi) + w(v\phi) = -2\,g(v,w)\,\phi, \tag{30.18}$$

since both sides are bilinear and symmetric in v, w, and coincide when $v = w$.

Example 30.4. Every almost Hermitian manifold (M, g, α) of real dimension four, endowed with the canonical orientation (see §9), carries a distinguished spin$_{\mathrm{c}}$-structure $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$. Specifically,we define $\mathcal{S}^-$ to be the tangent bundle TM treated as a complex vector bundle, with the Hermitian fibre metric (9.3). For $\mathcal{S}^+$ we choose the (real) subbundle of $\mathrm{Hom}_{\mathbf{R}}(TM, TM)$ spanned by $\Lambda^+ M$ (see (6.4)) along with the identity section $\mathrm{Id} : TM \to TM$. As for $\mathcal{K}$, we let it be the real-plane subbundle of $\Lambda^+ M$ obtained as the orthogonal complement of its real-line subbundle spanned by α. The real fibre dimension of $\mathcal{S}^+$ thus equals four, and $\mathcal{K}$ is a subbundle of $\mathcal{S}^+$. Moreover, $\mathrm{Hom}_{\mathbf{R}}(TM, TM)$ becomes a complex vector bundle if we declare the multiplication by i in each fibre $\mathrm{Hom}_{\mathbf{R}}(T_x M, T_x M)$, $x \in M$, to be the operator $\beta \mapsto \alpha\beta$ of the composition with $\alpha = \alpha(x)$. The subbundles $\mathcal{S}^+$ and $\mathcal{K}$ of $\mathrm{Hom}_{\mathbf{R}}(TM, TM)$ are both invariant under this multiplication by i, that is, $\mathcal{K}$ a complex-line subbundle of the complex-plane bundle $\mathcal{S}^+$. (To see this, note that, by Corollary 9.4, α is a section of $\Lambda^+ M$, and choose a basis $\alpha_1, \alpha_2, \alpha_3$ of $\Lambda^+_x M$, $x \in M$, as in Corollary 6.5, with $\alpha_1 = \alpha(x)$; the required invariance properties now are obvious from (6.12).) The Clifford multiplication by a vector $v \in T_x M$, $x \in M$, now is simply the *evaluation operator* $\mathcal{S}^+_x \to \mathcal{S}^-_x$ given by $\beta \mapsto \beta v$, which is complex-linear as $(\alpha\beta)v = \alpha(\beta v)$. Let $\langle\,,\rangle_\bullet$ be the fibre metric in $\mathrm{Hom}_{\mathbf{R}}(TM, TM)$ given by $4\langle\beta, \gamma\rangle_\bullet = \mathrm{Trace}\,\beta, \gamma^*$. (Thus, $\langle\,,\rangle_\bullet$ differs from the inner products we normally use for twice-covariant tensors: By a factor of two for skew-symmetric tensors, by a factor of four for symmetric ones.) Since $\alpha = -\alpha^* = -\alpha^{-1}$, we thus have $\langle\alpha\beta, \alpha\gamma\rangle_\bullet = \langle\beta, \gamma\rangle_\bullet$, and so (as in Remark 3.18), $\langle\,,\rangle_\bullet$ is the real part of a unique Hermitian complex-sesquilinear inner product, which we will denote $\langle\,,\rangle'$. Note that Id and α_2 thus form, at any $x \in M$, a $\langle\,,\rangle'$-orthonormal complex basis of $\mathcal{S}^+_x$, while α_2 is a unit vector spanning the complex line $\mathcal{K}_x$, and Id spans the complex line $\mathcal{K}^\perp_x$, i.e., the real span of Id and α, which also is the $\langle\,,\rangle'$-orthogonal complement of $\mathcal{K}_x$.

The $\mathcal{K}_x$-valued skew-symmetric multiplications in $\mathcal{S}^+_x$ and $\mathcal{S}^-_x$ are given by $(\beta, \gamma) \mapsto \beta\bullet\gamma)$ and $(v, w) \mapsto vw = 2\,\mathrm{pr}\,(v \wedge w)$, where $\mathrm{pr} : \mathrm{Hom}_{\mathbf{R}}(TM, TM) \to \mathcal{K}$ is the morphism of orthogonal projection, while the bivector $v \wedge w$ is treated, with the aid of g, as a (skew-adjoint) operator $T_x M \to T_x M$. As for $\beta \bullet \gamma$ (where we use $\bullet$ to distinguish it from the composite $\beta\gamma$ of β and γ), it is uniquely characterized by the requirement of being real-bilinear plus the conditions $\beta \bullet \gamma = \beta\gamma$ when $\beta \in \mathcal{K}_x$ and $\gamma \in \mathcal{K}^\perp_x$, $\beta \bullet \gamma = -\beta\gamma$ when $\beta \in \mathcal{K}^\perp_x$ and $\gamma \in \mathcal{K}_x$, and $\beta \bullet \gamma = 0$ when both β and γ are in $\mathcal{K}_x$ or $\mathcal{K}^\perp_x$ (where we use $\bullet$ to distinguish it from the composite $\beta\gamma$). Both skew-symmetric multiplications just described are complex-bilinear. Namely, $(\alpha\beta)\bullet\gamma = \beta\bullet(\alpha\gamma) = \alpha(\beta\bullet\gamma)$ since both $\mathcal{K}_x$ and $\mathcal{K}^\perp_x$ are complex subspaces, that is, are invariant under $\beta \mapsto \alpha\beta$. Similarly, $\mathrm{pr}\,[(\alpha v) \wedge w] = \mathrm{pr}\,[v \wedge (\alpha w)] = \alpha[\mathrm{pr}\,(v \wedge w)]$ for $v, w \in T_x M$, $x \in M$, as an immediate consequence of (6.33), (9.1), (2.23) and (2.14).

According to Remark 30.2, to establish the normalization conditions (30.3) for the skew-symmetric multiplications just described, it suffices to do it for the orthonormal bases $\psi^\pm, \phi^\pm$ of $\mathcal{S}^\pm_x$ defined by

$$(\psi^+, \phi^+, \psi^-, \phi^-) = (\mathrm{Id}\,,\, \alpha_2\,,\, e_1,\,,\, e_3)\,, \tag{30.19}$$

where $e_1, \ldots, e_4$ is a positive-oriented orthonormal basis of $T_x M$ such that

$$\alpha_1/\sqrt{2}\,,\; \alpha_2/\sqrt{2}\,,\; \alpha_3/\sqrt{2}$$

are given by (6.10) (cf. Remark 6.19). We then actually have $\mathrm{Id}\bullet\alpha_2=\alpha_2$ and $e_1e_3=2\,\mathrm{pr}\,(e_1\wedge e_3)=e_1\wedge e_3+e_4\wedge e_2=\alpha_2$, while $\langle\alpha_2,\alpha_2\rangle'=1$. Similarly, the Clifford product satisfies $\langle\beta v,\beta v\rangle_\bullet=|v|^2\langle\beta,\beta\rangle_\bullet$, as one sees writing $\beta=r+\gamma$ with $r\in\mathbf{R}$ standing for r times Id, and then using (6.38) to verify that $|(r+\gamma)v|^2=[r^2+\langle\gamma,\gamma\rangle/2]|v|^2$, as required.

Finally, the Clifford multiplication $F=F_v:\mathcal{S}_x^+\to\mathcal{S}_x^-$ by any vector $v\in T_xM$ has a real nonnegative determinant. Namely, writing $v_j=g(v,e_j)$, we have, in bases (30.19), $\mathrm{Id}\,v=ae_1+be_3$, $\alpha_2v=-\,\overline{b}e_1+\overline{a}e_3$, with $a=v_1+iv_2$, $b=v_3+iv_4$, i.e., F has the matrix representation (30.10). Hence $\det F=|a|^2+|b|^2$, and the corresponding basis (30.12) is nothing else than $e_1,\dots,e_4$.

Now, assigning to any $v\in T_xM$, $x\in M$, the Clifford multiplication $F=F_v$, we obtain a norm-preserving isomorphism and orientation-preserving vector-bundle isomorphism $TM\to\mathcal{V}$.

Example 30.5. In the case where $\mathcal{K}=M\times\mathbf{C}$ is the trivial product line bundle over the given oriented Riemannian 4-manifold (M,g) with the standard (constant) fibre metric $\langle\,,\rangle$ (that is, $\langle\rho,\xi\rangle=\rho\overline{\xi}$ for $\rho,\xi\in\mathcal{K}_x=\mathbf{C}$), a spin$_\mathrm{c}$-structure for (M,g) associated with $(\mathcal{K},\langle\,,\rangle)$ is called a *spin structure* for (M,g) (Milnor, 1963; Lawson and Michelsohn, 1989). Instead of $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$, such a spin structure will simply be denoted $(\mathcal{S}^+,\mathcal{S}^-)$. The distinguished unit section of $\mathcal{K}$ formed by the constant function 1 reduces the structure group from G given by (30.1) to the 4-dimensional group

$$\mathrm{Spin}(4)\;=\;\mathrm{SU}(2)\times\mathrm{SU}(2)\,. \tag{30.20}$$

The orthogonal direct sum $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$ then carries a natural Spin(4)-structure (§29). We call $\mathcal{S}$ the *spinor bundle* corresponding to the given spin structure, and sections of $\mathcal{S}$ are referred to as *spinor fields* on M. The Clifford multiplication by a vector $v\in T_xM$ then is, again, extended to a skew-Hermitian operator $v:\mathcal{S}_x\to\mathcal{S}_x$ as in (30.16), which satisfies the Clifford-algebra relations (30.17) and (30.18). Let

$$A\;=d \tag{30.21}$$

now stand for the standard flat connection in the product line bundle $\mathcal{K}=M\times\mathbf{C}$. (Our notation reflects the fact that, for vectors v tangent to M, the covariant derivative A_v then is nothing else than the directional derivative d_v acting on sections of $\mathcal{K}$, which may be treated as functions $M\to\mathbf{C}$.) Furthermore, let ∇^+ and ∇^- be the pair of U(2)-connections in $\mathcal{S}^+$ and $\mathcal{S}^-$, respectively, that corresponds under the bijection (30.13) to (A,∇), where $A=d$ and ∇ denotes the Levi-Civita connection of (M,g) (regarded, in view of (30.15), as a connection in $\mathcal{V}$). The direct-sum connection

$$\nabla\;=\;\nabla^+\oplus\nabla^- \tag{30.22}$$

(also denoted ∇) in the spinor bundle $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$ is called the *spinor connection* associated with the spin structure $(\mathcal{S}^+,\mathcal{S}^-)$ and the Riemannian metric g on M.

Remark 30.6. Let $(\mathcal{S}^+,\mathcal{S}^-)$ be a spin structure over a Riemannian manifold (M,g), and let $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$ be the corresponding spinor bundle. For any point $x\in M$, the only linear operators $Q:\mathcal{S}_x\to\mathcal{S}_x$ commuting with all Clifford multiplications by vectors $v \in T_xM$ are multiples of Id. In fact, every pair of nonzero spinors $\chi^\pm \in \mathcal{S}^\pm$, the plane $\mathrm{Span}\,\{\chi^+, \chi^-\}$ in $\mathcal{S}$ is Q-invariant. To see this, pick $\phi^\pm \in \mathcal{S}^\pm$ with $\chi^\pm\phi^\pm = 1$ and set $\psi^\pm = \chi^\pm \mp i\phi^\pm$. Thus, $\psi^\pm \in \mathcal{S}^\pm$ and $\psi^\pm\phi^\pm = 1$, so that the basis $(\psi^+, \phi^+, \psi^-, \phi^-)$ of $\mathcal{S}_x = \mathcal{S}_x^+ \oplus \mathcal{S}_x^-$ satisfies (30.5). Let the vectors $v, w \in T_xM$ given by $v = F_{10}$, $w = F_{01}$ (with (30.12), for that basis), with the identification (30.15)), act on $\mathcal{S}_x$ as in (30.17). The plane $\mathrm{Span}\,\{\chi^+, \chi^-\}$ is the eigenspace for the eigenvalue i of the composite vw of the Clifford multiplications by v and w (as one easily verifies using (30.10) with $\chi^\pm = \psi^\pm \pm i\phi^\pm$). Since Q commutes with vw, the plane must be Q-invariant. Furthermore, since every 1-dimensional vector subspace of $\mathcal{S}_x^+$ or $\mathcal{S}_x^+$ is the intersection of two such planes, every nonzero spinor in $\mathcal{S}_x^+$ or $\mathcal{S}_x^+$ must be an eigenvector of Q, and so $Q = \lambda^\pm$ on $\mathcal{S}_x^\pm$ for some $\lambda^\pm \in \mathbf{C}$ (where $\lambda^\pm$ stands for $\lambda^\pm$ times the identity). Finally, since $v\psi^\pm = \pm\,\psi^\mp$ (by (30.10) with $a = 1$, $b = 0$), we have $\lambda^+\psi^- = \lambda^+ v\psi^+ = vQ\psi^+ = Qv\psi^+ = Q\psi^- = \lambda^-\psi^-$, and so $\lambda^+ = \lambda^-$, i.e., Q equals λ^+ times the identity, as required.

Let (M, g) again denote an oriented Riemannian 4-manifold, and let $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ be a fixed spin$_\mathrm{c}$-structure for (M, g). Any given U(1)-connection A in $\mathcal{K}$ then gives rise to the ("twisted") *Dirac operator* $\mathcal{D}^A$, which assigns to any C^1 section ψ of $\mathcal{S}^+$ a continuous section $\mathcal{D}^A\psi$ of $\mathcal{S}^-$. To define $\mathcal{D}^A$, let us denote ∇ the Levi-Civita connection of (M, g), and let ∇^+ and ∇^- be the connections in $\mathcal{S}^+$ and $\mathcal{S}^-$ corresponding under the bijection (30.13) to A and ∇. (Note that $\mathcal{V} = TM$ by (30.15).) We now set

$$\left[\mathcal{D}^A\psi\right](x) \,=\, \mathrm{Trace}\,\{T_xM \ni (v, w) \mapsto v(\nabla_w^+\psi) \in \mathcal{S}_x^-\}\,, \tag{30.23}$$

for any $x \in M$, where Trace stands for the g_x-Trace (contraction) of a bilinear mapping, and v in $v(\nabla_w^+\psi)$ stands for the Clifford multiplication by v. In other words, $\left[\mathcal{D}^A\psi\right](x) = \sum_j e_j(\nabla_{e_j}^+\psi)$, e_j being an arbitrary orthonormal basis of T_xM.

More generally, we can also extend $\mathcal{D}^A$ to an operator taking C^1 sections of the direct-sum $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ bundle to continuous sections of $\mathcal{S}$, using an obvious modification of formula (30.23) (with subscripts $^\mp$ instead of $^\pm$, and with $v(\nabla_w^-\psi)$ defined as in (30.16)). Note that $\mathcal{D}^A$ then sends sections of $\mathcal{S}^\pm$ to sections of $\mathcal{S}^\mp$ (since so do the the Clifford multiplications; see (30.16)).

Example 30.7. In the case where the spin$_\mathrm{c}$-structure $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ just discussed is a spin structure $(\mathcal{S}^+, \mathcal{S}^-)$ for (M, g), one defines the ("untwisted") Dirac operator $\mathcal{D}$ to be $\mathcal{D}^A$, given by (30.23), where $A = d$ stands for the standard flat connection (30.21) in the product line bundle $\mathcal{K} = M \times \mathbf{C}$. As in the preceding paragraph, $\mathcal{D}$ acts on C^1 sections ψ of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, interchanging the subbundles $\mathcal{S}^+$ and $\mathcal{S}^-$, and formula (30.23) for $A = d$ can be rewritten as

$$[\mathcal{D}\psi]\,(x) \,=\, \mathrm{Trace}\,\{T_xM \ni (v, w) \mapsto v(\nabla_w\psi) \in \mathcal{S}_x\}\,, \tag{30.24}$$

or, more explicitly,

$$[\mathcal{D}\psi](x) \,=\, \sum_j e_j(\nabla_{e_j}\psi)\,, \tag{30.25}$$

where ∇ now denotes the spinor connection (30.22) in $\mathcal{S}$, and e_j is any orthonormal basis of T_xM. Those C^1 sections ψ of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ for which $\mathcal{D}\psi = 0$ are called *harmonic spinor fields* or, briefly, *harmonic spinors*.

The questions of whether and when a $\mathrm{spin_c}$-structure or a spin structure exists, for a given oriented Riemannian 4-manifold (M,g), are settled by the following classical results.

Theorem 30.8 (Hirzebruch, 1958)**.** *Let M be a compact manifold, and let G be the matrix group defined by* (30.1)*. Every G-bundle $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$ over M then satisfies the condition*

$$w_2(\mathcal{V}) \,=\, c_1(\mathcal{K}) \bmod 2\,. \tag{30.26}$$

Conversely, given a complex line bundle $\mathcal{K}$ with a $\mathrm{U}(1)$*-structure and a real* 4*-space bundle $\mathcal{V}$ with an* $\mathrm{SO}(4)$*-structure, both over M, for which* (30.26) *holds, there exists a G-bundle $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$ over M realizing $\mathcal{K}$ and $\mathcal{V}$ via* (30.4) *and* (30.8)*.*

Idea of proof. If one tries to construct $\mathcal{S}^+$ and $\mathcal{S}^-$ for any prescribed $\mathcal{K}$ and $\mathcal{V}$, using induction over increasing dimensions of skeleta of a fixed CW decomposition of M, condition (30.26) turns out to be the only obstruction. ■

Two further important facts are immediate from Theorem 30.8.

Corollary 30.9 (Hirzebruch, 1958)**.** *An oriented Riemannian* 4*-manifold (M,g) always admits a* $\mathrm{spin_c}$*-structure associated with some $(\mathcal{K},\langle\,,\rangle)$. Any such line bundle $\mathcal{K}$ must satisfy the relation*

$$w_2(M) \,=\, c_1(\mathcal{K}) \bmod 2\,. \tag{30.27}$$

■

Corollary 30.10 (Milnor, 1963)**.** *An oriented Riemannian* 4*-manifold (M,g) admits a spin structure if and only if $w_2(M)=0$.* ■

Remark 30.11. An oriented Riemannian 4-manifold (M,g) admitting a spin structure is referred to as a (4-dimensional) *spin manifold*. According to Corollary 30.10, being a spin manifold is a topological property of the tangent bundle TM and hence has nothing to do with the metric g or the orientation chosen (provided that M is orientable). This can also be seen directly as follows. Suppose that a spin structure $(\mathcal{S}^+,\mathcal{S}^-)$ does exist for some fixed metric g and a fixed orientation in M. The pair $(\mathcal{S}^-,\mathcal{S}^+)$ then becomes a spin structure for the same metric g and the opposite orientation, if we declare the Clifford multiplication $\mathcal{S}_x^- \to \mathcal{S}_x^+$ by any $v\in T_xM$, $x\in M$, to be the negative adjoint $-v^*$ of the original Clifford multiplication $v:\mathcal{S}_x^+\to\mathcal{S}_x^-$. In fact, the negative adjoints of the basis (30.12) obtained as in Remark 30.3 from any $(\psi^+,\phi^+,\psi^-,\phi^-)\in\mathcal{B}_x$ then form nearly the same thing as the basis (30.12) corresponding to the analogous basis $(\psi^-,\phi^-,\psi^+,\phi^+)$ of $\mathcal{S}_x^+\oplus\mathcal{S}_x^-$, the only difference being the sign of F_{10}. That is precisely what amounts to reversing the orientation in $\mathcal{V}_x$ (and T_xM).

As for a change of the metric, note that any two Riemannian metrics on M are congruent via an orientation-preserving vector-bundle isomorphism $TM\to TM$ (see Proposition 30.14 below), and so, to obtain a spin structure for the new metric, it suffices to replace the identification (30.15) by its composite with F, while leaving the rest of the structure unchanged.

Remark 30.12. Given a positive-definite Euclidean/Hermitian inner product $\langle\,,\rangle$ in a finite-dimensional real/complex vector space V, let $\mathcal{U}$ be the set of all positive self-adjoint linear operators $F : V \to V$. Then $\mathcal{U}$ is a manifold (namely, an open subset of the vector space of all self-adjoint operators in V), and the square mapping $\mathcal{U} \ni F \mapsto F^2 \in \mathcal{U}$ is a C^∞ diffeomorphism. More precisely, since F and F^2 commute, F is uniquely determined by F^2; specifically, F leaves each eigenspace of F^2 invariant, and the restriction of F to the λ-eigenspace of F^2 equals $\sqrt{\lambda}$ times the identity. Finally, the inverse of the square mapping, denoted $\varPhi \mapsto \sqrt{\varPhi}$, is of class C^∞ as a consequence of the inverse mapping theorem. (In fact, the differential of the square mapping at any $F \in \mathcal{U}$ is injective: If it sends some operator $\dot{F}$ to zero, i.e., $0 = (F^2)^{\cdot} = \dot{F}F + F\dot{F}$, then $\dot{F}[\mathrm{Ker}\,(F-\lambda)] \subset \mathrm{Ker}\,(F+\lambda)$ for any real λ, and hence $\dot{F} = 0$ since all eigenvalues of F are real and *positive.*)

Lemma 30.13. *For any finite-dimensional real/complex vector space V there exists a C^∞ mapping which assigns to each pair of positive-definite Euclidean/Hermitian inner products $\langle\,,\rangle$ and $\langle\,,\rangle'$ in V a linear isomorphism $F : V \to V$ sending $\langle\,,\rangle$ onto $\langle\,,\rangle'$ and such that $\det F$ is real and positive.*

Proof. .We have $\langle v, w\rangle' = \langle \varPhi v, w\rangle$ for all $v, w \in V$ with a unique linear operator $\varPhi : V \to V$, which, in addition, is self-adjoint relative to $\langle\,,\rangle$. Thus, we may set $F = \sqrt{\varPhi}$ (notations as in Remark 30.12), which completes the proof. ■

Applying Lemma 30.13 to individual fibres of a vector bundle, we now obtain

Proposition 30.14. *Given two positive-definite Riemannian/Hermitian fibre metrics $\langle\,,\rangle$ and $\langle\,,\rangle'$ in a real/complex vector bundle over a manifold M, there exists a C^∞ vector-bundle isomorphism $F : \mathcal{E} \to \mathcal{E}$ such that $\det F$ is real and positive at each point and F sends $\langle\,,\rangle$ onto $\langle\,,\rangle'$ in the sense that $\langle F\psi, F\phi\rangle' = \langle \psi, \phi\rangle'$ for all $x \in M$ and $\psi, \phi \in \mathcal{E}_x$.* ■

§31. Harmonic spinors and the Lichnerowicz theorem

Let $(\mathcal{S}^+, \mathcal{S}^-)$ be a spin structure for a given oriented Riemannian 4-manifold (M, g) (Example 30.5), and let ∇ stand both for the Levi-Civita connection in TM and the spinor connection it induces in the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. For any fixed local coordinate system x^j in M, denoting γ_j the operator of Clifford multiplication by the coordinate vector field e_j, let us set

$$\gamma^j \;=\; g^{jk}\gamma_k\,. \tag{31.1}$$

Thus, γ^j is the Clifford multiplication by the 1-form dx^j (treated, with the aid of the metric g, as a tangent vector field w, with the components $w^k = g^{jk}$). Also, letting ∇ stand for the spinor connection in $\mathcal{S}$, we can set

$$\nabla_j = \nabla_{e_j}\,. \tag{31.2}$$

In this way, both γ^j and ∇_j are operators acting on spinor fields defined on the coordinate domain: The former in a pointwise fashion, i.e., as a local section of the bundle $\mathrm{Hom}(\mathcal{S}, \mathcal{S})$, the latter as a differential operator. Furthermore, we will identify any complex-valued function f defined on an open subset U of M with the local section $f \cdot \mathrm{Id}$ of $\mathrm{Hom}(\mathcal{S}, \mathcal{S})$, i.e., with the operator acting on spinor fields via multiplication by f. From now on, the composites (written multiplicatively) and commutators involving the γ^j, ∇_j and various functions f, will all refer to this operator interpretation.

Lemma 31.1. *Given a spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ over an oriented Riemannian 4-manifold (M, g) and a local coordinate system x^j in M, let γ^j and ∇_j be the operators* (31.1), (31.2) *acting on spinor fields defined on the coordinate domain, and let Γ_{jl}^k, g^{jk}, $R_{jklm} = R_{jkl}{}^p g_{pm}$, $R_{jk} = R_{jlk}{}^l$ and $\mathrm{s} = g^{jk} R_{jk}$ stand, as usual, for the Christoffel symbols, the components of the reciprocal metric, curvature tensor, Ricci tensor, and the scalar curvature function of (M, g), with* (4.1), (2.8), (4.25), (4.36) *and* (4.40). *Then we have*

$$\gamma^j \gamma^k + \gamma^k \gamma^j = -2 g^{jk}, \tag{31.3}$$

$$R_{lm} \gamma^l \gamma^m = -\mathrm{s}, \tag{31.4}$$

$$R_{jklm} \gamma^j \gamma^k \gamma^l \gamma^m = -2\mathrm{s}, \tag{31.5}$$

as well as the commutation relations

$$[\nabla_j, \gamma^k] = -\Gamma_{jl}^k \gamma^l, \tag{31.6}$$

$$4 [\nabla_j, \nabla_k] = R_{jklm} \gamma^l \gamma^m. \tag{31.7}$$

Finally, the curvature tensor R^∇ of the spinor connection ∇ in $\mathcal{S}$ satisfies

$$R^\nabla(e_j, e_k) = -\frac{1}{4} R_{jklm} \gamma^l \gamma^m. \tag{31.8}$$

Proof. The left-hand side of (31.3) equals $g^{jl} g^{km} (\gamma_l \gamma_m + \gamma_m \gamma_l)$. Thus, (31.3) is immediate from $\gamma_j \gamma_k + \gamma_k \gamma_j = -2 g_{jk}$ (see (30.18)). Formula (31.4) is in turn obvious from (4.40) and (31.3) along with symmetry of the Ricci tensor (see (4.38)).

Furthermore, for any given permutation $(\lambda, \mu, \nu, \rho)$ of $\{1, 2, 3, 4\}$, let $S_{\lambda\mu\nu\rho}$ be the local section of $\mathrm{Hom}(\mathcal{S}, \mathcal{S})$ with

$$S_{\lambda\mu\nu\rho} = R_{j_\lambda j_\mu j_\nu j_\rho} \gamma^{j_1} \gamma^{j_2} \gamma^{j_3} \gamma^{j_4}. \tag{31.9}$$

By (4.32) and (4.33),

$$S_{\lambda\mu\nu\rho} = -S_{\lambda\mu\rho\nu}, \tag{31.10}$$

$$S_{\lambda\mu\nu\rho} + S_{\lambda\nu\rho\mu} + S_{\lambda\rho\mu\nu} = 0. \tag{31.11}$$

Moreover,

$$S_{\lambda\mu\nu\rho} + S_{\lambda\nu\mu\rho} = -2\mathrm{s} \qquad \text{whenever} \quad |\mu - \nu| = 1. \tag{31.12}$$

In fact, by (31.9), the left-hand side of (31.12) equals the sum over the indices $j_1, \ldots, j_4$ of the expressions

$$R_{j_\lambda j_\mu j_\nu j_\rho} \phi (\gamma^{j_\mu} \gamma^{j_\nu} + \gamma^{j_\nu} \gamma^{j_\mu}) \psi, \tag{31.13}$$

where Φ and Ψ are local sections of $\mathrm{Hom}(\mathcal{S},\mathcal{S})$ such that $\Phi\Psi = \gamma^k\gamma^l$, (k,l) being the ordered pair with $(k,l) = (j_\lambda, j_\rho)$ when $\lambda < \rho$, and $(k,l) = (j_\rho, j_\lambda)$ when $\rho < \lambda$. Thus, (31.12) is immediate from (31.3), (4.36) and (31.4). Consequently,

$$S_{1342} = S_{1234} + 2\,\mathrm{s}\,, \qquad S_{1423} = S_{1234} + 4\,\mathrm{s}\,. \tag{31.14}$$

To see this, note that (31.10) and (31.12) yield $S_{1342} = -\,S_{1324} = S_{1234} + 2\,\mathrm{s}$ and $S_{1423} = -\,S_{1432} = S_{1342} + 2\,\mathrm{s}$; combining these relations, we obtain (31.14). However, by (31.11) and (31.14), $S_{1234} = -\,S_{1342} - S_{1423} = -\,2S_{1234} - 6\,\mathrm{s}$, i.e., $S_{1234} = -\,2\,\mathrm{s}$. In view of (31.9) with $(\lambda,\mu,\nu,\rho) = (1,2,3,4)$, this yields (31.5).

On the other hand, since γ^k is the Clifford multiplication by dx^j (treated as a vector field), (31.6) is immediate from

$$\nabla_j\, dx^k \;=\; -\,\Gamma^k_{jl}\, dx^l \tag{31.15}$$

and the fact the Levi-Civita connection in TM and the spinor connection in $\mathcal{S}$ together make the Clifford multiplication parallel (which in turn is nothing else than the definition of the spinor connection).

Finally, to prove (31.7) and (31.8), first note that, by (4.23) with $[e_j, e_k] = 0$, we have

$$R^\nabla(e_j, e_k) \;=\; [\nabla_k, \nabla_j]\,. \tag{31.16}$$

Furthermore,

$$4\,[[\nabla_j, \nabla_k],\, \gamma^p] \;=\; [R_{jklm}\gamma^l\gamma^m, \gamma^p] \;=\; 4g^{pm}R_{jklm}\gamma^l\,, \tag{31.17}$$

and so both sides of (31.7) yield equal commutators with each γ^p. In fact, the Jacobi identity for the operator commutator gives $[[\nabla_j,\nabla_k],\gamma^p] = -\,[[\nabla_k,\gamma^p],\nabla_j] - [[\gamma^p,\nabla_j],\nabla_k]$. Repeatedly applying (31.6), we see that this equals $[\Gamma^p_{kl}\gamma^l,\nabla_j] - [\Gamma^p_{jl}\gamma^l,\nabla_k] = -\Gamma^p_{kl}[\nabla_j,\gamma^l] - (\partial_j\Gamma^p_{kl})\gamma^l + \Gamma^p_{jl}[\nabla_k,\gamma^l] + (\partial_k\Gamma^p_{jl})\gamma^l = [\partial_j\Gamma^p_{kl} - \partial_k\Gamma^p_{jl} + \Gamma^p_{js}\Gamma^s_{kl} - \Gamma^p_{ks}\Gamma^s_{jl}]\gamma^l$. Hence, by (4.25), $[[\nabla_j,\nabla_k],\gamma^p] = g^{pm}R_{jklm}\gamma^l$. On the other hand, we have, by (31.3), $R_{jklm}\gamma^l\gamma^m\gamma^p = -R_{jklm}\gamma^l\gamma^p\gamma^m + 2g^{pm}R_{jklm}\gamma^l$ and $R_{jklm}\gamma^l\gamma^p\gamma^m = -\,R_{jklm}\gamma^p\gamma^l\gamma^m + 2g^{lp}R_{jklm}\gamma^m$. We have thus obtained the formula $R_{jklm}\gamma^l\gamma^m\gamma^p = R_{jklm}\gamma^p\gamma^l\gamma^m + 4g^{pm}R_{jklm}\gamma^l$, which may also be rewritten as the commutator relation $[R_{jklm}\gamma^l\gamma^m,\gamma^p] = 4g^{pm}R_{jklm}\gamma^l$, and so (31.17) follows. Now (31.7) is an obvious consequence of (31.17) and the fact that any traceless linear operator $\mathcal{S}_x \to \mathcal{S}_x$ commuting with all Clifford multiplications by vectors $v \in T_xM$ must be zero (see Remark 30.6), while both sides of (31.7) do represent such traceless operators at each $x \in U$: Namely, $\mathrm{Trace}\,[R_{jklm}\gamma^l\gamma^m] = 0$ since $\mathrm{Trace}\,[\gamma^l\gamma^m]$ is symmetric as a function of j,k, and R_{jklm} is skew-symmetric; on the other hand, tracelessness (as well as the pointwise character) of $[\nabla_j,\nabla_k]$ is clear from (31.16) and Corollary 29.3.

Finally, (31.8) is obvious from (31.7) and (31.16). This completes the proof. ∎

For any (local) C^2 spinor field ψ, the covariant derivative $\nabla\psi$ of ψ is a C^1 section of $\mathrm{Hom}\,(TM,\mathcal{S}) = T^*M \otimes \mathcal{S}$, sending every tangent vector $v \in T_xM$ to $\nabla_v\psi \in \mathcal{S}_x$. Thus, since $\nabla_v\psi = v^j\nabla_j\psi$ (as $v = v^j\, e_j$) and $v^j = (dx^j)(v)$, we have

$$\nabla\psi \;=\; dx^j \otimes \nabla_j\psi\,. \tag{31.18}$$

Consequently, $\nabla\psi$ may be thought of as a differential 1-form of class C^1 valued in $\mathcal{S}$, and so its divergence $\mathrm{div}\,(\nabla\psi)$ can be defined using (4.41). The resulting composite operator $\mathrm{div} \circ \nabla$ sends spinor fields to spinor fields. We have

Lemma 31.2 (Lichnerowicz, 1963)**.** *Under the assumptions of* Lemma 31.1*, we have the local expressions*

$$\mathrm{div} \circ \nabla \;=\; g^{jk} \left(\nabla_j \nabla_k \;-\; \Gamma_{jk}^l \nabla_l\right) \,, \tag{31.19}$$

$$\mathcal{D} \;=\; \gamma^j \,\nabla_j \,, \tag{31.20}$$

for div $\circ\nabla$ *and the Dirac operator* $\mathcal{D}$ *of* $\mathcal{S}$*, and the coordinate-free* Lichnerowicz formula

$$\mathcal{D}^2 \;=\; -\,\mathrm{div} \circ \nabla \;+\; \frac{1}{4}\,\mathrm{s}\,. \tag{31.21}$$

Proof. Relation (31.20) is nothing else than the definition of $\mathcal{D}$ (§29). To obtain (31.19), first note that, treating $w = dx^j$, for a fixed j, as a vector field (with $w^k = g^{kj}$), we obtain, from (4.42) and (4.12), $\mathrm{div}\,(dx^j) = \partial_k g^{jk} + \Gamma_{kl}^k g^{jl}$. Thus, since the reciprocal metric is parallel, $\mathrm{div}\,(dx^j) = -\,g^{kl}\Gamma_{kl}^j$. On the other hand, for any (local) C^1 spinor field ϕ and a C^1 vector field w, (4.1) clearly gives $\mathrm{div}\,(w\otimes\phi) = (\mathrm{div}\,w)\phi + \nabla_w\phi$. Applying this to $\phi = \nabla_j\psi$ and $w = dx^j$ and noting that then $\nabla_w\phi = g^{jk}\nabla_k\phi = g^{jk}\nabla_k\nabla_j\psi$, we now get (31.19). Finally, (31.20) gives $\mathcal{D}^2 = (\gamma^j\nabla_j)\gamma^k\nabla_k = \gamma^j\gamma^k\nabla_j\nabla_k + \gamma^j[\nabla_j,\gamma^k]\nabla_k$. Thus, by (31.6), $\mathcal{D}^2 = (\gamma^j\nabla_j)\gamma^k\nabla_k = \gamma^j\gamma^k(\nabla_j\nabla_k - \Gamma_{jk}^l\nabla_l)$. Writing $\gamma^j\gamma^k = \frac{1}{2}(\gamma^j\gamma^k + \gamma^k\gamma^j) + \frac{1}{2}(\gamma^j\gamma^k - \gamma^k\gamma^j)$ and using (31.3) along with symmetry of Γ_{jk}^l in j,k (see (4.1)), we now obtain $\mathcal{D}^2 = g^{jk}(\nabla_j\nabla_k - \Gamma_{jk}^l\nabla_l) + \frac{1}{2}\gamma^j\gamma^k[\nabla_j,\nabla_k]$, and hence (31.21) is immediate from (31.19), (31.7) and (31.5). This completes the proof. ■

Theorem 31.3 (Lichnerowicz, 1963)**.** *Let* M *be a compact oriented* 4*-manifold which admits a spin structure and has a nonzero signature* $\tau(M)$*. Then* M *carries no Riemannian metric* g *whose scalar curvature* s *is positive everywhere. More generally, any metric* g *on* M *with* $\mathrm{s}\ge 0$ *must have* $\mathrm{s}=0$ *identically.*

Outline of proof. For any fixed metric g on M, the Atiyah-Singer index theorem, applied to the Dirac operator $\mathcal{D}$, states that $\tau(M) = 16\,(\mathrm{d}_+ - \mathrm{d}_-)$, where d_+ (or d_-) is the dimension of the space of all *positive* (or *negative*) *harmonic spinor fields*, that is, sections ψ of $\mathcal{S}^+$ (or, $\mathcal{S}^-$) with $\mathcal{D}\psi = 0$. (For details, see, e.g., Bourguignon, 1981.) Hence, if $\tau(M)\ne 0$, there exists a nonzero harmonic spinor field ψ. Let us now denote $(\;,\;)$ and $\|\;\|$ the L^2 inner product and the L^2 norm. Using (31.21) and integration by parts (Theorem 24.3), we obtain $0 = 4\,(\mathcal{D}^2\psi,\psi) = 4\,\|\nabla\psi\|^2 + (\mathrm{s}\psi,\psi)$. If $\mathrm{s}\ge 0$, this implies that $\psi = 0$ at all points with $\mathrm{s}\ne 0$ $\nabla\psi = 0$ everywhere. Since $\psi\ne 0$ somewhere in M, we thus have $\psi\ne 0$ everywhere, and so $\mathrm{s}=0$ identically on M. ■

Remark 31.4. Lichnerowicz proved this result for manifolds of any dimension, with an appropriate definition of a spin structure and with the signature replaced by the *"A-roof genus"* $\hat{A}(M)$. For spin manifolds of dimension four, $\hat{A}(M) = \tau(M)/16$.

§32. Non-Kähler Hermitian Einstein metrics

We have the following partial classification result for those compact oriented Riemannian Einstein four-manifolds (M,g) which satisfy condition (20.2), i.e., whose self-dual Weyl tensor W^+, acting as a self-adjoint bundle morphism $W^+ : \Lambda^+M \to \Lambda^+M$, has fewer than three distinct eigenvalues at every point.

Theorem 32.1 (Derdziński, 1983)**.** *Let* (M, g) *be a compact oriented Riemannian Einstein four-manifold with* (20.2) *and such that* W^+ *is not parallel. The following assertions then hold for* (M, g) *replaced, if necessary, by a two-fold Riemannian covering*:

(i) M *is diffeomorphic to* $S^2 \times S^2$ *or to a connected sum* $\mathbf{C}\mathrm{P}^2 \# \, k \, \overline{\mathbf{C}\mathrm{P}^2}$ *with* $0 \le k \le 8$.

(ii) $W^+ \ne 0$ *everywhere in* M *and the metric* $|W^+|^{2/3} g$ *on* M*, conformal to* g*, is a Kähler metric for some complex structure on* M*, which is compatible with the given orientation.*

(iii) *With that complex structure,* M *is biholomorphic to a compact complex surface obtained by blowing up* k *points* $(0 \le k \le 8)$ *in* $\mathbf{C}\mathrm{P}^2$*, or by blowing up* k *points* $(0 \le k \le 7)$ *in a holomorphic* $\mathbf{C}\mathrm{P}^1$ *bundle over* $\mathbf{C}\mathrm{P}^1$.

(iv) *The complex surface* M *admits a nontrivial holomorphic vector field, which is a Killing field for both metrics* g *and* $|W^+|^{2/3} g$*, and whose flow commutes with every transformation in the identity component of the isometry group of* (M, g).

Proof. Assertions (ii) and (iv) are immediate from Propositions 20.1 and 22.4 and the fact that the almost complex structure of any Kähler manifold is integrable. Now (i) and (iii) follow from the classification theorem for compact complex surfaces admitting nontrivial holomorphic vector fields, due to Carrell, Howard and Kosniowski (1973). ■

Remark 32.2. The assumptions of Theorem 32.1 are automatically satisfied by any compact oriented Riemannian Einstein 4-manifold (M, g) which is not locally symmetric and admits an effective isometric action of a Lie group G with $\dim G \ge 4$, provided that

(a) We change the orientation of M, if necessary; or,

(b) Neither (M, g) itself nor any two-fold Riemannian covering of (M, g) admits a parallel bivector field α that makes it into a Kähler manifold whose canonical orientation coincides with the original orientation. (This last condition amounts to requiring that α be a self-dual bivector field, cf. Corollary 9.4.)

In fact, the principal (i.e., highest-dimensional) orbits of the action cannot be of dimension four; otherwise, (M, g) would be locally homogeneous and hence, by Jensen's theorem (Corollary 7.3), locally symmetric. According to the last clause of Lemma 20.9, condition (20.2) holds for either local orientation. Since (M, g) is not locally symmetric, W is not parallel (see (5.10)), and so W^+ and W^- cannot be both parallel, which establishes case (a). Also, by Berger's Theorem 26.1, $\chi(M) > 0$ (or else (M, g) would be flat and hence locally symmetric). Thus, in view of Bochner's Theorem 24.8(ii), the scalar curvature s of (M, g) is positive; otherwise, M would admit a continuous vector field without zeros (namely, any nontrivial Killing field, which exists according to Lemma 17.16). Let us now suppose that, for the original orientation, W^+ *is* parallel. We cannot have $W^+ = 0$ identically (in fact, as $\mathrm{s} > 0$, Hitchin's Theorem 33.4, applied to the opposite orientation, then would imply that (M, g) is locally symmetric). Consequently, assertion (i)b) of Corollary 9.10 holds, for some bivector field α which satisfies $\alpha^2 = -1$ due to the '(c) implies (a)' assertion in Lemma 9.3. Now Corollary

9.10(ii) contradicts the "non-Kähler" assumption in (b) above. In other words, in case (b), W^+ is not parallel, as required.

Remark 32.3. The *Page manifold* (M, g) consists of the compact complex surface $M = \mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}\mathrm{P}^2}$ obtained by blowing up a point in $\mathbf{C}\mathrm{P}^2$, along with a Riemannian Einstein metric (M, g) invariant under an effective action of $\mathrm{U}(2)$, which was found by Page (1978). This (M, g) is the only known example that satisfies the assumptions of Theorem 32.1 (other than the obvious modifications of the Page manifold obtained by passing to finite isometric quotients or rescaling the metric). The fact that the assumptions of Theorem 32.1 hold for the Page manifold is clear from Remark 32.2(b), since $M = \mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}\mathrm{P}^2}$ is simply connected and admits no Riemannian Kähler-Einstein metric (Example 36.9; of course, one can also verify directly that the Page metric is not Kähler). The reason why the Page metric, whatever it looks like, cannot be locally symmetric is that, as M is simply connected, M then would have to be diffeomorphic to one of the standard examples (Theorem 14.7), that is, to $\mathbf{R}^4$, S^4, or $\mathbf{C}\mathrm{P}^2$, which $\mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}\mathrm{P}^2}$ is not.

According to Theorem 32.1(ii), the Page metric is globally conformal to a Kähler metric. The latter metric was independently discovered by Calabi (1982). See also Chave and Valent (1996).

PART III: FURTHER GLOBAL RESULTS

The following four sections contain brief descriptions of several important results concerning conditions necessary (or, in some cases, sufficient) for a given compact 4-manifold to admit a Riemannian Einstein metric. Included are Hitchin's theorems on the equality case in Thorpe's inequality and on the structure of compact self-dual Einstein manifolds whose scalar curvature is positive (Theorems 33.3, 33.4); LeBrun's result on nonexistence of Einstein metrics on some complex surfaces, based on the Seiberg-Witten theory (Corollary 34.2); Gromov's estimate on the simplicial volume of compact Einstein 4-manifold (Theorem 35.1); the celebrated results of Aubin and Yau which establish *existence* of Kähler-Einstein metrics on certain complex manifolds via the Calabi conjectures (Theorem 36.6); and, finally, theorems of Matsushima, Lichnerowicz and Futaki on Kähler-Einstein manifolds with a positive first Chern class (Theorems 36.8, 36.10).

§33. Hitchin's theorems on compact Einstein 4-manifolds

Given a compact oriented Riemannian 4-manifold (M,g), let us denote $b^{\pm}$ the dimension of the space of all harmonic 2-forms on (M,g) that, viewed as bivector fields, are sections of $\Lambda^{\pm}M$. According to Hodge's theorem on the representation of real cohomology by harmonic forms (see Wells, 1979), we then have

$$b_2 \,=\, b^+ +\, b^- \,, \tag{33.1}$$

$b_k = b_k(M)$ being the kth Betti number of M. In fact, for a differential form α of any degree, being harmonic means that

$$d\alpha \,=0\,, \qquad \operatorname{div}\alpha \,=\, 0\,. \tag{33.2}$$

Since $\operatorname{div} \,=\, -\,*\,d\,*$, it follows that harmonicity of a 2-form α is equivalent to that of $*\alpha$; thus, by (6.5), a 2-form α is harmonic if and only if so are its $\Lambda^{\pm}M$ components α^+ and α^-. Furthermore, the intersection form Q in $H^2(M,\mathbf{R})$ is, in terms of harmonic 2-forms $\alpha,\ \beta$, given by $Q(\alpha,\beta)=\int_M[\alpha\wedge\beta]$, i.e., $Q(\alpha,\beta)=\int_M(\alpha,\beta)\,\mathrm{vol}_g$, where stands $(\ ,\ )$ for the wedge form (cf. formula (37.8) in §37); since Λ^+M and Λ^-M are wedge-orthogonal at each point (§37), the sign pattern of Q consists of b^+ pluses and b^- minuses. The (co)homological definitions of the Euler characteristic $\chi=\chi(M)$ and signature $\tau=\tau(M)$ of the compact 4-manifold M now lead to the relations

$$\tau \,=\, b^+ -\, b^- \,, \qquad \chi \,=\, 2\,-\,2b_1\,+\,b^+ +\, b^- \,. \tag{33.3}$$

Lemma 33.1. *Suppose that a compact, oriented,* 4*-dimensional Riemannian manifold* (M,g) *is anti-self-dual in the sense that* $W^+ = 0$ *and has a nonnegative scalar curvature function* s, *and let* $b^{\pm}=b^{\pm}(M)$ *be as in* (33.3).

(i) *If* s *is positive somewhere in* M, *then* $b^+=0$ *and* $\tau(M)=-\,b^-\le 0$.
(ii) *If* $\mathrm{s}\,=\,0$ *identically on* M, *then* $b^+\in\{0,1,3\}$ *and, for global* C^2 *sections of* Λ^+M, *harmonicity is equivalent to being parallel.*
(iii) *If* $\mathrm{s}=0$ *identically and* M *is simply connected, then* $b^+=3$, $\tau(M)=3-b^-$ *and* $\chi(M)=5+b^-$.

Proof. Let α be a harmonic section of Λ^+M. As $W^+ = 0$, (6.15) gives $W\alpha = 0$ and so, in view of (33.2), relation (5.17) with $n = 4$ becomes $3\,\mathrm{div}\,(\nabla\alpha) = \mathrm{s}\alpha$. Using integration by parts (Theorem 24.3), we thus obtain

$$\begin{aligned} 0 \,&=\, \int_M \langle\alpha,\, 2\,\mathrm{s}\alpha \,-\, 6\,\mathrm{div}\,(\nabla\alpha)\rangle\,\mathrm{vol}_g \\ &=\, \int_M \alpha^{jk}\left[\mathrm{s}\alpha_{jk} \,-\, 3\,\alpha_{jk,l}{}^{l}\right]\mathrm{vol}_g \,=\, \int_M \left[2\,\mathrm{s}\,|\alpha|^2 \,+\, 6\,|\nabla\alpha|^2\right]\mathrm{vol}_g\,. \end{aligned} \tag{33.4}$$

Since $\mathrm{s} \ge 0$, this implies $\nabla\alpha = 0$. As the fibre dimension of Λ^+M is 3, we thus obtain $b^+ \le 3$. On the other hand, we cannot have $b^+ = 2$, since the bundle Λ^+M is oriented (and so two independent parallel sections would lead to a third one via a vector-product construction). Therefore, assertion (ii) follows. Also, as $\mathrm{s}\,|\alpha|^2 = 0$, positivity of s at some $x \in M$ yields $\alpha(x) = 0$ and so α must vanish identically in virtue of being parallel. Along with (33.3), this proves (i).

Finally, suppose that $\mathrm{s} = 0$ and M is simply connected. By Lemma 6.16, the Levi-Civita connection ∇ in Λ^+M is flat, and so Λ^+M has a three-dimensional space of global parallel sections. Thus, (iii) follows from (ii) and (33.3). This completes the proof. ■

By a $K3$ *surface* we mean any compact simply connected 4-manifold M which admits a Riemannian metric g and a bivector field α such that (M, g, α) is a Ricci-flat Kähler manifold (as defined in §10). About the existence of $K3$ surfaces, see §36.

Remark 33.2. Our definition of a $K3$ surface, although not standard, is equivalent to those normally used in the literature. (See also §36.) It is well-known that all $K3$ surfaces are mutually diffeomorphic as real 4-manifolds. What we can easily see is that, for a $K3$ surface M, the invariants $b_k = b_k(M)$, $b^\pm = b^\pm(M)$, $\chi = \chi(M)$ and $\tau = \tau(M)$, appearing in (33.3), have specific unique values. Namely, choosing g, α and an orientation of M so as to make (M, g, α) a Ricci-flat Kähler manifold for which α is a section of Λ^+M, we have, for any $K3$ surface M,

$$\chi \,=\, 24\,, \quad \tau \,=\, -\,16\,, \quad b_1 \,=\, b_3 \,=\, 0\,, \quad b_2 = 22\,, \quad b^+ \,=\, 3\,, \quad b^- \,=\, 19\,. \tag{33.5}$$

To see this, note that Proposition 9.8 shows that then $W^+ = 0$, and, as $\mathrm{s} = 0$, both equalities in (26.4) are in fact equalities. (By (25.6), the orientation chosen as above also ensures that $\tau(M) \le 0$.) Hence $[M] = 0$ in (26.3). In other words, *for $K3$ surfaces, the Thorpe inequality* (26.5) *becomes an equality.* That equality, i.e., $3\tau + 2\chi = 0$, combined with Lemma 33.1(iii) and (33.1), now yields (33.5).

As we just observed, the equality case in (26.5) occurs for $K3$ surfaces with Ricci-flat Kähler metrics (and, obviously, for flat Riemannian 4-manifolds; cf. (25.1) and (25.6), (25.7)). The following result of Hitchin states that these are, basically, the only such examples.

Theorem 33.3 (Hitchin, 1974)**.** *Let g be an Einstein metric on a compact* 4-*manifold M with*

$$3\,|\tau(M)| \,=\, 2\,\chi(M)\,. \tag{33.6}$$

Then either g is flat, or, for some $r \in \{1, 2, 4\}$, (M, g) admits an r-fold Riemannian covering by a $K3$ surface with a Ricci-flat Kähler metric.

Proof. Suppose that (M,g) is not flat. By Berger's Theorem 26.1, we then have $\chi(M) > 0$. Thus, in view of (33.6) and (25.7), M is orientable and $|\tau(M)| > 0$. Also, as $\chi(M) > 0$, Corollary 28.3(i) shows that the first Betti number b_1 of M must be zero. The same is obviously true for any finite Riemannian covering of (M,g); therefore, the Cheeger-Gromoll theorem (Cheeger and Gromoll, 1971) implies that M has a finite fundamental group. Choosing the orientation of M so that $\tau(M) < 0$ and combining (26.3) and (26.4) with (33.6), we now see that $W^+ = 0$ and $\mathrm{s} = 0$ everywhere in M.

Applying Lemma 33.1(ii), (iii) to the Riemannian universal covering $(\tilde{M},g)$ of (M,g), we see that $\Lambda^+\tilde{M}$ has a 3-dimensional space of parallel sections. One such nonzero section α, normalized so as to satisfy $\langle\alpha,\alpha\rangle = 2$, thus makes $(\tilde{M},g)$ into a Ricci-flat Kähler manifold $(\tilde{M},g,\alpha)$. Hence $\tilde{M}$ is a $K3$ surface.

From (33.6) and (33.3) with $b_1 = 0$ and $\tau = \tau(M) < 0$, we now obtain

$$b^- = 5\,b^+ + 4\,, \qquad \chi = 6\,b^+ + 6\,, \qquad \tau = -4\,(b^+ + 1)\,. \tag{33.7}$$

On the other hand, by Lemma 33.1(ii), $b^+ = b^+(M) \in \{0,1,3\}$. Since $\chi(\tilde{M}) = r\,\chi(M)$, where r is the order of $\pi_1 M$, (33.7) shows that the possible cases (b^+ equal to 0, 1 or 3) lead to our assertion, with $r \in \{1,2,4\}$. This completes the proof. ■

Theorem 33.4 (Hitchin, 1981; Friedrich and Kurke, 1982)**.** *Let g be a self-dual Einstein metric on a compact oriented 4-manifold M.*

- (a) *If g has a positive scalar curvature, then (M,g) is isometric to the sphere S^4, or the complex projective plane $\mathbf{C}\mathrm{P}^2$ with a multiple of the standard metric.*
- (b) *If g is Ricci-flat, then (M,g) satisfies the assertion of* Theorem 33.3.

For a proof, see, e.g., Hitchin (1981) or subchapter 13 C in Besse (1987). ■

§34. The Seiberg-Witten equations and LeBrun's theorem

Let $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$ be a fixed $\mathrm{spin_c}$-structure (§30) for an oriented Riemannian 4-manifold (M,g). A simple alternative description of the bundles $\Lambda^\pm M$ of self-dual and anti-self-dual bivectors in M (§6) now can be obtained as follows. For any $x\in M$, let us set

$$\mathcal{L}_x^\pm = \{B \in \mathrm{Hom}\,(\mathcal{S}_x^\mp,\mathcal{S}_x^\mp):\ B^* = -B\,,\ \ \mathrm{Trace}\,B = 0\}\,. \tag{34.1}$$

Diagonalizing such B, we see that $\mathcal{L}_x^\pm$ is a 3-dimensional real vector space contained in the space $\mathcal{V}_x^\mp$ defined as in (30.7) for the G-bundle $(\mathcal{S}^\mp,\mathcal{S}^\mp,[\mathcal{S}^\mp]^{\wedge 2})$ (rather than $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$). Thus, $\mathcal{L}_x^\pm$ carries a Euclidean norm $|\ |$ satisfying (30.7) and (30.9) for all $F\in\mathcal{L}_x^\pm$ and $\phi\in\mathcal{S}_x^\mp$. The actions of $\mathcal{L}_x^+$ and $\mathcal{L}_x^-$ on $\mathcal{V}_x$ by, respectively, right and left multiplications (that is, compositions) now clearly consist of skew-adjoint operators (since, by (30.9), $2\langle F,F'\rangle = \mathrm{Trace}\;F^*F'$ for $F,F'\in\mathcal{V}_x$). As the metric g provides an identification $\mathfrak{so}(\mathcal{V}_x) = [TM]^{\wedge 2}$ between skew-adjoint operators and bivectors, we thus obtain linear operators

$$\mathcal{L}_x^\pm \to \mathfrak{so}(\mathcal{V}_x) = [TM]^{\wedge 2} = \Lambda_x^+ M \oplus \Lambda_x^- M\,. \tag{34.2}$$

These operators are isomorphic identifications

$$\mathcal{L}_x^{\pm} \,=\, \Lambda_x^{\pm} M\,, \tag{34.3}$$

as one easily sees using Lemma 6.2 and the matrix representation of $\mathcal{L}_x^{\pm}$ and $\mathcal{V}_x$ based on any fixed pair of bases with (30.5).

We can now describe a natural quadratic function $\mathcal{S}_x^+ \ni \phi \mapsto \phi\overline{\phi} \in \mathcal{L}_x^+ = \Lambda_x^+ M$ by letting $\phi\overline{\phi}$ be $|\phi|^2 i$ times the traceless part of the orthogonal projection operator $\mathcal{S}_x^+ \to \mathbf{C}\phi$ (if $\phi \neq 0$) and and setting $\phi\overline{\phi} = 0$ if $\phi = 0$. In other words,

$$\phi\overline{\phi} \,=\, i\left[\langle\,\cdot\,,\phi\rangle\,\phi \,-\, \frac{1}{2}\,|\phi|^2\cdot \mathrm{Id}\right], \tag{34.4}$$

and so this is the quadratic function associated with the sesquilinear mapping $(\phi,\psi) \mapsto \phi\overline{\psi}$ given by $\phi\overline{\psi} = i\left[\langle\,\cdot\,,\psi\rangle\,\phi \,-\, \frac{1}{2}\,\langle\phi,\psi\rangle\cdot\mathrm{Id}\right]$.

Finally, using the notations of (30.23) and (34.4), we can write the *Seiberg-Witten equations*

$$\mathcal{D}^A\psi \,=\, 0\,, \qquad\qquad \psi\overline{\psi} \,=\, F_+^A\,. \tag{34.5}$$

Thus, (34.5) is a system of partial differential equations imposed on a pair (ψ, A) consisting of a C^∞ section ψ of $\mathcal{S}^+$ and a U(1)-connection A in $\mathcal{K}$. Here F^A denotes $-i$ times the curvature R^A of A, which makes F^A a (real-valued) 2-form on M, and F_+^A stands for the self-dual part of F^A, i.e., its $\Lambda^+ M$ component relative to the decomposition $[TM]^{\wedge 2} = \Lambda^+ M \oplus \Lambda^- M$. (See (6.4); as usual, we use the metric g to identify 2-forms and bivectors.)

By a *gauge transformation* of the given $\mathrm{spin_c}$-structure $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$ over a 4-manifold (M,g) we mean a pair of C^∞ vector-bundle isomorphisms $\varPhi^\pm : \mathcal{S}^\pm \to \mathcal{S}^\pm$ which preserve the inner product $\langle\,,\rangle$ in each fibre and, in addition, have the property that $\det \varPhi^+ = \det \varPhi^-$ at every point of M. (Clearly, the gauge transformations are nothing else than arbitrary C^∞ sections of a bundle of Lie groups over M with fibres isomorphic to the group G given by (30.1).) On the other hand, any C^∞ vector-bundle isomorphism $\varPhi : \mathcal{E} \to \mathcal{E}$ in a vector bundle $\mathcal{E}$ naturally acts on connections ∇ in $\mathcal{E}$ by transforming ∇ into a new connection $\tilde\nabla$ with $\tilde\nabla_v\psi = \varPhi\left[\nabla_v(\varPhi^{-1}\psi)\right]$ for tangent vectors v and local C^1 sections ψ and, if $\varPhi$ preserves some given G-structure in $\mathcal{E}$, it will transform each G-connection into a G-connection. By letting a gauge transformation $(\varPhi^+,\varPhi^-)$ in our $\mathrm{spin_c}$-structure $(\mathcal{S}^+,\mathcal{S}^-,\mathcal{K})$ act on a pair $\nabla^\pm$ of U(2)-connections in $\mathcal{S}^\pm$ and using the the bijection (30.13), where ∇ now denotes the Levi-Civita connection of (M,g), we thus make $(\varPhi^+,\varPhi^-)$ act on the U(1)-connection A in $\mathcal{K}$, transforming it into another U(1)-connection. At the same time, $\varPhi$ acts in an obvious way on C^∞ sections ψ of $\mathcal{S}^+$. Applied to a solution (ψ, A) of (34.5), our gauge transformation will thus transform it into another such pair which, for obvious reasons of naturality, will again be a solution to (34.5). Consequently, solutions to (34.5) are plentiful, forming an infinite-dimensional space, and rather than the solution space itself, it is much more interesting to study the *moduli space* of solutions to (34.5), that is, the set of their equivalence classes modulo the relation of being congruent under a gauge transformation.

It turns out that in this case, for *generic* metrics (where genericity can always be achieved by a small perturbation of the original metric), the moduli space is a finite set and each of its elements has a well defined "sign" ± 1. The algebraic sum of these signs is called the *Seiberg-Witten invariant* of M and the spin$_c$-structure $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$, and denoted $n_c(M)$. (See Witten, 1994.)

For a compact oriented 4-manifold M, let us set

$$[\![M]\!] \,=\, 2\,\chi(M) \,+\, 3\,\tau(M)\,. \tag{34.6}$$

For instance, if M is obtained by blowing up $k \geq 0$ points in a compact complex surface N, we have

$$[\![M]\!] \,=\, [\![N]\!] \,-\, k\,. \tag{34.7}$$

Theorem 34.1 (LeBrun, 1996)**.** *Let M be a compact complex surface obtained by blowing up $k \geq 1$ points in a minimal complex algebraic surface N of general type. Every Riemannian metric g on M then satisfies the estimates*

$$\|\mathrm{s}\|^2 \,>\, 32\,\pi^2\,[\![N]\!]\,, \tag{34.8}$$

$$24\,\pi^2\,k \,<\, 3\,\|\mathrm{E}\|^2 \,+\, 16\,\pi^2\,[\![N]\!]\,, \tag{34.9}$$

where $\|\mathrm{s}\|$ and $\|\mathrm{E}\|$ stand for the L^2 norms of the scalar curvature function s and, respectively, the traceless Ricci tensor E, of the Riemannian 4-manifold (M, g). In particular, if M carries an Einstein metric, we have

$$3\,k \,<\, 2\,[\![N]\!]\,. \tag{34.10}$$

Idea of proof. One can always find a bivector field α on M such that $(\tilde{M}, g, \alpha)$ is an almost Hermitian manifold and the Seiberg-Witten invariant $n_c(M)$ of M and the spin$_c$-structure $(\mathcal{S}^+, \mathcal{S}^-, \mathcal{K})$ associated with α as in Example 30.4 satisfies $n_c(M) \neq 0$. The assertion then follows from a Weitzenböck-formula argument. ■

Corollary 34.2 (LeBrun, 1996)**.** *Let M be a compact complex surface obtained by blowing up $k \geq 1$ points in a minimal complex algebraic surface N of general type. If*

$$k \,\geq\, \frac{2}{3}\,[\![N]\!]\,, \tag{34.11}$$

then M does not admit a Riemannian Einstein metric.

In fact, set $[M]_+ = 2\chi(M) + 3\tau(M)$. For an Einstein metric on M, (34.15) and (34.1) yield $c_1^2(N) = [N]_+ = [M]_+ + k < (32\pi^2)^{-1}\,\|\mathrm{s}\|^2$, so by (25.8), $c_1^2(N) < (32\pi^2)^{-1}\,\|\mathrm{s}\|^2 \leq 3[M]_+ = 3c_1^2(N) - 3k$, as required. ■

Note that, in view of (34.6) and (25.9), relation (34.11) can also be rewritten as

$$k \,\geq\, \frac{2}{3}\,c_1^2(N)\,. \tag{34.12}$$

§35. Gromov's estimate for Einstein 4-manifolds

The *simplicial volume* of a compact orientable manifold M (Gromov, 1981) is the infimum of all sums $\sum_r |a_r|$ over all combinations $\sigma = \sum_r a_r \sigma_r$ of n-dimensional singular simplices in M, with $a_r \in \mathbf{R}$ and $n = \dim M$, such that σ is a cycle (that is, $\partial\sigma = 0$) and its homology class in $H_n(M, \mathbf{R})$ is the fundamental class of M (for whichever fixed orientation). We have

Theorem 35.1 (Gromov, 1981)**.** *The simplicial volume $\|M\|$ of every compact 4-manifold M carrying an Einstein metric satisfies the inequality*

$$\|M\| \leq 2592\,\pi^2\,\chi(M)\,. \tag{35.1}$$

For a proof, see Gromov (1981). ■

Gromov's Theorem 35.1 leads to examples of compact 4-manifolds M admitting no Einstein metrics, for which the nonexistence of such a metric does not follow from the Thorpe inequality (26.5) combined with Hitchin's Theorem 33.3. Specifically, such M can be constructed by doubling the manifold with boundary obtained by removing a suitable number of disjoint balls from the product $\Sigma \times \Sigma$ for a suitable closed surface Σ. For details, see Gromov (1981) or Besse (1987).

§36. Kähler-Einstein metrics on compact complex surfaces

Let us recall that a differential 2-form β on a manifold M is called *closed* (or, *exact*) if β is C^∞-differentiable and $d\beta = 0$ (or, respectively, $\beta = d\vartheta$ for some C^∞-differentiable 1-form ϑ on M). The second *de Rham cohomology space* $H^2(M, \mathbf{R})$ of M is the quotient real vector space Z^2/B^2, where Z^2 consists of all closed 2-forms, and B^2 consists of all exact 2-forms on M. Every closed 2-form β thus gives rise to the element

$$[\beta] \in H^2(M, \mathbf{R}) \tag{36.1}$$

represented by β, which is called the *cohomology class* of β.

Suppose now that M is a *complex manifold* (see §23), and g is a Kähler metric on M (defined as in Remark 23.4). The Kähler form α of g then is the differential 2-form corresponding to g under the isomorphism (23.11); in other words, α is the complex structure tensor J viewed, with the aid of g, as a twice-covariant tensor field. The metric g gives rise to two important cohomology classes. The first is $\omega = [\alpha] \in H^2(M, \mathbf{R})$, the class of α (sometimes termed the *Kähler class* of g). The other, denoted $c_1(M)$ or simply c_1, and called the *first Chern class of the complex manifold* M, is given by

$$c_1 = \frac{1}{2\pi}\,[\rho] \in H^2(M, \mathbf{R})\,, \tag{36.2}$$

where ρ is the Ricci form of g, defined by (23.13). Although ρ itself obviously depends on g, the Chern class c_1 does not, i.e., it is the same for all Kähler metrics on the given complex manifold M. To see this, note that we have relation (23.24):

$$\rho = -\,i\,\partial\bar\partial \log|\det g|\,.$$

Even though $\det g$ is defined (as a function) only locally, the ratio $f = \det \tilde{g}/\det g$ of two such expressions, for two Kähler metrics g and $\tilde{g}$, is a well-defined C^∞ function on M, and so the corresponding Ricci forms ρ and $\tilde{\rho}$ differ by an exact 2-form; in fact,

$$\tilde{\rho} = \rho - i\,\partial\overline{\partial}\log|f|\,, \tag{36.3}$$

while $\partial\overline{\partial} = d\,\overline{\partial}$, as $d = \partial + \overline{\partial}$ and $\overline{\partial}\,\overline{\partial} = 0$ (cf. §23).

The first Chern class can be defined in a much more general situation (e.g., for all complex vector bundles over arbitrary real manifolds). However, for our purposes it suffices to introduce it just in this particular context, that is, for (the tangent bundles of) complex manifolds admitting Kähler metrics.

Let us now suppose that, in addition, our complex manifold M is *compact*. One then says that a cohomology class $\omega \in H^2(M, \mathbf{R})$ is *positive* ($\omega > 0$), or *negative* ($\omega < 0$), if ω can be represented by a closed 2-form β which is anti-Hermitian and such that the corresponding Hermitian tensor field h is positive definite (or, negative definite) at every point of M.

Note that, according to Lemma 23.7, a positive definite Hermitian tensor field h corresponding (under (23.11)) to a *closed* 2-form is nothing else than a Riemannian Kähler metric g on the complex manifold M. In other words, M admits a Riemannian Kähler metric if and only if there exists a cohomology class $\omega \in H^2(M, \mathbf{R})$ with $\omega > 0$.

Lemma 36.1. *Given a compact complex manifold M admitting a Kähler-Einstein metric g, let* s *and c_1 denote the constant scalar curvature of g and the first Chern class of M.*

(a) *If g is Ricci-flat, we have $c_1 = 0$.*
(b) *If $\mathrm{s} < 0$, then $c_1 < 0$.*
(c) *If $\mathrm{s} > 0$, then $c_1 > 0$.*

This is obvious from (36.2) with (23.27) and (23.28). ∎

The assertion of Lemma 36.1 is, as stated, a purely formal (and trivial) consequence of the definitions. It is the next result that adds some real flesh to it showing, in effect, that the binary relation $<$ in $H^2(M, \mathbf{R})$ associated with positivity/negativity of cohomology classes $\omega \in H^2(M, \mathbf{R})$ (so that $\omega < \omega'$ stands for $\omega' - \omega > 0$) is a strict partial ordering: For instance, it shows that the three cases of Lemma 36.1 are mutually exclusive, i.e., no two of them can simultaneously occur in any given compact complex manifold.

Proposition 36.2. *Let there be given a compact complex manifold M and a cohomology class $\omega \in H^2(M, \mathbf{R})$.*

(i) *If $\omega > 0$ or $\omega < 0$, then $\omega \neq 0$.*
(ii) *We cannot simultaneously have $\omega > 0$ and $\omega < 0$.*

Proof. In (i), changing the sign of ω, if necessary, we may assume that $\omega > 0$. A closed anti-Hermitian 2-form α of class C^∞ with $[\alpha] = \omega$ now may be chosen so as to be the Kähler form of a Kähler metric g on M. For any given point $x \in M$, Lemma 9.3 now implies equality (9.5) with $\alpha = \alpha(x)$, where n denotes the real dimension of M and $e_1, \ldots, e_n$ is a suitable positive-oriented $g(x)$-orthonormal basis of T_xM. The mth exterior power of α, with $m = n/2$, is obviously given by

$$\alpha^{\wedge m} = m!\, e_1 \wedge e_2 \wedge \ldots \wedge e_{2m-1} \wedge e_{2m} = m! \,\mathrm{vol}\,, \tag{36.4}$$

vol being the volume form of the canonically oriented manifold (M, g) (see Remark 24.1). Hence its oriented integral is positive:

$$\int_M \alpha^{\wedge m} > 0\,. \tag{36.5}$$

Consequently, $\omega = [\alpha] \neq 0$ in $H^2(M, \mathbf{R})$. In fact, if we had $\alpha = d\zeta$, it would clearly follow that $\alpha^{\wedge m} = d[\zeta \wedge \alpha^{\wedge(m-1)}]$ and, from Stokes's formula (24.11), we would have $\int_M \alpha^{\wedge m} = 0$, contradicting (36.5). Now (i) follows. As for (ii), note that, if we had $\omega = [\alpha] = -[\beta]$ for anti-Hermitian forms α, β corresponding to positive definite Hermitian tensors g and h, then $\alpha + \beta$ would be exact, even though it corresponds to the positive definite tensor $g + h$, contrary to (i). This completes the proof. ■

Corollary 36.3. *For any compact complex manifold M admitting a Kähler metric, we have*

$$H^2(M, \mathbf{R}) \neq \{0\}\,. \tag{36.6}$$

This is clear since, by Lemma 36.2(i), the corresponding Kähler form α then has $[\alpha] \neq 0$. ■

Corollary 36.4. *There exist compact complex manifolds M of any even real dimension $n \geq 4$ which do not admit a Kähler metric.*

Proof. Examples are provided by the *Hopf manifolds* obtained as quotients of $U = \mathbf{C}^{n/2} \smallsetminus \{0\}$ under the action of $\mathbf{Z}$ consisting of the transformations $x \mapsto a^k x$, $k \in \mathbf{Z}$, with a fixed real number $a > 1$. Since the action is holomorphic, the obvious complex-manifold structure of U descends to the quotient, here denoted M. On the other hand, U can be identified with $N = \mathbf{R} \times S^{n-1}$ via the diffeomorphism $N \ni (t, u) \mapsto e^t u \in U$, which makes the above $\mathbf{Z}$ action appear as $(t, u) \mapsto (t + ck, u)$, $k \in \mathbf{Z}$, with $c = \log a > 0$. Hence the quotient M is C^∞-diffeomorphic to $S^1 \times S^{n-1}$. Thus (e.g., using Künneth's formula) we obtain $H^2(M, \mathbf{R}) = \{0\}$ and, in view of Corollary 36.3, M admits no Kähler metric. ■

Remark 36.5. The existence of a Kähler metric on a complex manifold M is, however, guaranteed whenever M is a complex submanifold of $\mathbf{C}\mathrm{P}^m$, in any complex dimension m; for instance, the submanifold metric of M induced by the Fubini-Study metric of $\mathbf{C}\mathrm{P}^m$ (Example 10.6) is automatically Kähler.

Lemma 36.1 also rises the question of whether the conditions on the first Chern class c_1, necessary for the existence of a Kähler-Einstein metric on the given compact complex manifold M (assumed to admit a Kähler metric) are also *sufficient*. This is well-known not to be the case when $c_1 > 0$; see Theorem 36.8 below. The proposition that the answer is 'yes' when $c_1 < 0$ or $c_1 = 0$, is known as *Calabi's conjecture* (Calabi, 1954).

Calabi's conjecture was proved by Aubin (case $c_1 < 0$) and Yau (cases $c_1 < 0$ and $c_1 = 0$). The uniqueness assertions below were already established by Calabi (1954).

Theorem 36.6 (Aubin, 1976; Yau, 1977). *Let M be a compact complex manifold admitting a Kähler metric, and let c_1 be the first Chern class of M.*

(i) *If $c_1 < 0$, then M also admits a Kähler-Einstein metric g, and such a metric is unique up to a constant factor.*
(ii) *If $c_1 = 0$, then every positive cohomology class $\omega \in H^2(M, \mathbf{R})$ contains a unique Kähler form representing a Ricci-flat Kähler metric.*

A proof of Theorem 36.6 requires analysis techniques that are far beyond the scope of this article. The reader is referred to one of many existing detailed presentations, such as *Séminaire Palaiseau* (1978). ■

As already mentioned in §33, by a $K3$ *surface* we mean any compact simply connected 4-manifold M which admits a Riemannian metric g and a bivector field α such that $(\tilde{M}, g, \alpha)$ is a Ricci-flat Kähler manifold. (This definition, although not standard, is equivalent to those normally used in the literature.) In view of (36.1), the (real) first Chern class $c_1(M)$ of any $K3$ surface $(\tilde{M}, g, \alpha)$ must be zero. On the other hand, Yau's Theorem 36.6(ii) guarantees that every simply connected compact complex surface M admitting a Kähler metric and satisfying the condition $c_1(M) = 0$ is a $K3$ surface. An example of such a complex surface M is the *Kummer surface* $M \subset \mathbf{C}\mathrm{P}^3$, given by the equation $x^2 + y^2 + z^2 = 0$ in homogenous coordinates x, y, z. (Cf. Remark 36.5.) Thus, we have

Corollary 36.7. *In every real dimension $n \geq 4$ there exists a compact Ricci-flat Riemannian manifold which is not flat.*

In fact, such examples are provided by products of $K3$ surfaces and tori, involving at least one $K3$-surface factor, with product metrics obtained from Ricci-flat Kähler metrics on $K3$ surfaces and flat metrics on tori. Note that such manifolds M do not admit a flat metric since the $K3$ surfaces are simply connected, and so the universal covering space of M cannot be diffeomorphic to $\mathbf{R}^n$. ■

The analogue of the Calabi conjecture in the case where $c_1 > 0$ is false. For such manifolds, there exist further obstructions to the existence of a Kähler-Einstein metric. One such obstruction stems from results of Lichnerowicz (1957) and Matsushima (1957), the other from an invariant discovered by Futaki (1983). For a detailed exposition of the subject, see Bourguignon (1997); all we can present here is a brief outline of just a few facts. First, we have

Theorem 36.8 (Lichnerowicz, 1957, and Matsushima, 1957)). *Given a compact Kähler manifold (M, g, α), let $\mathfrak{h} = \mathfrak{hol}(M)$ be the vector space of all holomorphic vector fields on M, and let $\mathfrak{g} = \mathfrak{isom}(M, g)$ be the Lie algebra of all Killing fields on (M, g). If the scalar curvature s of g is constant, we have*

$$\mathfrak{h} \;=\; \mathfrak{g} \;+\; \alpha\,\mathfrak{g}\,, \tag{36.7}$$

that is, every holomorphic vector field u can be written as $u = v + \alpha w$ for some Killing fields v and w. ■

Since the complex Lie algebra $\mathfrak{hol}(M)$ of all holomorphic vector fields on M depends only on the complex-manifold structure of M, Theorem 36.8 provides a necessary condition for a compact complex manifold M to admit a Kähler-Einstein metric: Namely, $\mathfrak{h} = \mathfrak{hol}(M)$ then is Lie-algebra isomorphic to the complexification of $\mathfrak{isom}(M, g)$ for some Kähler metric g and, consequently, $\mathfrak{h}$ must be reductive (that is, be the direct sum of its center and its *commutator ideal* $[\mathfrak{h}, \mathfrak{h}]$). As a consequence, we obtain

Example 36.9. The compact complex manifold $M = \mathbf{C}\mathrm{P}^2 \# k\,\overline{\mathbf{C}\mathrm{P}^2}$ obtained by blowing up k points in $\mathbf{C}\mathrm{P}^2$, where $1 \le k \le 8$, admits a Kähler metric and satisfies the condition $c_1(M) > 0$, but does not admit a Kähler-Einstein metric. (The reason is that $\mathfrak{hol}(M)$ is not reductive.)

Let g be a Kähler metric on a compact complex manifold M and, again, let $\mathfrak{h} = \mathfrak{hol}(M)$. The *Futaki invariant* of M and g (Futaki, 1983) is the real-linear function $\mathcal{F} : \mathfrak{h} \to \mathbf{R}$ given by

$$\mathcal{F}(w) = \int_M d_w F \,\mathrm{vol}_g\,, \tag{36.8}$$

where $F : M \to \mathbf{R}$ is the C^∞ function uniquely characterized by the Hodge decompostion

$$\rho = \beta + i\,\partial\overline{\partial} F\,, \tag{36.9}$$

of the Ricci form ρ (which is a *closed* anti-Hermitian 2-form) into a *harmonic* anti-Hermitian 2-form β and an *exact* anti-Hermitian 2-form; the latter then necessarily equals $i\,\partial\overline{\partial}F$ for some F, unique up to an additive constant. To make F completely unique, one requires in addition that $\int_M F\,\mathrm{vol}_g = 0$.

Theorem 36.10 (Futaki, 1983). *Let M be a compact complex manifold which admits a Kähler-Einstein metric and satisfies $c_1(M) > 0$, and let $\mathcal{F}$ denote the Futaki invariant of any Kähler metric on M whose Kähler form belongs to the cohomology class $c_1(M)$. Then $\mathcal{F} = 0$.* ■

Theorem 36.10 can be used to establish nonexistence of Kähler-Einstein metrics on some compact complex manifolds M that admit Kähler metrics and have $c_1(M) > 0$, and for which such a conclusion cannot be obtained from Theorem 36.8. See Futaki (1983), Besse (1987) or Bourguignon (1997).

PART IV: THE INDEFINITE-METRIC CASE

The next thirteen sections deal with local properties of indefinite Einstein metrics in dimension four. We begin with a quick presentation of Petrov's classification of curvature types for such metrics (sections 37 through 40 and, later in the text, §42 and §47). The techniques developed there then are used to present a classification of all possible local-isometry types of locally symmetric Einstein 4-manifolds. The classification theorem itself is due to Cartan (1926) in the Riemannian case, Petrov (1969) for Lorentzian metrics, and Cahen and Parker (1980) for the neutral sign pattern $- - + +$; in our presentation, it takes the form of Theorems 41.4, 41.5 and 41.6, stated in §41 and proved in sections 43 through 46.

In contrast with the Riemannian case, there exists 4-dimensional indefinite Einstein metrics essentially different from the "obvious" examples which consist of spaces of constant curvature, spaces of constant holomorphic sectional curvature, and products of two surface metrics with equal constant Gaussian curvatures. Those "exotic" metrics form a particularly large collection of examples in the case of the neutral sign pattern $- - + +$ (see Theorem 41.6).

The last two sections contain brief comments on the rôle of Einstein metrics in general relativity and, respectively, examples of Ricci-flat pseudo-Riemannian 4-manifolds with the neutral sign pattern $- - + +$ which are curvature-homogeneous, but not locally homogeneous.

§37. Geometry of bivectors

Throughout this section, let us assume that $\mathcal{T}$ is a four-dimensional real vector space carrying a fixed *pseudo-Euclidean inner product* (that is, a nondegenerate symmetric bilinear form, cf. §3), which we will denote $\langle\,,\rangle$. The possible sign patterns of $\langle\,,\rangle$ are, up to an overall sign change,

$$+ + + + \,, \qquad - - + + \,, \qquad \text{and} \qquad - + + + \,. \tag{37.1}$$

When dealing with the exterior product of two vectors $a, b \in \mathcal{T}$, we will often skip the wedge symbol and write

$$ab \,=\, a \wedge b \,\in\, \mathcal{T}^{\wedge 2}\,. \tag{37.2}$$

The inner product $\langle\,,\rangle$ in $\mathcal{T}$ induces a pseudo-Euclidean inner product (also denoted $\langle\,,\rangle$) in the six-dimensional bivector space $\mathcal{T}^{\wedge 2}$ characterized by

$$\langle ab, cd\rangle \,=\, \langle a, c\rangle\langle b, d\rangle \,-\, \langle b, c\rangle\langle a, d\rangle \tag{37.3}$$

for $a, b, c, d \in \mathcal{T}$. (This is well-defined on $\mathcal{T}^{\wedge 2}$ due to bilinearity and skew-symmetry in both pairs a, b and c, d.) Setting

$$\varepsilon_a \,=\, \langle a, a\rangle \tag{37.4}$$

for vectors $a \in \mathcal{T}$, we thus have

$$\langle ab, ab\rangle \,=\, \varepsilon_a\varepsilon_b \quad \text{whenever} \quad a, b \in \mathcal{T} \quad \text{and} \quad \langle a, b\rangle \,=\, 0\,. \tag{37.5}$$

Note that $\langle\,,\rangle$ in $\mathcal{T}^{\wedge 2}$ is actually nondegenerate; in fact,

$$ab\,,\ ac\,,\ ad\,,\ cd\,,\ db\,,\ bc \quad \text{is an} \ \langle\,,\rangle\text{-orthonormal basis of} \quad \mathcal{T}^{\wedge 2}\,, \tag{37.6}$$

with the sign pattern

$$+ + + + + + \,, \quad - - - + + + \,, \quad \text{and} \quad + - - + - - \,, \tag{37.7}$$

whenever a, b, c, d is an orthonormal basis of $\mathcal{T}$ having, respectively, the first, second or third sign pattern in (37.1).

If, in addition, $\mathcal{T}$ is *oriented*, the inner product $\langle\,,\rangle$ in $\mathcal{T}$ gives rise to a distinguished *volume element* $\mathrm{vol} \in \mathcal{T}^{\wedge 4}$, which is the nonzero 4-vector given by $\mathrm{vol} = ab \wedge cd$ (i.e., $a \wedge b \wedge c \wedge d$) for any positive-oriented orthonormal basis a, b, c, d of $\mathcal{T}$. (See (3.34).) The exterior multiplication of bivectors (§3) now gives rise to the *wedge form* $(\,,\,)$, which is a real-valued symmetric bilinear form in the bivector space $\mathcal{T}^{\wedge 2}$, characterized by

$$\alpha \wedge \beta \,=\, (\alpha, \beta) \cdot \mathrm{vol} \tag{37.8}$$

for any $\alpha, \beta \in \mathcal{T}^{\wedge 2}$.

The pseudo-Euclidean inner product $\langle\,,\rangle$ in the bivector space $\mathcal{T}^{\wedge 2}$ allows us, as usual (see (3.28)) to identify linear operators $F : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ with bilinear forms A on $\mathcal{T}^{\wedge 2}$, via the relation $A(\alpha, \beta) = \langle F\alpha, \beta\rangle$ for $\alpha, \beta \in \mathcal{T}^{\wedge 2}$. For the wedge form $A = (\,,\,)$, the corresponding operator $\mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ is denoted $*$ and called the *Hodge star acting on bivectors.* Explicitly, $(\alpha, \beta) = \langle *\alpha, \beta\,\rangle$, i.e.,

$$\alpha \wedge \beta \,=\, \langle *\alpha, \beta\,\rangle \cdot \mathrm{vol} \tag{37.9}$$

for all $\alpha, \beta \in \mathcal{T}^{\wedge 2}$. Since the wedge form $(\,,\,)$ symmetric, we have

$$\langle *\alpha, \beta\rangle \,=\, \langle \alpha, *\beta\rangle \tag{37.10}$$

for $\alpha, \beta \in \mathcal{T}^{\wedge 2}$, that is, $* : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ is self-adjoint relative to the inner product (2.17).

Proposition 37.1. *Let $a, b \in \mathcal{T}$ be two linearly independent vectors a 4-dimensional oriented real vector space $\mathcal{T}$ with a pseudo-Euclidean inner product $\langle\,,\rangle$.*

(i) *If $\langle a, a\rangle = \langle a, b\rangle = \langle b, b\rangle = 0$, then*

$$*(ab) \,=\, \pm\,(ab)\,, \tag{37.11}$$

the sign $\pm$ being $+$ or $-$ depending on whether a basis a, b, c, d of $\mathcal{T}$, formed by a, b and any vectors $c, d \in \mathcal{T}$ with $\langle c, c\rangle = \langle c, d\rangle = 0$, $\langle a, c\rangle = \langle b, d\rangle = 1$, $\langle a, d\rangle = \langle b, c\rangle = 0$, is positive or negative oriented. Note that such c, d exist by Lemma 3.14.

(ii) *If $\langle a, a\rangle = \varepsilon_a \in \{1, -1\}$ and $\langle a, b\rangle = \langle b, b\rangle = 0$, then*

$$*(ab) \,=\, \pm\,\varepsilon_c\, cb\,, \tag{37.12}$$

where $c \in \mathcal{T}$ is any vector with $\langle c, c\rangle = \varepsilon_c \in \{1, -1\}$ and $\langle a, c\rangle = \langle b, c\rangle = 0$, and the sign $\pm$ is $+$ or $-$ depending on whether a basis a, b, c, d of $\mathcal{T}$, consisting of a, b, c and any vector $d \in \mathcal{T}$ with $\langle c, d\rangle > 0$ is positive or negative oriented. Such c, d must exist by Lemma 3.15.

(iii) *If a, b are orthonormal, that is, $\langle a, a\rangle = \langle b, b\rangle = 1$ and $\langle a, b\rangle = 0$, then*

$$*(ab) \,=\, \varepsilon_c \varepsilon_d\, cd\,, \tag{37.13}$$

with ε_c, ε_d as in (37.4)*, for any vectors $c, d \in \mathcal{T}$ such that a, b, c, d is a positive-oriented orthonormal basis of $\mathcal{T}$.*

Proof. To verify (37.13) for any given a positive-oriented orthonormal basis a, b, c, d of $\mathcal{T}$, it suffices to take the $\langle\, ,\rangle$-inner products of both sides with all elements of (37.6) and use (37.3), (37.9) and (37.5). This proves (iii). Let us now consider any three vectors a, b, c having the inner-product properties listed in (ii), that is, $\langle a, a\rangle = \varepsilon_a$, $\varepsilon_a, , \varepsilon_c \in \{1, -1\}$, $\langle c, c\rangle = \varepsilon_c$ and $\langle a, b\rangle = \langle b, b\rangle = \langle a, c\rangle = \langle b, c\rangle = 0$. Such a, b, c must be linearly independent, since a, c are orthonormal and b is orthogonal to them, while $b \neq 0$ (as a, b are assumed linearly independent). Consequently, $\mathrm{Span}\,\{a, b, c\} = b^\perp$, and the set $\varOmega = \mathcal{T} \smallsetminus b^\perp$ consists precisely of all $d \in \mathcal{T}$ such that a, b, c, d is a basis of $\mathcal{T}$. Its subset $\varOmega^+ = \{d \in \mathcal{T} : \langle c, d\rangle \neq 0\}$ is convex. We may assume in (ii) that, in addition, d is null and orthogonal to a, b, and $\langle c, d\rangle = 1$ (see Lemma 3.15(b)); convexity of $\varOmega^+$ guarantees that this modification of d will not affect the orientation represented by a, b, c, d (cf. Remark 3.6). Similarly, in (i), we may require, in addition, that $\langle d, d\rangle = 0$, since that will be the case if we replace d with $d - \langle d, d\rangle b/2$ and leave a, b, c unchanged. To establish (i) and (ii), let us now define $\bar a, \bar b, \bar c, \bar d \in \mathcal{T}$ by either $\sqrt{2}\,\bar a = a - c$, $\sqrt{2}\,\bar b = b - d$, $\sqrt{2}\,\bar c = a + c$, $\sqrt{2}\,\bar d = b + d$ (in (i)), or $\bar a = a$, $\sqrt{2}\,\bar b = d + b$, $\bar c = c$, $\sqrt{2}\,\bar d = d - b$ (in (ii)). Thus, $\bar a, \bar b, \bar c, \bar d$ is an orthonormal basis of $\mathcal{T}$ with the sign pattern $- - + +$ (in (i)) or $(\varepsilon_a, +1, \varepsilon_c, +1)$ (in (ii)) and, in both cases, representing the same orientation as a, b, c, d (as one sees evaluating $\bar a \wedge \bar b \wedge \bar c \wedge \bar d$). Let us now reverse the orientation of $\mathcal{T}$, if necessary, so as to make the basis a, b, c, d positive oriented; this will of course change the sign of $*$ as well. We have $\sqrt{2}\,a = \bar a + \bar c$, $\sqrt{2}\,b = \bar b + \bar d$ (in (i)) and $\sqrt{2}\,b = \bar b - \bar d$ (in (ii)), and so, applying (iii) to the new orthonormal basis $\bar a, \bar b, \bar c, \bar d$, we obtain, in (i), $2 * (ab) = *(\bar a\bar b) + *(\bar a\bar d) + *(\bar c\bar b) + *(\bar c\bar d) = \bar c\bar d - \bar b\bar c - \bar d\bar a + \bar a\bar b = (\bar a + \bar c)(\bar b + \bar d) = 2ab$, so that $*(ab) = ab$, while, in (ii), $\sqrt{2} *(ab) = *(\bar a\bar b) - *(\bar a\bar d) = \varepsilon_c\,\bar c(\bar b - \bar d) = \sqrt{2}\,\varepsilon_c\,cb$, which gives $*(ab) = \varepsilon_c\,cb$. This completes the proof. ■

The pseudo-Euclidean inner product $\langle\, ,\rangle$ in our real 4-space $\mathcal{T}$ leads to natural isomorphic identifications

$$\mathcal{T}^{\wedge 2} \,=\, L_{\mathrm{skew}}(\mathcal{T}, \mathcal{T}; \mathbf{R})\,, \tag{37.14}$$

$$\mathcal{T}^{\wedge 2} \,=\, \mathfrak{so}(\mathcal{T})\,, \tag{37.15}$$

of the bivector space $\mathcal{T}^{\wedge 2}$, first with the space of all skew-symmetric bilinear forms $\mathcal{T} \times \mathcal{T} \to \mathbf{R}$ and then with the Lie algebra of all skew-adjoint operators $\mathcal{T} \to \mathcal{T}$ (notation as in §3). Specifically, (37.14) and (37.15) identify every bivector $\alpha \in \mathcal{T}^{\wedge 2}$ with the bilinear form $(b, c) \mapsto \alpha(b, c)$ and the operator $b \mapsto \alpha b$ characterized by

$$\alpha(b, c) \,=\, \langle \alpha, bc\rangle \tag{37.16}$$

and

$$\langle \alpha b, c\rangle \,=\, \langle \alpha, bc\rangle \tag{37.17}$$

for all $c \in \mathcal{T}$ (cf. also (2.20); here $bc = b \wedge c$, as in (37.2)). Note that (37.17) describes precisely the operator corresponding to the form (37.16) via the pseudo-Euclidean inner product $\langle\, ,\rangle$ in $\mathcal{T}$ (cf. (3.28)), so that skew-adjointness of the operator is clear from skew-symmetry of the form. Also, the fact that the identifications (37.14) and (37.15) are really isomorphic is clear for dimensional reasons, since the assignments sending $\alpha \in \mathcal{T}^{\wedge 2}$ to the corresponding operator and form are injective in view of (37.6).

Recall (§3) that a bivector $\alpha \in \mathcal{T}^{\wedge 2}$ is called decomposable if $\alpha = bc$ for some $b, c \in \mathcal{T}$. As an operator $\mathcal{T} \to \mathcal{T}$, such a bivector $\alpha = bc$ is given by

$$(bc)d \,=\, \langle d, b\rangle c \,-\, \langle d, c\rangle b\,. \tag{37.18}$$

In fact, by (37.17) and (37.3), the inner products of both sides with any vector $a \in \mathcal{T}$ equal $\langle bc, da\rangle$. (Cf. also (2.22).) For a nonzero decomposable bivector $\alpha = bc$, the factors b, c must be linearly independent so, by (37.18),

$$\mathrm{Span}\,\{b, c\} \,=\, \alpha(\mathcal{T})\,, \tag{37.19}$$

that is, the plane $\mathrm{Span}\,\{b, c\}$ in $\mathcal{T}$ spanned by them coincides with the image of α treated as an operator $\mathcal{T} \to \mathcal{T}$. In particular, the plane $\mathrm{Span}\,\{b, c\}$ then is uniquely determined by α (see also (3.10)). We will refer to $\mathrm{Span}\,\{b, c\}$ as the *plane associated with the nonzero decomposable bivector* $\alpha = bc$.

As before, let $\langle\,,\rangle$ be a pseudo-Euclidean inner product in a 4-dimensional real vector space $\mathcal{T}$. We define the *sign factor* ε of $\langle\,,\rangle$ by

$$\varepsilon \,=\, \varepsilon_a\varepsilon_b\varepsilon_c\varepsilon_d \,=\, \pm 1\,, \tag{37.20}$$

with (37.4), for any orthonormal basis a, b, c, d of $\mathcal{T}$. Thus, $\varepsilon = +1$ except for the Lorentz-like sign patterns $-\,+\,+\,+$ and $-\,-\,-\,+$. Now (37.13) implies

$$*^2 \,=\, \varepsilon \cdot \mathrm{Id}\,, \tag{37.21}$$

so that the inverse of $*$ equals $*$ times the sign factor ε. From (37.10), we now obtain

$$\langle *\alpha, *\beta\,\rangle \,=\, \varepsilon\,\langle \alpha, \beta\,\rangle \tag{37.22}$$

whenever $\alpha, \beta \in \mathcal{T}^{\wedge 2}$.

Equality (37.21) accounts for one of the most crucial formal differences between the Riemannian (Euclidean) case on the one hand, and the Lorentzian case on the other. That difference has profound consequences for the geometry of the bivector space and, ultimately, the classification of the possible curvature types. Specifically, if the inner product $\langle\,,\rangle$ in $\mathcal{T}$ has the Riemannian sign pattern $+\,+\,+\,+$ (or the neutral sign pattern $-\,-\,+\,+$), (37.21) says that $*$ is an involution, and so, according to Remark 3.2, it leads to a direct-sum decomposition of $\mathcal{T}^{\wedge 2}$ into the (± 1)-eigenspaces $\mathcal{B}^{\pm}$ of $*$, with

$$\mathcal{T}^{\wedge 2} \,=\, \mathcal{B}^+ \oplus\, \mathcal{B}^-\,, \qquad \mathcal{B}^- \,=\, [\mathcal{B}^+]^\perp\,, \qquad \dim \mathcal{B}^{\pm} \,=\, 3\,. \tag{37.23}$$

In fact, the summands $\mathcal{B}^{\pm}$ are mutually orthogonal due to self-adjointness of $*$ (see (37.10) and Remark 3.17), and they are both 3-dimensional since, for any positive-oriented orthonormal basis a, b, c, d of $\mathcal{T}$ representing the sign pattern $+\,+\,+\,+$ or $-\,-\,+\,+$, the bivectors

$$\sqrt{2}\,\alpha^{\pm} \,=\, \pm\, ab \,+\, cd\,, \qquad \sqrt{2}\,\beta^{\pm} \,=\, \pm\, ac \,+\, db\,, \qquad \sqrt{2}\,\gamma^{\pm} \,=\, \pm\, ad \,+\, bc \tag{37.24}$$

or, respectively,

(37.25) $\sqrt{2}\,\alpha^\pm = \pm\, ab + cd\,, \qquad \sqrt{2}\,\beta^\pm = \pm\, ac - db\,, \qquad \sqrt{2}\,\gamma^\pm = \pm\, ad - bc$

form an $\langle\,,\rangle$-orthonormal basis of $\mathcal{B}^\pm$ with the sign pattern $+++$ (or, respectively, $--+$), as one easily sees using (37.6), (37.7) and (37.13).

In the case where $\mathcal{T} = T_xM$ for an oriented pseudo-Riemannian 4-manifold (M,g) whose metric g has one of the sign patterns $++++$ or $--++$, the decomposition (37.23) is written as

$$[T_xM]^{\wedge 2} = \Lambda^+_x M \oplus \Lambda^-_x M\,. \tag{37.26}$$

See also (6.4).

On the other hand, if $\langle\,,\rangle$ has the Lorentzian sign pattern $-+++$, (37.21) states that $*$ is a *complex structure* in $\mathcal{T}^{\wedge 2}$, that is, endowes $\mathcal{T}^{\wedge 2}$ with the structure of a 3-dimensional complex vector space for which $*$ is the multiplication by i. (See Remark 3.9.) Furthermore, by (37.22) with $\varepsilon = -1$ and Remark 3.18, $\langle\,,\rangle$ is the real part of a unique complex-bilinear inner product $(\,,)_{\mathbf{c}}$ in the complex 3-space $\mathcal{T}^{\wedge 2}$, which can explicitly be written as

$$(\alpha,\beta\,)_{\mathbf{c}} = \langle\alpha,\beta\,\rangle - i\,\langle\alpha,*\beta\,\rangle \tag{37.27}$$

for $\alpha,\beta\in\mathcal{T}^{\wedge 2}$ (see (3.35)). By (37.6), (37.13) and (37.27),

$$bc\,,\quad bd\,,\quad cd \tag{37.28}$$

is a $(\,,)_{\mathbf{c}}$-orthonormal basis of $\mathcal{T}^{\wedge 2}$ as defined in §3, whenever a,b,c,d is a Lorentz-orthonormal basis of $\mathcal{T}$ with the sign pattern $-+++$.

We now proceed to explore some relations between two aspects of the geometry of $\mathcal{T}^{\wedge 2}$, one related to the Hodge star operator $*:\mathcal{T}^{\wedge 2}\to\mathcal{T}^{\wedge 2}$, the other to the Lie-algebra structure of $\mathcal{T}^{\wedge 2} = \mathfrak{so}(\mathcal{T})$ (represented by the commutator of operators).

Proposition 37.2. *Given an oriented 4-dimensional real pseudo-Euclidean vector space $\mathcal{T}$ and bivectors $\alpha,\beta\in\mathcal{T}^{\wedge 2}$, we have*

$$[*\alpha,\beta] = *[\alpha,\beta] = [\alpha,*\beta] \tag{37.29}$$

and

$$[\alpha,\beta] = 0 \qquad\text{whenever}\qquad *\,\alpha = \alpha \quad\text{and}\quad *\,\beta = -\,\beta\,. \tag{37.30}$$

Here $[\alpha,\beta]$ is the bivector corresponding to the commutator of α and β under the identification (37.17) *between bivectors in $\mathcal{T}$ and operators $\alpha\in\mathfrak{so}(\mathcal{T})$.*

Proof. Let a,b,c,d be a positive-oriented orthonormal basis of $\mathcal{T}$. Using (37.13) and (2.28) we now obtain $[ab,*(ab)] = 0 = *[ab,ab]$, $[ab,*(cd)] = 0 = *[ab,cd]$, and $[ab,*(ac)] = \varepsilon_d\varepsilon_b[ab,db] = \varepsilon_d ad = \varepsilon_a*(bc) = *[ab,ac]$, where the signs $\varepsilon_a,\varepsilon_b,\varepsilon_c,\varepsilon_d$ are defined as in (37.4). The first equality in (37.29), that is, $[*\alpha,\beta] = *[\alpha,\beta]$ thus holds whenever α and β are exterior products of pairs of vectors belonging to a fixed orthonormal basis of $\mathcal{T}$. (Permuting a,b,c,d as needed, we see that the three relations just established correspond to the cases where the two exterior products have two, zero or, respectively, one factor in common.) The first equality of (37.29) now follows for all α and β in view of bilinearity of both sides in α and β. As for the second equality in (37.4), it is a consequence of the first one combined with skew-symmetry of $[\alpha,\beta]$ in α and β. Finally, for α, β as in (37.30), relation (37.29) gives $*[\alpha,\beta] = [*\alpha,\beta] = [\alpha,\beta] = [\alpha,-*\beta] = -*[\alpha,\beta]$, which in turn implies (37.30) since $*$ is an isomorphism (cf. (37.21)). This completes the proof. ■

As immediate consequences, we have

Corollary 37.3. *Let the inner product* $\langle\,,\rangle$ *of an oriented* 4*-dimensional pseudo-Euclidean vector space* $\mathcal{T}$ *have the Riemannian sign pattern* $++++$ *or the neutral sign pattern* $--++$. *Then the summands* $\mathcal{B}^{\pm}$ *of* (37.23) *are mutually commuting ideals in the Lie algebra* $\mathcal{T}^{\wedge 2} = \mathfrak{so}(\mathcal{T})$, *so that* (37.23) *also represents a direct sum of Lie algebras.* ■

Corollary 37.4. *Let* $\mathcal{T}$ *be an oriented* 4*-dimensional pseudo-Euclidean vector space with an inner product* $\langle\,,\rangle$ *of the Lorentzian sign pattern* $-+++$. *Then* $\mathcal{T}^{\wedge 2} = \mathfrak{so}(\mathcal{T})$ *is a complex Lie algebra, that is, its commutator pairing is complex-bilinear for the complex structure in* $\mathcal{T}^{\wedge 2}$ *introduced by the Hodge star* $*$. ■

It turns out that the anticommutators have some interesting properties as well: Relation (37.31) below indicates that vectors in $b \in \mathcal{T}$ may be thought of as "spinors" on which bivectors $\alpha \in \mathcal{B}^{\pm}$ operate as "vectors", via the "Clifford multiplication" given by the evaluation pairing $(\alpha, b) \mapsto \alpha b$. (Cf. Example 30.5.)

Proposition 37.5. *Let* $\mathcal{T}$ *be an oriented* 4*-dimensional real vector space with an inner product* $\langle\,,\rangle$ *which is Euclidean or neutral, that is, has the sign pattern* $++++$ *or* $--++$. *Any bivectors* $\alpha, \beta \in \mathcal{T}^{\wedge 2}$ *such that* $\alpha, \beta \in \mathcal{B}^{\pm}$ *for some sign* $\pm$ *then satisfy the* Clifford-algebra relations

$$\alpha\beta + \beta\alpha = -\langle\alpha, \beta\rangle\,, \tag{37.31}$$

where $\alpha\beta$ *is the composite of* α *and* β *treated, with the aid of* $\langle\,,\rangle$, *as skew-adjoint operators* $\mathcal{T} \to \mathcal{T}$, *while the real number* $\lambda = \langle\alpha, \beta\rangle$ *stands for* λ *times the identity. In particular, for any* $\alpha \in \mathcal{B}^{\pm}$ *we have*

$$2\,\alpha^2 = -\langle\alpha, \alpha\rangle\,. \tag{37.32}$$

Proof. From (37.18) we obtain $\gamma(bc)d = \langle d, b\rangle\gamma c - \langle d, c\rangle\gamma b$ for all bivectors $\gamma \in \mathcal{T}^{\wedge 2}$ and vectors $b, c, d \in \mathcal{T}$. Applying this to the case where γ is decomposable (cf. (2.27)) and using (37.18) along with (37.13), it is easy to verify that (37.31) and (37.32) holds when $\alpha = ab + *(ab)$ and $\beta = ac + *(ac)$ for any orthonormal vectors a, b, c. This shows that (37.31) holds when α, β is any pair of vectors of a basis of $\mathcal{B}^{\pm}$ having the form (37.24) or (37.25). In view of bilnearity of (37.31) in α and β,

Finally, (37.32) implies (37.31) according to Remark 3.12. This completes the proof. ■

Lemma 37.6. *Given a* 4*-dimensional real vector space* $\mathcal{T}$ *with an inner product* $\langle\,,\rangle$ *of the Lorentzian sign pattern* $-+++$ *and a nonzero bivector* $\alpha \in \mathcal{T}^{\wedge 2}$, *we have*

(i) $(\alpha, \alpha)_{\mathbf{c}} \in \mathbf{R}$ *if and only if* α *is decomposable.*

(ii) $(\alpha, \alpha)_{\mathbf{c}} = 1$ *if and only if* $\alpha = bc$ *for some vectors* $b, c \in \mathcal{T}$ *with* $\langle b, b\rangle = \langle c, c\rangle = 1$ *and* $\langle b, c\rangle = 0$.

(iii) $(\alpha, \alpha)_{\mathbf{c}} = 0$ *if and only if* $\alpha = be$ *for some vectors* $b, e \in \mathcal{T}$ *such that* $\langle b, e\rangle = \langle e, e\rangle = 0$ *and* $\langle b, b\rangle = 1$.

Proof. (i) is immediate from (37.27), (37.9) and Lemma 3.7(b). Moreover, the 'if' parts of both (ii) and (iii) are obvious from (37.27) and (37.5) with (37.4). Let us now assume that $(\alpha, \alpha)_{\mathbf{c}}$ equals 1 or 0. Thus, by (i), $\alpha = bc$ for some vectors

$b, c \in \mathcal{T}$. Since $\alpha \neq 0$, b and c are linearly independent and so they span a 2-dimensional subspace P of $\mathcal{T}$, which depends only on α (see (37.19)). We may therefore choose b, c as above which are also orthogonal. By (i), (37.27) and (37.5), the numbers ε_b, ε_c defined as in (37.4) satisfy $\varepsilon_b \varepsilon_c = q$ with $q = 1$ in (ii) and $q = 0$ in (iii). Rescaling both b and c, we may assume that $\varepsilon_b, \varepsilon_c \in \{1, 0, -1\}$. Then, in (ii), $\varepsilon_b = \varepsilon_c = \pm 1$ and the sign $\pm$ cannot be a minus since the sign pattern of $\langle\,,\rangle$ in $\mathcal{T}$ is $-+++$, with just one minus (see Remark 3.13). This proves (ii). Similarly, in (iii), one of $\varepsilon_b, \varepsilon_c$ must be 0, while the other cannot be 0 or -1 since, according to Remark 3.13, that would similarly contradict our assumption about the Lorentzian sign pattern of $\langle\,,\rangle$. This completes the proof. ■

Lemma 37.7. *Let $\beta, \gamma \in \mathcal{T}^{\wedge 2}$ be two linearly independent bivectors in a 4-dimensional real vector space with an inner product $\langle\,,\rangle$ of the Lorentzian sign pattern $-+++$. The following two conditions are equivalent*:

(a) *$\langle\beta, \beta\rangle = \langle\beta, \gamma\rangle = \langle\gamma, \gamma\rangle = 0$ and $\gamma = *\beta$ for some orientation of $\mathcal{T}$.*
(b) *$\beta = be$ and $\gamma = ce$ for some linearly independent vectors $b, c, e \in \mathcal{T}$ such that $\langle b, b\rangle = \langle c, c\rangle = 1$ and $\langle b, c\rangle = \langle b, e\rangle = \langle c, e\rangle = \langle e, e\rangle = 0$.*

Furthermore, β and γ then determine the vector e in (b) *uniquely up to a sign.*

Proof. Assume (b). According to Lemma 3.15(ii), we may choose a null vector $d \in \mathcal{T}$ orthogonal to b, c, and such that $\langle d, e\rangle = 1$. Applying Proposition 37.1(ii) (with the rôles of a, b, c, d in Proposition 37.1(ii) now played by b, c, e, d), we obtain $*(be) = ce$ for the orientation which the basis b, c, e, d positive. Since the inner-product relations in (a) are immediate from (37.5) and (37.6), this proves (a).

Conversely, suppose that (a) holds. From (37.27) we now have $(\beta, \beta)_{\mathbf{c}} = 0$ and so, by Lemma 37.6(iii), $\beta = be$ for some $b, e \in \mathcal{T}$ with $\langle b, e\rangle = \langle e, e\rangle = 0$ and $\langle b, b\rangle = 1$. Let us now choose $c \in \mathcal{T}$ which along with those b and e satisfies inner-product relations in (b), and a null vector $d \in \mathcal{T}$ orthogonal to b, c, and such that $\langle d, e\rangle = 1$. (They exist in view of Lemma 3.15; note that we cannot have $\langle c, c\rangle = -1$, since d, e span a nondegenerate plane P on which $\langle\,,\rangle$ has the sign pattern $-+$, and so its sign pattern on $P^\perp = \mathrm{Span}\,\{b, c\}$ must consists of the remaining signs $++$.) Changing the sign of c if necessary, we may assume that the basis b, c, e, d is positive for the orientation used in (a). Applying Proposition 37.1(ii) (with the rôles of a, b, c, d in Proposition 37.1(ii) again played by b, c, e, d), we obtain $\gamma = *\beta = *(be) = ce$. This completes the proof. ■

Lemma 37.8. *For a nonzero bivector $\beta \in \mathcal{T}^{\wedge 2}$ in a 4-dimensional real vector space $\mathcal{T}$ with an inner product $\langle\,,\rangle$ of the neutral sign pattern $--++$, the following two conditions are equivalent*:

(i) *$\langle\beta, \beta\rangle = 0$, and $*\beta = \beta$ for some orientation of $\mathcal{T}$.*
(ii) *There exist linearly independent vectors $c, d \in \mathcal{T}$ with $\beta = cd$ and $\langle c, c\rangle = \langle c, d\rangle = \langle d, d\rangle = 0$.*

Proof. (ii) implies (i) in view of (37.5) and Proposition 37.1(i). Conversely, let us assume (i). By (37.9), $\beta \wedge \beta = 0$, so (see Lemma 3.7) we have $\beta = cd$ for some linearly independent vectors $c, d \in \mathcal{T}$. By (37.19), $\mathrm{Span}\,\{c, d\}$ depends only on β, so that we may choose c, d as above which are orthogonal. By (37.19), one of c, d is null, for instance, d. Now c must be null as well, for otherwise we could rescale both c and d so as to have $\beta = cd$ and $\langle c, c\rangle = \pm 1$, $\langle c, d\rangle = \langle d, d\rangle = 0$ and, by

Proposition 37.1(ii), β and $*\beta$ would be linearly independent. This completes the proof. ∎

Lemma 37.9. *Let $\beta, \gamma \in \mathcal{T}^{\wedge 2}$ be nonzero bivectors in a 4-dimensional real vector space $\mathcal{T}$ with an inner product $\langle\,,\rangle$ of the neutral sign pattern $--++$. The following two conditions are equivalent*:

(a) $\langle \beta, \beta \rangle = \langle \gamma, \gamma \rangle = 0$ *and* $*\beta = \beta$, $*\gamma = -\gamma$ *for some orientation of* $\mathcal{T}$.

(b) *There exist linearly independent vectors* $b, c, e \in \mathcal{T}$ *such that* $\beta = be$, $\gamma = ce$ *and* $\langle b, c \rangle = 1$, $\langle b, b \rangle = \langle b, e \rangle = \langle c, c \rangle = \langle c, e \rangle = \langle e, e \rangle = 0$.

Furthermore, β and γ then determine the vector e in (b) *uniquely up to a sign.*

Proof. Assume (b). Since the sign pattern of $\langle\,,\rangle$ restricted the plane $\operatorname{Span}\{b, c\}$ is $-+$, the same must be the case for its orthogonal complement $P^\perp$. As $e \in P^\perp$ is nonzero and null, we may choose a null vector $d \in P^\perp$ such that $\langle d, e \rangle = 1$. Applying Proposition 37.1(i) (with the rôles of a, b, c, d in Proposition 37.1(i) now played by b, e, c, d), we obtain $*(be) = be$ for the orientation which makes the basis b, e, c, d positive. Similarly, from Proposition 37.1(i) a, b, c, d which now are c, e, b, d, we obtain $*(ce) = -ce$ (for the same orientation). In view of (37.5), this proves (a).

Conversely, assume (a). By Lemma 37.8, both β and γ are decomposable and the planes associated with them as in (37.19) are null (cf. (3.26)). However, choosing vectors $a, b, c, d \in \mathcal{T}$ with $\beta = ab$ and $\gamma = cd$, we have $a \wedge b \wedge c \wedge d = 0$ as $\beta \wedge \gamma = \langle *\beta, \gamma \rangle \cdot \mathrm{vol} = 0$ by (37.9) and (a). Therefore, a, b, c, d are linearly dependent (see §3). The planes $\beta(\mathcal{T})$, $\gamma(\mathcal{T})$ thus have a nontrivial intersection L which must be 1-dimensional, or else β and γ would be linearly dependent, contradicting (37.23) and (a). Consequently, $\beta = be$ and $\gamma = ce$ for some linearly independent vectors $b, c, e \in \mathcal{T}$. Since the planes $\beta(\mathcal{T})$, $\gamma(\mathcal{T})$ are both null, we have $\langle b, c \rangle \neq 0$ (or else b, c, e would span a null subspace, contradicting (3.27)). Another choice of b, c, e with these properties amounts to replacing b, c, e with $\lambda b + \ldots$, $\mu c + \ldots$ and νe, where λ, μ, ν are nonzero scalars and each $\ldots$ stands for some multiple of e. (Note that e is unique up to a factor since it spans the line L.) One now easily sees that λ, μ, ν leading to new b, c, e with all properties listed in (b) are unique up to an overall sign change. This also proves uniqueness of $\pm e$, and hence completes the proof. ∎

§38. Weyl tensors acting on bivectors

An obvious first step towards understanding the local structure of pseudo-Riemannian Einstein manifolds (M, g) in dimension four consists in classifying the algebraic types of the pairs $(g(x), R(x))$ consisting of the metric $g(x)$ and the curvature tensor $R(x)$ at any given point $x \in M$. In view of (5.10), $R(x)$ is completely described by the scalar curvature s (constant by Schur's Theorem 5.1) and the Weyl conformal curvature tensor $W(x)$ at x. The problem thus is reduced to understanding the structure of the analogous pairs $(g(x), W(x))$.

The following discussion of the structure of $W(x)$ is valid for all pseudo-Riemannian 4-manifolds (and not just Einstein spaces); however, it is only for Einstein metrics that W and a constant scalar-curvature function s give a complete description of the curvature. In this section, we will provide an "intrinsic" characterization of these Weyl tensors $W(x)$, treated as operators acting on bivectors. This characterization goes back to Petrov (1950) and Singer and Thorpe (1969). Our next step will be Petrov's classification of such *Weyl-tensor operators*, presented later in §39.

To simplify the discussion, let us replace the tangent space T_xM and the metric $g(x)$ with an arbitrary four-dimensional real vector space $\mathcal{T}$ carrying a fixed *pseudo-Euclidean inner product* (that is, a nondegenerate symmetric bilinear form), denoted $\langle\,,\rangle$, and having one of the sign patterns (37.1). Similarly, instead of $W(x)$ we consider here an arbitrary *algebraic Weyl tensor* in $\mathcal{T}$, that is, a quadrilinear mapping

$$(a,b,c,d)\,\mapsto\, abcd\,\in\,\mathbf{R}\,, \tag{38.1}$$

sending vectors $a,b,c,d\in\mathcal{T}$ to a real number denoted $abcd$, which satisfies conditions analogous to (5.23), (5.24) and (5.25). Those conditions are, explicitly

$$abcd\,=\,-\,bacd\,=\,-\,abdc\,, \tag{38.2}$$

$$abcd\,+\,bcad\,+\,cabd\,=\,0\,, \tag{38.3}$$

and

$$\sum_{e\,\in\,\mathbf{E}}\varepsilon_e\,aebe\,=\,0 \tag{38.4}$$

with $\varepsilon_e=\langle e,e\rangle=\pm 1$, for any unordered orthonormal basis $\mathbf{E}$ of $\mathcal{T}$.

Remark 38.1. Compared to (5.24), one symmetry, namely

$$abcd\,=\,cdab\,, \tag{38.5}$$

seems to be missing from (38.3). The reason is that (38.5) is automatically true for any quadrilinear mapping (38.1) satisfying conditions (38.3) and (38.3). In fact, $abcd=-abdc=dabc+bdac=-dacb-bdca=(acdb+cdab)+(dcba+cbda)=-acbd+(cdab+cdab)-cbad=2cdab-acbd-cbad=2cdab+bacd=2cdab-abcd\,$, as required.

Lemma 38.2. *Let* (38.1) *be any algebraic Weyl tensor in a four-dimensional pseudo-Euclidean vector space* $\mathcal{T}$*. For any orthonormal basis* $\mathbf{E}=\{a,b,c,d\}$ *of* $\mathcal{T}$ *we then have*

$$\varepsilon_c\,acbc\,=\,\varepsilon_d\,dbad\,,\qquad\quad cdcd\,=\,\varepsilon\,abab\,, \tag{38.6}$$

where $\varepsilon=\pm 1$ *denotes the sign factor of* $\langle\,,\rangle$ *introduced in* (37.20)*, and* ε_c*,* ε_d *are defined as in* (37.4) *or* (38.4)*.*

Proof. Combining (38.4) with the (skew)symmetry relations (38.2) and (38.5), we obtain $\varepsilon_c\,acbc-\varepsilon_d\,dbad=\sum_{e\,\in\,\mathbf{E}}\varepsilon_e\,aebe=0$. Similarly, with cancellations due to (38.2) and (38.5), $2\varepsilon_c\varepsilon_d\,cdcd-2\varepsilon_a\varepsilon_b\,abab=\varepsilon_d(\varepsilon_a\,dada+\varepsilon_b\,dbdb+\varepsilon_c\,dcdc)+\varepsilon_c(\varepsilon_a\,caca+\varepsilon_b\,cbcb+\varepsilon_d\,cdcd)-\varepsilon_a(\varepsilon_b\,abab+\varepsilon_c\,acac+\varepsilon_d\,adad)-\varepsilon_b(\varepsilon_a\,baba+\varepsilon_c\,bcbc+\varepsilon_d\,bdbd)$, which equals 0, since so does, by (38.4), each of the parenthesized three-term sums. This completes the proof. ∎

As in §37, we denote $\mathcal{T}^{\wedge 2}$ the bivector space of $\mathcal{T}$ and use the notational convention (37.2) for the exterior product $ab = a \wedge b \in \mathcal{T}^{\wedge 2}$ of vectors $a, b \in \mathcal{T}$. Also, we will use the same symbol $\langle\,,\rangle$ the inner product of bivectors induced by the original inner product $\langle\,,\rangle$ in $\mathcal{T}$ (see (37.3)). By a *Weyl-tensor operator* in the 4-dimensional pseudo-Euclidean vector space $\mathcal{T}$ we will mean any real-linear operator $\mathsf{W} : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ which is self-adjoint relative to the inner product $\langle\,,\rangle$ of $\mathcal{T}^{\wedge 2}$ and satisfies the relations

$$\mathsf{W}* \,=\, *\mathsf{W}\,, \tag{38.7}$$

$$\mathrm{Trace}\ \mathsf{W} \,=\, 0\,, \tag{38.8}$$

$$\mathrm{Trace}\ [*\mathsf{W}] \,=\, 0\,. \tag{38.9}$$

Note that these relations mean that W commutes with the Hodge star, while both W and its composite with the Hodge star are traceless. Recall that the Hodge star operator $* : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$, given by (37.9), is well-defined only when a fixed orientation is chosen in $\mathcal{T}$. However, since $*$ changes sign when the orientation of $\mathcal{T}$ is reversed, the above definition of a Weyl-tensor operator does not depend on the orientation used.

The following lemma establishes a natural correspondence between Weyl-tensor operators and algebraic Weyl tensors in $\mathcal{T}$.

Lemma 38.3. *For any real-linear operator* $\mathsf{W} : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$*, let us consider the quadrilinear mapping assigning to any four vectors* $a, b, c, d \in \mathcal{T}$ *the real number*

$$abcd \,=\, \langle \mathsf{W}(ab), cd\,\rangle \tag{38.10}$$

with $ab = a \wedge b$*. Then the following two conditions are equivalent*:

(a) *The quadrilinear mapping* (38.10) *is an algebraic Weyl tensor*;
(b) W *is a Weyl-tensor operator.*

Proof. Step (i): Since exterior products ab span $\mathcal{T}^{\wedge 2}$, self-adjointness of W is clearly equivalent to the symmetry (38.5) for the quadrilinear mapping (38.10).

Step (ii): Let us now fix an orientation of $\mathcal{T}$ (thus making $*$ well-defined). By (37.21), relation (38.7) means nothing else than $*\mathsf{W}* \,=\, \varepsilon\,\mathsf{W}$, where $\varepsilon = \pm 1$ is the sign factor (37.20). In view of (37.10), this can further be rewritten as $\langle \mathsf{W}(*\alpha), *\beta\rangle = \varepsilon\,\langle \mathsf{W}\alpha, \beta\rangle$ for all $\alpha, \beta \in \mathcal{T}^{\wedge 2}$. The last condition holds for all α, β if and only if it does for those α, β which are exterior products of pairs of different vectors from a fixed positive-oriented orthonormal basis a, b, c, d of $\mathcal{T}$. These two products may in turn have zero, one, or two factors in common. Applying (37.13) in each of these three cases (i.e., with $\alpha = \beta = ab$; or $\alpha = ac$, $\beta = bc$; or $\alpha = ab$, $\beta = cd$), we see (using (37.20)) that (38.7) is equivalent to requiring (38.5) and (38.6) to hold for every positive-oriented orthonormal basis a, b, c, d of $\mathcal{T}$.

Step (iii): Computing the traces of both W and the composite $*\mathsf{W}$ in the orthonormal basis (37.6) corresponding to any fixed positive-oriented orthonormal basis $\mathbf{E}$ of $\mathcal{T}$, we obtain, from (38.10), (37.5) and (37.10),

$$2\,\mathrm{Trace}\ \mathsf{W} \;=\; \sum_{a, b\,\in\,\mathbf{E}} \varepsilon_a \varepsilon_b \langle\, \mathsf{W}(ab), ab\rangle \;=\; \sum_{a\,\in\,\mathbf{E}} \varepsilon_a \sum_{b\,\in\,\mathbf{E}} \varepsilon_b\, abab\,, \tag{38.11}$$

$$
\begin{aligned}
2\,\mathrm{Trace}\,[*\,\mathsf{W}] \;&=\; \sum_{a,b\,\in\,\mathbf{E}} \varepsilon_a\varepsilon_b\langle *[\mathsf{W}(ab)],ab\rangle \\
&=\; \langle\,\mathsf{W}(ab),*(ab)\rangle \;=\; \varepsilon \textstyle\sum_{\mathrm{even}} abcd\,,
\end{aligned} \tag{38.12}
$$

where $\sum_{\mathrm{even}}$ denotes summation over all even permutations (a,b,c,d) of the (ordered) basis $\mathbf{E}$.

Let us now assume (a). According to steps (i) and (ii) above, self-adjointness of W and relation (38.7) then are immediate consequences of (38.5) and Lemma 38.2. Furthermore, (38.8) and (38.9) now are obvious from (38.11) and (38.12) along with (38.4) and, respectively, the fact that

$$
\begin{aligned}
\textstyle\sum_{\mathrm{even}} abcd \;&=\; (abcd+acdb+adbc) \;+\; (badc+bdca+bcad) \\
&\quad+\; (cabd+cbda+cdab) \;+\; (dcba+dbac+dacb)\,,
\end{aligned} \tag{38.13}
$$

with each of the parenthesized three-term sums vanishing in view of (38.3). This yields (b).

Conversely, let us suppose that (b) holds. Thus, for all $a,b,c,d\in\mathcal{T}$, we have (38.5) (in view of step (i) above) and (38.2) (from (38.10) with $ab=a\wedge b$). Using (38.2) and (38.5), we can now rewrite the right-hand side of (38.13) so that each of the product-like terms begins with the factor a, i.e., $\sum_{\mathrm{even}} abcd = (abcd+acdb+adbc)+(abcd+acdb+adbc)+(acdb+adbc+abcd)+(abcd+acdb+adbc) = 3(abcd+acdb+adbc)$. Combined with (38.9), this yields (38.3) whenever a,b,c,d form a positive-oriented orthonormal basis of $\mathcal{T}$. On the other hand, (38.2) and (38.5) clearly imply that (38.3) holds whenever $a,b,c,d\in\mathcal{T}$ are elements of such a fixed basis and two or more of them coincide. Due to quadrilinearity of (38.10) in $a,b,c,d\in\mathcal{T}$, this proves (38.3) for all $a,b,c,d\in\mathcal{T}$. Finally, the left-hand side of (38.4) is a bilinear function ρ of $a,b\in\mathcal{T}$, independent of the orthonormal basis $\mathbf{E}$ used (since the summation involved is a contraction, i.e., amount to taking the trace of a linear operator). Moreover, by (38.5), ρ is symmetric. On the other hand, using (38.8) and step (ii), we obtain (38.6) for every positive-oriented orthonormal basis a,b,c,d of $\mathcal{T}$. The second relation in (38.6) now allows us to rewrite the right-hand side of (38.11), replacing $cdcd=dcdc$, $bdbd=dbdb$ and $bcbc=cbcb$ with $\varepsilon abab$, $\varepsilon acac$ and, respectively, $\varepsilon adad$. Thus, by (38.8), $0=2\,\mathrm{Trace}\,\mathsf{W}=4\varepsilon_a(\varepsilon_b abab+\varepsilon_c acac+\varepsilon_d adad)$. In other words, $\rho(a,a)=0$ for any unit vector $a\in\mathcal{T}$ (since a then may be completed to a basis as above), and so, due to bilinearity, $\rho(a,a)=0$ for every non-null vector a. Thus, $\rho=0$ in view of symmetry of ρ and the fact that non-null vectors form a dense set in $\mathcal{T}$. Consequently, we obtain (38.4), which completes the proof. ■

Remark 38.4. For $\mathcal{T}$ as above, algebraic Weyl tensors in $\mathcal{T}$ are in a natural isomorphic correspondence with Weyl-tensor operators. More precisely, the set of all Weyl-tensor operators in $\mathcal{T}$ obviously forms a vector space, which will be denoted $\mathrm{We}\,(\mathcal{T})$. By assigning to each $\mathsf{W}\in\mathrm{We}\,(\mathcal{T})$ the quadrilinear mapping (38.10) we now obtain a linear isomorphism $\varPhi$ between $\mathrm{We}\,(\mathcal{T})$ and the space $\mathcal{W}$ of all algebraic Weyl tensors in $\mathcal{T}$. To see this, first note that, in view of Lemma 38.3, $\varPhi$ sends $\mathrm{We}\,(\mathcal{T})$ into $\mathcal{W}$. Moreover, $\varPhi$ is injective, as a consequence of (38.10), since $\mathcal{T}^{\wedge 2}$ admits an orthonormal basis of the form (37.6). Finally, to show that $\varPhi$ is surjective, let us fix an algebraic Weyl tensor (38.1). Since $abcd$ then is bilinear and skew-symmetric in a, b and, separately, in c, d, there is a real-valued bilinear form χ on $\mathcal{T}^{\wedge 2}$ with $\chi(\alpha,\beta) = abcd$ whenever $\alpha = ab$ and $\beta = cd$. On the other hand, there must exist a linear operator $\mathsf{W} : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ with $\langle \mathsf{W}\alpha,\beta\rangle = \chi(\alpha,\beta)$ for all $\alpha,\beta \in \mathcal{T}^{\wedge 2}$ (since the assignment of χ to W via this formula is injective, and hence isomorphic). Now $\mathsf{W} \in \mathrm{We}\,(\mathcal{T})$ in view of Lemma 38.3, as required.

According to Remark 38.4, our quest to understand algebraic Weyl tensors in dimension 4 has been reduced to studying Weyl-tensor operators. The structure of the latter can in turn be easily described if we consider separate cases based on the sign pattern of $\langle\,,\rangle$ in $\mathcal{T}$ (assumed to be one of (37.1)).

First, for the sign patterns $++++$ and $--++$, condition (38.7) is equivalent to requiring that W leave the eigenspaces $\mathcal{B}^{\pm}$ of $*$ invariant. Thus, providing an operator W with (38.7) amounts to prescribing its restrictions

$$\mathsf{W}^{\pm} : \mathcal{B}^{\pm} \to \mathcal{B}^{\pm} \tag{38.14}$$

to the subspaces $\mathcal{B}^{\pm}$ (which are direct summands of $\mathcal{T}^{\wedge 2}$, cf. (37.23)). Conditions (38.8) and (38.9) then can be rewritten as

$$\mathrm{Trace}\,\mathsf{W}^{\pm} \,=\, 0 \tag{38.15}$$

for both signs $\pm$. In fact, $\mathrm{Trace}\,\mathsf{W}^{\pm} = \mathrm{Trace}\,[\mathsf{W}\cdot\mathrm{pr}^{\pm}]$, $\mathsf{W}\cdot\mathrm{pr}^{\pm}$ being the composite of W with the projection $\mathrm{pr}^{\pm} : \mathcal{T}^{\wedge 2} \to \mathcal{B}^{\pm}$; on the other hand, by (37.21) with $\varepsilon = 1$, $2\,\mathrm{pr}^{\pm} = * \pm \mathrm{Id}$. This shows that (38.15) is equivalent to (38.8) plus (38.9). Finally, since the subspaces $\mathcal{B}^{\pm}$ are mutually orthogonal (as $*$ is self-adjoint, cf. (37.10)), self-adjointness of W means that $\mathsf{W}^{\pm}$ are both self-adjoint.

On the other hand, for the Lorentzian sign pattern $-+++$, $\mathcal{T}^{\wedge 2}$ is a complex 3-space in which the operator of the multiplication by i is $*$. (See the paragraph preceding formula (37.27) in §37.) Therefore, conditions (38.7) – (38.9) imposed on a real-linear operator $\mathsf{W} : \mathcal{T}^{\wedge 2} \to \mathcal{T}^{\wedge 2}$ mean that W is complex-linear and its *complex* trace is zero. (Note that, according to Lemma 3.3(ii), $\mathrm{Trace}_{\mathbf{R}}\mathsf{W} = 2\,\mathrm{Re}\,[\mathrm{Trace}_{\mathbf{C}}\mathsf{W}]$ and $\mathrm{Trace}_{\mathbf{R}}[*\,\mathsf{W}] = \mathrm{Trace}_{\mathbf{R}}[i\,\mathsf{W}] = -2\,\mathrm{Im}\,[\mathrm{Trace}_{\mathbf{C}}\mathsf{W}]$.) Also, as one easily verifies, self-adjointness of W relative to the (real) inner product $\langle\,,\rangle$ of $\mathcal{T}^{\wedge 2}$ then amounts to requiring that it be self-adjoint relative to the complex-bilinear inner product $(\,,)_{\mathbf{c}}$ given by (37.27).

This discussion can be summarized as follows.

Proposition 38.5. *Let* $\mathrm{We}\,(\mathcal{T})$ *be the space of Weyl-tensor operators in a* 4*-space* $\mathcal{T}$ *with an inner product* $\langle\,,\rangle$ *of one of the sign patterns* (37.1)*. Then*

$$\dim_{\mathbf{R}}\mathrm{We}\,(\mathcal{T}) \,=\, 10 \tag{38.16}$$

and

(i) *For the first two sign patterns* $\pm\pm++$*,* $\mathrm{We}\,(\mathcal{T})$ *is the direct sum of the five-dimensional subspaces* $\mathcal{W}^{+}$ *and* $\mathcal{W}^{-}$ *which consist, respectively, of all traceless operators* $\mathsf{W}^{+} : \mathcal{B}^{+} \to \mathcal{B}^{+}$ *or* $\mathsf{W}^{-} : \mathcal{B}^{-} \to \mathcal{B}^{-}$ *that are self-adjoint relative to the* $(\pm\pm+)$ *inner product obtained by restricting* $\langle\,,\rangle$ *to the respective three-dimensional summand space* $\mathcal{B}^{+}$ *or* $\mathcal{B}^{-}$ *of* (37.23)*.*

(ii) *For the Lorentz sign pattern* $-+++$*,* $\mathrm{We}\,(\mathcal{T})$ *is the five-dimensional complex vector space of all self-adjoint traceless operators* W *in the complex* 3*-space* $\mathcal{T}^{\wedge 2}$ *with the multiplication by* i *provided by the Hodge star, and with the complex-bilinear inner product* $(\,,)_{\mathbf{c}}$ *defined by* (37.27)*.* ■

§39. The Petrov-Segre classes of Weyl-tensor operators

This section deals with an algebraic classification, due to Petrov (1950), of the pairs $(g(x), W(x))$ consisting of the values, at any point x, of the metric and the Weyl conformal tensor W of a pseudo-Riemannian 4-manifold (M, g). In the case where g is an Einstein metric, this (plus a choice of the constant scalar curvature) will also classify the analogous pairs $(g(x), R(x))$ formed by the metric and the curvature tensor R at x.

First, let us simplify the notation, just as we did at the beginning of §38. Specifically, we replace the metric $g(x)$ in the tangent space T_xM with a fixed *pseudo-Euclidean inner product* (that is, a nondegenerate symmetric bilinear form), denoted $\langle\,,\rangle$, in an arbitrary four-dimensional real vector space $\mathcal{T}$. The symbol $\langle\,,\rangle$ will also be used for the inner product induced by $\langle\,,\rangle$ in the bivector space $\mathcal{T}^{\wedge 2}$. Without much loss of generality, we will also assume that the sign pattern of $\langle\,,\rangle$ in $\mathcal{T}$ is one of those listed in (37.1).

At the same time, the Weyl tensor $W(x)$ is replaced with an arbitrary *Weyl-tensor operator* $\mathsf{W} : \mathcal{B} \to \mathcal{B}$, analogous to the one in (38.10), acting in a suitably chosen vector space $\mathcal{B}$ endowed with a bilinear form h. The space $\mathcal{B}$ and the form h are in turn defined as follows. In the case where $\langle\,,\rangle$ has the Lorentzian sign pattern $(-\,+\,+\,+)$, the space $\mathcal{B}$ will be the whole bivector space $\mathcal{T}^{\wedge 2}$, and h will denote the complex-bilinear inner product $(\,,)_{\mathbf{c}}$ with (37.27). (For (M, g) and $x \in M$ as above we thus have, in this case, $\mathcal{B} = [T_xM]^{\wedge 2}$ and $\mathsf{W} = W(x)$.) On the other hand, if the sign pattern of $\langle\,,\rangle$ is Riemannian $(+\,+\,+\,+)$ or neutral $(-\,-\,+\,+)$, the symbol $\mathcal{B}$ will stand for one of the subspaces $\mathcal{B}^{\pm}$ of $\mathcal{T}^{\wedge 2}$ appearing in (37.23), and h will be the inner product $\langle\,,\rangle$ of bivectors, restricted to $\mathcal{B} = \mathcal{B}^{\pm}$. Thus, in the latter case, $\mathsf{W} : \mathcal{B} \to \mathcal{B}$ really stands for the restriction of the Weyl-tensor operator to $\mathcal{B}^{\pm}$ (which was previously denoted $\mathsf{W}^{\pm}$, cf. (38.14)). Applying this to a pseudo-Riemannian 4-manifold (M, g) and $x \in M$, we now obtain $\mathcal{B} = \Lambda_x^{\pm}M$ and $\mathsf{W} = W^{\pm}(x)$.)

As a result, we end up with three objects, denoted $\mathcal{B}$, h, W, and assumed, according to Proposition 38.5, to have the following properties:

(39.1) $\mathcal{B}$ is a real/complex 3-dimensional vector space;

(39.2) h is a nondegenerate real/complex bilinear symmetric form on $\mathcal{B}$ and, if $\mathcal{B}$ is real, h has the sign pattern $+\,+\,+$ or $-\,-\,+$;

(39.3) $\mathsf{W} : \mathcal{B} \to \mathcal{B}$ is an h-self-adjoint traceless real/complex linear operator.

Thus, denoting $\mathbf{K}$ the scalar field ($\mathbf{R}$ or $\mathbf{C}$), we have three possibilities, which account, in this order, for the three sign patterns of $\langle\,,\rangle$ in (37.1):

(39.4)
a) $\mathbf{K} = \mathbf{R}$, h is positive definite;
b) $\mathbf{K} = \mathbf{R}$, h has the sign pattern $-\,-\,+$;
c) $\mathbf{K} = \mathbf{C}$.

Example 39.1. Let $\mathcal{B}$ be the numerical space $\mathbf{K}^3$ (so that h and W in (39.2), (39.3) may both be viewed as 3×3 matrices over the field $\mathbf{K}$), and let $\delta = \pm 1$. Conditions (39.1) – (39.3), with the appropriate choice among the options a), b) and c) in (39.4), are satisfied by

$$\text{I)} \quad h = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathsf{W} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}, \ \lambda, \mu, \nu \in \mathbf{K}, \ \lambda + \mu + \nu = 0\,.$$

$$(39.5) \quad \text{II)} \quad h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathsf{W} = \begin{bmatrix} \lambda & 0 & \varepsilon \\ 0 & -2\lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad \text{with} \quad \lambda \in \mathbf{K}, \ \varepsilon = \pm 1\,.$$

$$\text{III)} \quad h = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \delta & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathsf{W} = \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$(39.6) \qquad h = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{W} = \begin{bmatrix} -2p & 0 & 0 \\ 0 & p & q \\ 0 & -q & p \end{bmatrix}, \quad p, q \in \mathbf{K} = \mathbf{R}, \ q \neq 0\,.$$

Cases I), II), III) of (39.5) with $\mathbf{K} = \mathbf{C}$ and $\delta = 1$ are usually referred to as (the canonical forms of) *Petrov's types* I, II, III. See Remark 40.3 below.

The above examples describe, up to an isomorphism, all possible cases. Namely, we have

Proposition 39.2 (Petrov, 1950)**.** *For every triple* $(\mathcal{B}, h, \mathsf{W})$ *satisfying conditions* (39.1) – (39.3)*, there exists an isomorphic identification* $\mathcal{B} = \mathbf{K}^3$*, i.e., a basis of* $\mathcal{B}$*, which makes* h *and* W *appear as one of the examples* (39.5) *or* (39.6) *with some* $\delta = \pm 1$*. Specifically, in case* (39.4)a) *we have* $\delta = 1$ *and* (39.5)I)*, in case* (39.4)b) $\delta = -1$ *and one of* (39.5)I)*,* (39.5)II)*,* (39.5)III)*,* (39.6)*, while in case* (39.4)c) *we have* $\delta = 1$ *and* (39.5)I)*,* (39.5)II) *with* $\varepsilon = 1$*, or* (39.5)III)*.*

Proof. The assertion for case (39.4)a) is clear. In cases (39.4)b) and (39.4)c), let us first suppose that every eigenvector of W is null. Hence W has exactly one eigenvector α, up to a factor, or else there would exist two independent, orthogonal null eigenvectors, contradicting nondegeneracy of h. For the same reason, there exists β in the h-orthogonal complement $\alpha^\perp$ with $h(\beta,\beta) = \delta$ for some $\delta \in \{1, -1\}$. Also, we can find $\gamma \in \beta^\perp$ such that α, β, γ is a basis of $\mathcal{B}$ for which the matrix of h is as in (39.5)III). (In fact, h is indefinite when restricted to the plane $\beta^\perp$, since $\alpha \in \beta^\perp$; thus, $\beta^\perp$ contains a nonzero null vector other than a multiple of α, which, due to nondegeneracy of h, cannot be orthogonal to α.) From self-adjointness of W it follows that $\alpha^\perp$ is W-invariant, and so $\mathsf{W}\alpha = \lambda\alpha$, $\mathsf{W}\beta = \mu\beta + \xi\alpha$ and $\mathsf{W}\gamma = \nu\gamma + \rho\beta + \sigma\alpha$ for some $\lambda, \mu, \nu, \xi, \rho, \sigma \in \mathbf{K}$. Since λ, μ, ν then must be the roots of the characteristic polynomial of W, they are all equal (as W is assumed here to have just one line of eigenvectors) and so $\lambda = \mu = \nu = 0$, since $\operatorname{Trace} \mathsf{W} = 0$ by (39.3). Again using self-adjointness of W, we now obtain $\rho = \delta h(\mathsf{W}\gamma, \beta) = \delta h(\gamma, \mathsf{W}\beta) = \delta\xi$, and so $\rho \neq 0$, or else W would have two independent eigenvectors. Replacing α, β, γ with $\tilde\alpha = \rho\alpha$, $\tilde\beta = \beta + \rho^{-1}\sigma\alpha/2$, $\tilde\gamma = \rho^{-1}\gamma - \delta\rho^{-2}\sigma\beta/2 - \delta\rho^{-3}\sigma^2\alpha/8$, we obtain a basis of $\mathcal{B}$ satisfying (39.5)III), as required.

Let us now consider the remaining case where W does admit a non-null eigenvector β. If the same is true for the restriction of W to the h-orthogonal complement $\beta^\perp$, a non-null eigenvector of W in $\beta^\perp$, along with its complement in $\beta^\perp$, plus β itself, will provide three orthogonal, non-null, W-invariant lines, which easily leads to case (39.5)I). If, on the other hand, all eigenvectors of W in $\beta^\perp$ are null, either (**A**) there are none of them, or (**B**) we can pick one, say α. Assuming (**A**), we must have (39.4)b) with h restricted to $\beta^\perp$ having the sign pattern $-+$, and so $h(\beta,\beta) < 0$. A basis consisting of normalized β and two (suitably normalized) null vectors in $\beta^\perp$ then yields (39.6). (Again, note that, by (39.3), $\operatorname{Trace} \mathsf{W} = 0$.) Finally, let us assume (**B**). Completing α, β to a basis α, β, γ of $\mathcal{B}$ in which h is represented as in (39.5)II), we have $\mathsf{W}\beta = \mu\beta$, $\mathsf{W}\alpha = \lambda\alpha$ and $\mathsf{W}\gamma = \nu\gamma + \rho\beta + \sigma\alpha$ for some $\lambda, \mu, \nu, \rho, \sigma \in \mathbf{K}$. Since W leaves $\beta^\perp$ invariant and is self-adjoint, we have $\rho = 0$ and $\nu = \lambda$, while relation $0 = \operatorname{Trace} \mathsf{W} = \lambda + \mu + \nu$ then yields $\mu = -2\lambda$. Hence, if $\sigma = 0$, we may replace α, γ with an orthonormal basis of $\beta^\perp$ and thus obtain a special case of (39.5)I). On the other hand, if $\sigma \neq 0$, replacing α, γ with α/c, $c\gamma$ for a scalar $c \neq 0$ causes σ to be replaced by $c^2\sigma$, and so we can make σ equal to 1 (when $\mathbf{K} = \mathbf{C}$) or to ± 1 (when $\mathbf{K} = \mathbf{R}$). This leads to (39.5)II), and completes the proof. ■

The parameters λ, μ, ν, p, q appearing in the matrix form of W in Example 39.1 obviously classify the triples $(\mathcal{B}, h, \mathsf{W})$ with (39.1) – (39.3) up to an isomorphic equivalence (while $\delta = 1$ in cases (39.4)a), (39.4)c), and $\delta = -1$ in case (39.4)b)). In fact, these parameters are in an obvious relation with the roots of the characteristic polynomial of W. Another equivalence relation, much cruder than this isomorphic equivalence, turns out to be quite useful for our subsequent discussion. It consists in dividing all such triples $(\mathcal{B}, h, \mathsf{W})$ into the following seven *Petrov-Segre classes*. To keep track of the classes, we label them by listing the dimensions of the different eigenspaces of W. These dimensions are listed in slanted boldface, in decreasing order, without any separating commas; the appearance of dimension ***0*** on the list indicates that $\mathcal{B}$ is a real space, but the characteristic polynomial of W has a nonreal complex root. We thus have

(39.7)

Class ***1***: Case (39.5)III).
Class ***100***: Case (39.6).
Class ***11***: Case (39.5)II) with $\lambda \neq 0$.
Class ***111***: Case (39.5)I) with $\lambda \neq \mu \neq \nu \neq \lambda$.
Class ***2***: Case (39.5)II) with $\lambda = 0$.
Class ***21***: Case (39.5)I) with $\nu = \lambda \neq 0$ or $\mu = \lambda \neq 0$.
Class ***3***: $\mathsf{W} = 0$ (case (39.5)I) with $\lambda = \mu = \nu = 0$).

(As for the last line, note that if W has a 3-dimensional eigenspace, it must vanish as $\operatorname{Trace} \mathsf{W} = 0$ by (39.3).)

Due to the meaning of the slanted-boldface numbers labeling each Petrov-Segre class, the operator $\mathsf{W} : \mathcal{B} \to \mathcal{B}$ in (39.3) is diagonalizable if and only if these numbers add up to $\dim \mathcal{B} = 3$. We thus have the following division:

(39.8)

W diagonalizable: Classes ***3***, ***21*** and ***111***.
W nondiagonalizable: Classes ***2***, ***11***, ***100*** and ***1***.

Since self-adjoint operators in Euclidean spaces are all diagonalizable, this indicates that not all Petrov-Segre class occur in every one of the cases (39.4)a) – c). In fact, as an immediate consequence of Proposition 39.2, the possibilities are limited as follows:

(39.9)
a) $\mathbf{K} = \mathbf{R}$, h positive definite; classes ***3***, ***21***, ***111*** only.
b) $\mathbf{K} = \mathbf{R}$, h of the sign pattern $-\,-\,+$; all seven classes occur.
c) $\mathbf{K} = \mathbf{C}$; all classes except ***100*** are possible.

In case b) of (39.4) it is convenient to decompose the classes ***2*** and ***21*** into subclasses, characterized as follows:

(39.10)
Subclass $\boldsymbol{2}^+$: Case (39.5)II) with $\lambda = 0$ and $\varepsilon = +1$.
Subclass $\boldsymbol{2}^-$: Case (39.5)II) with $\lambda = 0$ and $\varepsilon = -1$.
Subclass $\boldsymbol{21}^+$: Case (39.5)I) with $\delta = -1$ and $\mu = \lambda \neq 0$, $\nu = -2\lambda$.
Subclass $\boldsymbol{21}^-$: Case (39.5)I) with $\delta = -1$ and $\nu = \lambda \neq 0$, $\mu = -2\lambda$.

The meaning of the signs $\pm$ in (39.10) varies with the class involved: In class ***2***, $\varepsilon = +1$ or $\varepsilon = -1$ depending on whether the symmetric form on $\mathcal{B}$ sending α, β to $h(\mathsf{W}\alpha, \beta)$ is positive or negative semidefinite. (In fact, by (39.5)II) with $\lambda = 0$, the matrix of that form is $\operatorname{diag}(0, 0, \varepsilon)$.) As for class ***21***, the sign $\pm$ accounts for positive or negative semidefiniteness of h restricted to the unique 1-dimensional eigenspace of W.

The reason why we introduce these subclasses *only* in case b) of (39.4) is obvious: In case (39.4)a) neither of the classes ***2*** and ***11*** occurs (see (39.9)a)), while class ***21*** does occur, but only with $\delta = +1$ in (39.5)I). On the other hand, in case (39.4)c) the "subclasses" listed in (39.10) are nothing else than their respective ambient classes ***2***, ***11*** and ***21***. In fact, by Proposition 39.2, we then may always assume that $\varepsilon = +1$ in (39.5)II). On the other hand, subclass $\boldsymbol{21}^+$ then coincides with $\boldsymbol{21}^+$ as one easily verifies by rearranging the order of a basis

$$\alpha\,,\ \beta\,,\ \gamma \tag{39.11}$$

of $\mathcal{B}$ that casts h and W into a canonical form (39.5)I) and multiplying some of the vectors of (39.11) by i.

Remark 39.3. Given a triple $(\mathcal{B}, h, \mathsf{W})$ with (39.1) – (39.3), let us consider a basis (39.11) of $\mathcal{B}$ that makes h and W assume one of the canonical forms listed in (39.5) and (39.6).

(i) If $\mathsf{W} \neq 0$, i.e., $(\mathcal{B}, h, \mathsf{W})$ belongs to one of the six Petrov-Segre classes ***1***, ***100***, ***11***, ***111***, ***2*** and ***21***, then, depending on the class, *either one, or all three of the elements* α, β, γ *of* (39.11) *are unique up to a finite number of choices.*

(ii) More precisely, for the classes ***1***, ***100***, ***11*** and ***111***, such a basis is unique up to changing some signs (provided that, in the case of class ***111***, the order of the eigenvalues λ, μ, ν in (39.5)I) is fixed, thus precluding a rearrangement of (39.11)). See (iv)a) – d) below.

(iii) For classes ***2*** and ***21***, one element of (39.11) is unique up to a sign. In the case of ***2***, that unique-up-to-a-sign element is α, and, according to

(39.5)II) with $\lambda = 0$, it is an eigenvector of W corresponding to the zero eigenvalue. (See the line preceding (iv)a).)

(iv) To verify the above claims, we will use, without further comments, the definitions (39.7) of the Petrov-Segre classes. Let us first note that, in the diagonalizable cases ***111*** and ***21***, assertion (ii) or, respectively, (iii), is obvious from (39.5)I), since an h-unit eigenvector of W corresponding to any specific simple eigenvalue is clearly unique up to a sign. In the remaining four cases ***1***, ***100***, ***11*** and ***2***, let us suppose that α', β', γ' is another basis of $\mathcal{B}$ in which h and W have a canonical form (39.5) or (39.6), a required by Proposition 39.2. Then some scalar c satisfies

$$\alpha' = c\alpha\,, \quad c \neq 0\,. \tag{39.12}$$

In fact, uniqueness of α up to a nonzero factor is clear as α spans a 1-dimensional subspace of $\mathcal{B}$, which is either a suitable eigenspace of W (for classes ***1***, ***100***, ***11***), or the set of all null vectors in $\operatorname{Ker}\mathsf{W}$ (class ***2***). Let us also write

$$\gamma' = r\gamma + \ldots\,, \tag{39.13}$$

for some scalar r, where '$\ldots$' stands for a combination of α and β. We will now proceed to verify that, in each of the four cases ***1***, ***100***, ***11*** and ***2***, c and r appearing in (39.12) and (39.13) also satisfy

$$c, r \in \{1, -1\}\,. \tag{39.14}$$

Note that this, along with (39.12), proves claim (iii) for class ***2***.

(a) In class ***100***, we have $c = \pm 1$ in (39.12), since $h(\alpha,\alpha) = h(\alpha',\alpha') = -1$ (cf. (39.6)). Moreover, β,γ are null vectors forming a basis of $\alpha^\perp$ with $h(\beta,\gamma) = 1$, and similarly for β',γ'; thus, $\mathbf{R}\beta' = \mathbf{R}\beta$ and $\mathbf{R}\gamma' = \mathbf{R}\gamma$ and, consequently, $\gamma' = r\gamma$ and $\beta' = \beta/r$, with $r \neq 0$. (Note that we cannot have $\mathbf{R}\beta' = \mathbf{R}\gamma$ and $\mathbf{R}\gamma' = \mathbf{R}\beta$ instead, i.e., $\gamma' = \rho\beta$ and $\beta' = \gamma/\rho$ for some $\rho \neq 0$, since that would imply $-\rho q\beta = -q\gamma' = (\mathsf{W} - p)\beta' = (\mathsf{W} - p)\gamma/\rho = q\beta/\rho$, which is impossible as $q \neq 0$ by(39.6).) Now, by (39.6), $rq\gamma = r(p\beta - \mathsf{W}\beta) = p\beta' - \mathsf{W}\beta' = q\gamma' = q\gamma/r$, so that $r = \pm 1$. This implies both (39.14) and assertion (ii) for class ***100***.

(b) For classes ***1*** and ***2***, relation (39.14) can be established as follows. By (39.5)III) and (39.5)II), $1 = h(\alpha,\gamma) = h(\alpha',\gamma') = cr$ (as $h(\alpha,\alpha) = h(\alpha,\beta) = 0$). Thus, $r = 1/c$. Also, by (39.5)III) and (39.5)II) with $\lambda = 0$, setting $\mathsf{A} = \mathsf{W}^2$ and $\eta = \delta$ (for class ***1***) or $\mathsf{A} = \mathsf{W}$ and $\eta = \varepsilon$ (for class ***2***), we obtain $\mathsf{A}\gamma = \eta\alpha$ and $\mathsf{A}\beta = \mathsf{A}\alpha = 0$, and so $\eta c\alpha = \eta\alpha' = \mathsf{A}\gamma' = r\mathsf{A}\gamma = \eta r\alpha = \eta\alpha/c$. This yields $c = r = \pm 1$, and hence (39.14).

(c) Continuing the discussion in (b) for class ***1***, we now obtain $\beta' = \eta\beta + t\alpha$ with some scalars η and t; in fact, $h(\alpha',\beta') = 0$, and so, by (39.12), $\beta \in \alpha^\perp = \operatorname{Span}\{\alpha,\beta\}$. Moreover, as $\mathsf{W}\beta = \mathsf{W}\alpha = 0$, we have $\delta\eta\alpha = \eta\,\mathsf{W}\beta = \mathsf{W}\beta' = \delta\alpha' = \delta c\alpha$, and so $\eta = c$. Thus, by (39.12) – (39.14) and (b), $\eta = c = r = \pm 1$ and

$$\alpha' = c\alpha\,, \quad \beta' = c\beta + t\alpha\,, \quad \gamma' = c\gamma + \rho\beta + \sigma\alpha\,, \qquad c = \pm 1 \tag{39.15}$$

for some scalars t, ρ, σ. Also, $0 = h(\beta, \gamma) = h(\beta', \gamma') = c(\delta\rho + t)$, while $0 = \mathsf{W}\gamma' - \beta' = c\beta + \delta\rho\alpha - (c\beta + t\alpha) = (\delta\rho - t)\alpha$, i.e., $\delta\rho + t = \delta\rho - t = 0$, which gives $t = \rho = 0$. Now, (39.15) with $t = \rho = 0$ yields $0 = h(\gamma, \gamma) = h(\gamma', \gamma') = 2c\sigma$, so that $\sigma = 0$. Thus, (39.15) with $t = \rho = \sigma = 0$ shows that the basis (39.11) is unique up to an overall change of sign, and assertion (ii) follows for class ***1***.

(d) In class ***11***, we have $\beta' = \eta\beta$ with $\eta = \pm 1$ since, according to (39.5)II) with $\lambda \neq 0$, β is an h-unit eigenvector of W corresponding to the simple eigenvalue -2λ. Now, again by (39.5)II), α, γ are null vectors forming a basis of $\beta^\perp$ with $h(\alpha, \gamma) = 1$, and the same holds for α', γ'. Hence, as in (a), we have $\mathbf{R}\alpha' = \mathbf{R}\alpha$ and $\mathbf{R}\gamma' = \mathbf{R}\gamma$ and, consequently, $\gamma' = r\gamma$ and $\alpha' = \alpha/r$, with $r \neq 0$. (Again, we cannot have $\mathbf{R}\alpha' = \mathbf{R}\gamma$ and $\mathbf{R}\gamma' = \mathbf{R}\alpha$ instead, since α, α' are eigenvectors of W, while γ, γ' are not.) Therefore, (39.12) and (39.13) give $\gamma' = r\gamma$ and $r = 1/c$. Now $\varepsilon c\alpha = \varepsilon\alpha' = (\mathsf{W} - \lambda)\gamma' = r(\mathsf{W} - \lambda)\gamma = \varepsilon r\alpha = \varepsilon\alpha/c$. This yields $c = r = \pm 1$, and hence (39.14), as well as assertion (ii), for class ***11***.

§40. Classes and genera of Weyl tensors

Let (M, g) be a pseudo-Riemannian 4-manifold, and let $x \in M$. If g has the Lorentzian sign pattern $- + + +$, we will speak of the *Petrov-Segre class of its Weyl tensor* W at x, obtain by applying the appropriate case of the definition (39.7) to the triple $(\mathcal{B}, h, \mathsf{W})$ with $\mathsf{W} = W(x)$, described in the third paragraph of §39. If, on the other hand, the sign pattern of g is Riemannian $(+ + + +)$ or neutral $(- - + +)$, the self-dual and anti-self-dual Weyl tensors $W^\pm(x)$ have their separate Petrov-Segre (sub)classes (again, given by (39.7) and (39.10) for $(\mathcal{B}, h, \mathsf{W})$ with $\mathsf{W} = W^\pm(x)$, as in the paragraph just quoted. The unordered pair formed by these two (sub)classes will be called the *Petrov-Segre genus at* x of the Weyl tensor W of g. The prefix '(sub)' indicates here that in the neutral case we will use the subclasses ***2***$^+$, ***2***$^-$, ***2***$^+$, ***21***$^-$ instead of the full classes ***2*** and ***21***. Our symbol for each genus will consists of its two constituent (sub)classes, separated by a slash, and listed in the reverse of the lexicographic order that is used in (39.7).

In this way, as a consequence of (39.9), there exist three Riemannian Petrov-Segre genera:

(40.1) ***3/3***, ***3/21***, ***21/21***,

six Lorentzian Petrov-Segre classes:

(40.2) ***3***, ***21***, ***2***, ***111***, ***11***, ***1***,

and a huge number (forty-five) of neutral genera, examples of which are

(40.3) ***3/3***, ***3/21***$^+$, ***21***$^-$/***2***$^-$, ..., ***1/1***.

Remark 40.1. The Petrov-Segre genus/class of a pseudo-Riemannian metric g on a 4-manifold M may of course vary with the point $x \in M$. It is, however, independent of x, if W is assumed parallel. (Similarly, in the Riemannian and neutral cases. the Petrov-Segre class of $W^\pm(x)$ is the same for all x, whenever $W^\pm$ is parallel.) Moreover, if W or, respectively, $W^\pm$ is parallel, then every point $x \in M$ has a neighborhood U admitting C^∞ bivector fields α, β, γ which, at each $y \in U$, form a complex basis of $[T_y M]^{\wedge 2}$ (or, respectively, a real basis of $\Lambda^\pm_y M$), in which W (or, $W^\pm$) appears as one the canonical forms (39.5), (39.6). In fact, such a basis chosen at x can be spread "radially" away from x using parallel transports; see Remark 4.6.) More importantly, those among α, β, γ which are, at each point, determined uniquely up to a sign by the conditions just mentioned, necessarily are *parallel* bivector fields (cf. the final clause in Remark 4.6). Thus, using Remark 39.3(ii), (iii), and letting the symbols W and h stand for $W(x)$ and for the complex-bilinear inner product $(\,,)_{\mathbf{c}}$ of bivectors at any fixed point $x \in M$, characterized by (37.27), we conclude that

(a) For the classes ***1***, ***100***, ***11*** and ***111***, α, β, γ are all parallel and h-orthonormal.
(b) In class ***2***, α is parallel and $\mathsf{W}\alpha = 0$, while $(\alpha, \alpha) = 0$ and $\alpha \neq 0$.
(c) In class ***21***, W is diagonalizable at each point with the eigenvalues -2λ, λ, λ for some scalar λ, and rearranging α, β, γ if necessary we may assume that $\mathsf{W}\alpha = -2\lambda\alpha$, while α is h-unit and parallel and, by (5.19), $\lambda = -\mathrm{s}/12$, where s is the scalar curvature.

In particular, the Petrov-Segre genus/class of a *locally symmetric* pseudo-Riemannian 4-manifold must be the same at all points. However, Remark 40.1(a), (b) along with the Weitzenböck formula (5.19) easily show that not all genera/classes are represented. More precisely, we have the following result, which quickly eliminates most possibilities, leaving only those which, as we will see later, are actually realized.

Proposition 40.2. *Suppose that (M, g) is a pseudo-Riemannian 4-manifold such that either*

(a) *g has the Lorentzian sign pattern $- + + +$ and its Weyl tensor W is parallel, or*
(b) *g has the Riemannian sign pattern $+ + + +$ or the neutral sign pattern $- - + +$, while M is oriented, and the self-dual Weyl tensor W^+ of (M, g) is parallel.*

Let $W^{(+)}$ denote W in case (a) *and W^+ in case* (b)*. The Petrov-Segre class of $W^{(+)}$ at each point of M then must be one of*

$$\textbf{\textit{3}}, \quad \textbf{\textit{21}}, \quad \textbf{\textit{2}}. \tag{40.4}$$

Furthermore,

(i) *If $W^{(+)}$ is of class* ***2****, then g is indefinite and its scalar curvature vanishes identically.*
(ii) *If g is neutral, while W^+ is of class* ***21*** *and, in addition, W^- is parallel, then W^- cannot be of class* ***2****.*

Proof. Let us suppose, on the contrary, that $W^{(+)}$ represents one of the remaining four classes ***1***, ***100***, ***11*** and ***111***. According to Remark 40.1(a), there exist parallel local bivector fields α, β, γ, locally trivializing the vector bundle $\mathcal{E} = [TM]^{\wedge 2}$ (case (a)), or $\mathcal{E} = \Lambda^+ M$ (case (b)). From (5.19) it now follows that $W^{(+)}$, as a bundle morphism $\mathcal{E} \to \mathcal{E}$ equals the multiplication by $1/6$ times the scalar-curvature function s. Since $W^{(+)}(x)$ is traceless at each point $x \in M$ (Proposition 38.5), this in turn implies that $\mathrm{s} = 0$ and $W^{(+)} = 0$ identically, so that $W^{(+)}$ is of class ***3*** rather than ***1***, ***100***, ***11*** or ***111***. This contradiction proves our assertion about the classes (40.4).

Finally, if $W^{(+)}$ is of class ***2***, we have, locally, $W\alpha = 0$ for a nonzero parallel bivector field α (Remark 40.1(b)), while (5.19) gives $W\alpha = \mathrm{s}\alpha/6$. Hence $\mathrm{s} = 0$. The rest of assertion (i) is clear since, in the *Riemannian* case, the self-adjoint operators $W^+(x)$, $x \in M$, are all diagonalizable, and hence cannot be of class ***2*** (cf. (39.8)). As for (ii), let us assume, on the contrary, that W^+ is of class ***21*** and W^- is of class ***2***, while $W^\pm$ are both parallel. Applying assertion (i) to the opposite orientation, we obtain $\mathrm{s} = 0$. On the other hand, according to Remark 40.1(c) with $\mathrm{s} = 0$, we have $W^+ = 0$, i.e., W^+ is of class ***3*** rather than ***21***. This contradiction completes the proof. ■

Remark 40.3. The classes listed in (40.2) form a slighly refined version of *Petrov's types* of the Weyl tensors $W(x)$, at points $x \in M$ of 4-dimensional Lorentz manifolds (M, g) (Petrov, 1950). Specifically, Petrov's type I comprises the Petrov-Segre classes ***3***, ***21*** and ***111*** (diagonalizable Weyl tensors $W(x)$; cf. (39.8)), Petrov's type II is formed by classes ***2*** and ***11***, and Petrov's type III consists of the Petrov-Segre class ***1*** alone. Thus, treating the Roman numerals I, II, III as the integers $k = 1, 2, 3$, it is clear from (39.7) and (39.5) that Petrov's type of $\mathsf{W} = W(x)$, treated as an operator acting on bivectors, is the least exponent k such that W^k is diagonalizable.

For more on Petrov's types, see Petrov (1969) and Chapter 3 of Besse (1987).

§41. Locally symmetric pseudo-Riemannian Einstein 4-manifolds

The class of locally symmetric pseudo-Riemannian Einstein 4-manifolds contains the obvious examples provided by spaces of constant curvature, spaces of constant holomorphic sectional curvature, and products of two surface metrics with equal constant Gaussian curvatures. (See §10.) If we assume that the metric in question is, in addition, positive definite, Theorem 14.7 (due to Cartan, 1926) states that these three types are, up to local isometries, the only possible examples.

A similar assertion fails in the case of indefinite metrics, where further, "exotic" examples exist. Such examples can easily be obtained using the construction described in Lemma 41.1 and Corollary 41.2 below, as well as further constructions described in §45 and §46 (see Examples 45.5 and 46.8). However, the "obvious" examples along with the "exotic" ones, mentioned above, together represent all possible local-isometry types of locally symmetric Einstein 4-manifolds. This is the content of classification theorems due to Petrov (1969) and Cahen and Parker (1980), which are stated at the end of this section, and proved later in sections 43 through 46.

Lemma 41.1. *Suppose that we are given any symmetric* 2×2 *real matrix* $\mathfrak{G}$ *with* $\det \mathfrak{G} \neq 0$ *and any* C^∞ *function* f *of two real variables* x^1, x^2*, defined on an open subset of* $\mathbf{R}^2$*. Let* x^j *be a coordinate system in a* 4*-manifold, and let* e_j*,* $j = 1, \dots, 4$*, be the corresponding coordinate vector fields. Using the ranges of indices given by*

$$j, k, l, m \in \{1, 2, 3, 4\}, \qquad a, b, c \in \{1, 2\}, \qquad \lambda, \mu \in \{3, 4\}, \tag{41.1}$$

we now define an indefinite metric g on the coordinate domain by requiring its component functions $g_{jk} = g(e_j, e_k)$ to form the block matrix

$$[g_{jk}] = \begin{bmatrix} \mathfrak{G} & \mathbf{0} \\ \mathbf{0} & \mathfrak{M} \end{bmatrix} \tag{41.2}$$

consisting of the 2×2 matrices

$$[g_{ab}] = \mathfrak{G}, \qquad [g_{\lambda\mu}] = \mathfrak{M} = \begin{bmatrix} 0 & 1 \\ 1 & -f \end{bmatrix} \tag{41.3}$$

with $f = f(x^1, x^2)$ treated as a function of (x^1, x^2, x^3, x^4). In other words, g is any metric such that

$$\begin{aligned} &g_{11}\,,\ g_{22} \quad \text{and} \quad g_{12} = g_{21} \quad \text{are constant,} \\ &g_{33} = 0\,, \quad g_{34} = g_{43} = 1\,, \\ &g_{a\lambda} = g_{\lambda a} = 0 \quad \text{for} \quad a \in \{1,2\} \quad \text{and} \quad \lambda \in \{3,4\}\,, \\ &\partial_3 g_{44} = \partial_4 g_{44} = 0\,. \end{aligned} \tag{41.4}$$

Then

(i) *The bivectors $e_a \wedge e_3$, $a = 1, 2$, are null and orthogonal, at every point, relative to the inner product $\langle\,,\rangle$ of bivectors induced by g.*

(ii) *The vector field e_3, the 2-forms $\beta^a = dx^a \wedge dx^4$, and the bivector fields $e_a \wedge e_3$, $a = 1, 2$, are all parallel relative to the Levi-Civita connection ∇ of g, and*

$$e_a \wedge e_3 = g_{ac}\beta^c\,, \quad \beta^a = g^{ac}\, e_c \wedge e_3\,, \quad (\text{summed over} \quad c = 1,2)\,, \tag{41.5}$$

where $[g^{ab}] = \mathfrak{G}^{-1}$, while bivectors and 2-forms being identified with the aid of the metric g.

(iii) *The curvature tensor R and Ricci tensor* Ric *of g are given by*

$$R = \frac{1}{2}\,(\partial_a\partial_c f)\,\beta^a \otimes \beta^c \qquad (\text{summed over} \quad a, c = 1, 2)\,, \tag{41.6}$$

$$\mathrm{Ric} = \frac{1}{2}\,\varPhi\, dx^4 \otimes dx^4\,, \quad \text{with} \quad \varPhi = g^{ac}\,\partial_a\partial_c f\,. \tag{41.7}$$

Proof. By (2.21), $\langle e_a \wedge e_3, e_b \wedge e_3\rangle = g_{ab}g_{33} - g_{a3}g_{b3}$, while (41.4) gives $g_{13} = g_{23} = g_{33} = 0$. This yields (i). To prove (ii), let us first note that

$$\begin{aligned} &\varGamma_{a44} = \varGamma_{4a4} = -\frac{1}{2}\,\partial_a f\,, \quad \varGamma_{44a} = \frac{1}{2}\,\partial_a f \quad \text{for} \quad a = 1, 2\,, \quad \text{and} \\ &\varGamma_{jkl} = 0 \quad \text{otherwise.} \end{aligned} \tag{41.8}$$

in view of (4.9) and the fact that, by (41.2) – (41.4), $\partial_j g_{kl} = 0$ unless $j \in \{1, 2\}$ and $k = l = 4$. From (4.7) and (41.4), it is now obvious that e_3 is ∇-parallel, while the ∇-covariant derivatives of e_1 and e_2 in any direction, at any point, are

g-orthogonal to e_1, e_2 and e_3, and hence (by (41.4)) must be multiples of e_3. This immediately implies that the bivector fields $e_1 \wedge e_3$ and $e_2 \wedge e_3$ are parallel. The rest of assertion (ii) now is clear since

$$[g^{\lambda\mu}] = \mathfrak{M}^{-1} = \begin{bmatrix} f & 1 \\ 1 & 0 \end{bmatrix}, \qquad (41.9)$$

and so the vector fields corresponding via g to the 1-forms dx^4 and dx^a, $a = 1, 2$, are the g-gradients $\nabla x^4 = e_3$ and $\nabla x^a = g^{ab} e_b$ (while the g^{ab} are constant by (41.4)).

On the other hand, formula (4.31) becomes, in our case, $R_{jklm} = \partial_k \Gamma_{jlm} - \partial_j \Gamma_{klm}$, as (by (41.8)) we have $g^{pq}\Gamma_{jkp}\Gamma_{lmq} = 0$ unless $j = k = l = m = 4$. Similarly, for $\partial_j \Gamma_{klm}$ to be nonzero, we must have $j \in \{1, 2\}$ and one of k, l, m must be 1 or 2, while the other two must equal 4. Consequently, $R_{jklm} = 0$ unless either $\{j, k\} = \{l, m\} = \{1, 4\}$, or $\{j, k\} = \{l, m\} = \{2, 4\}$. It now follows from (41.8) that all components R_{jklm} of the curvature tensor R of g are zero except, possibly, those related via the algebraic symmetries (4.32) to

$$R_{a4c4} = \frac{1}{2}\,\partial_a \partial_c f \quad \text{for} \quad a, c = 1, 2\,. \qquad (41.10)$$

The only nonzero components of $\beta = \beta^a$, $a = 1, 2$, obviously are $\beta_{a4} = 1$, $\beta_{4a} = -1$. Furthermore, for any fixed pair (a, c) with $a, c \in \{1, 2\}$, the four-times covariant tensor Z^{ac} given by

$$2\, Z^{ac} = \beta^a \otimes \beta^c + \beta^c \otimes \beta^a \qquad (41.11)$$

shares the symmetries (4.32) of R, and the only nonzero components of Z^{ac}, equal to 1 (or -1), occur for the indices $a4c4$ and $4a4c$ (or, respectively, $4ac4$ and $a44c$). This implies $2R = (\partial_a \partial_c f)\, Z^{ac}$, i.e., (41.6). Relation (41.7) is in turn immediate from (41.6), as $R_{jl} = R_{jklm} g^{km}$ (see (4.37)) while, by (41.2) and (41.9), $g^{44} = 0$ and $g^{a\lambda} = g^{\lambda a} = 0$ for $a \in \{1, 2\}$ and $\lambda \in \{3, 4\}$. Hence $R_{44} = \Phi$ and $R_{jl} = 0$ otherwise. This completes the proof. ■

The construction described in Lemma 41.1 leads to many examples of Ricci-flat metrics which will be useful later (see Remark 41.3 below, Example 43.1 in §43, Example 44.1 in §44, and Corollary 49.2 in §49). Namely, we have

Corollary 41.2. *Let g be an indefinite metric on an open subset of $\mathbf{R}^4$, defined as in* Lemma 41.1*, for some fixed function $f = f(x^1, x^2)$ and a symmetric nonsingular 2×2 matrix $\mathfrak{G} = [g_{ac}]$.*

(a) *g is locally symmetric if and only if f is a quadratic polynomial in x^1, x^2.*
(b) *g is Einstein or, equivalently, Ricci-flat, if and only if f is* $\mathfrak{G}$-harmonic*, that is, $g^{ac}\,\partial_a \partial_c f = 0$.*

In fact, (a) is clear from (41.6) combined with the relation $\nabla \beta^a = 0$ (Lemma 41.1(ii)) and the fact that the tensors Z^{11}, Z^{12}, Z^{22} given by (41.11) are linearly independent at each point, while (b) is obvious from (41.7). ■

Remark 41.3. According to Corollary 41.2(a), all metrics g on $\mathbf{R}^4$ obtained as in Lemma 41.1 with a function f that is a *quadratic homogeneous polynomial* in

x^1, x^2 are locally symmetric. Another interesting feature of those metrics is that each of them admits a one-parameter group is *nonisometric homotheties.* In fact, for any real $r > 0$, the diffeomorphism $F_r : \mathbf{R}^4 \to \mathbf{R}^4$ with $F_r(x^1, x^2, x^3, x^4) = (rx^1, rx^2, r^2x^3, x^4)$ then satisfies $F_r^* g = r^2 g$ (notation as in (2.29)). Consequently, multiplying g by a positive constant produces a metric which is still isometric to g.

We conclude this section with statements of three theorems which together provide a complete local classification locally symmetric pseudo-Riemannian Einstein manifolds (M, g) of dimension four. Since the Weyl tensor W of g then is parallel, its Petrov-Segre genus/class must be of the same class at all points of M. The possible types of such metrics g are listed below by their sign patterns and Petrov-Segre genera or classes.

Note that the first of our three theorems is nothing else than Theorem 14.7, rephrased so as to account for the possible Petrov-Segre genera:

Theorem 41.4 (Cartan, 1926)**.** *Let W denote the Weyl tensor of a locally symmetric Riemannian Einstein* 4*-manifold* (M, g)*. Then, one and only one of the following three cases occurs*:

(a) *W is of the Petrov-Segre genus **3/3** and (M, g) is a space of constant curvature, locally isometric to S^4, H^4 or $\mathbf{R}^4$ endowed with a constant multiple of its standard metric*;
(b) *W belongs to the genus **3/21** and (M, g) is a nonflat space of constant holomorphic sectional curvature, locally isometric to $\mathbf{C}\mathrm{P}^2$ or $(\mathbf{C}\mathrm{P}^2)^*$ with a constant multiple of its standard metric*;
(c) *W represents the genus **21/21** and g is, locally, a product of two surface metrics with equal nonzero constant Gaussian curvatures.*

Proof. In view of Theorem 14.7, all we need to verify is that the Petrov-Segre genera have been assigned correctly to cases (a) – (c). This in turn is obvious from (10.18), (10.20) and, respectively, (10.22), along with the definition of the genera (40.1) (see §40 and (39.7), (39.5)). Note that the scalar curvature s appearing in (10.20) and (10.22) must be nonzero in cases (b), (c), for otherwise we would have $W = 0$ and, consequently, g would be flat (as $R = W = 0$ by (5.10)). This would contradict the hypothesis in (b), and, in view of (16.28), also the assumptions about the factor Gaussian curvatures in (c). This completes the proof. ∎

Theorem 41.5 (Petrov, 1969)**.** *Let (M, g) be a locally symmetric four-dimensional Lorentzian Einstein manifold. Denoting W the Weyl tensor of (M, g), we then have one and only one of the following three cases*:

(a) *W is of the Petrov-Segre class **3** and (M, g) is a space of constant curvature, locally isometric to one of the manifolds listed in* Examples 10.3 *and* 10.4;
(b) *W is of class **21** and (M, g) is, locally, a Riemannian product of two pseudo-Riemannian surfaces having equal nonzero constant Gaussian curvatures*;
(c) *W is of class **2** and g is locally isometric to the Petrov metric described in* Example 43.1 *of* §43*.*

For a proof, see end of §43. ∎

Theorem 41.6 (Cahen and Parker, 1980)**.** *Let (M,g) be a locally symmetric Einstein 4-manifold with a metric g of the neutral sign pattern $--++$, and let W be the Weyl tensor of (M,g). Then, one and only one of the following cases occurs*:

(i) *W is of the Petrov-Segre genus **3/3** and (M,g) is a space of constant curvature, locally isometric to one of the manifolds described in* Examples 10.3 *and* 10.4;

(ii) *W has the genus $\boldsymbol{3}/\boldsymbol{21}^+$ and (M,g) is a nonflat space of constant holomorphic sectional curvature, locally isometric to one of the manifolds in* Example 10.6;

(iii) *W is of genus $\boldsymbol{3}/\boldsymbol{21}^-$ and (M,g) is locally isometric to a pseudo-complex projective space, as defined in* Example 46.8 *of* §46;

(iv) *W represents the genus $\boldsymbol{21}^+/\boldsymbol{21}^+$ and g is, locally, a product of two pseudo-Riemannian surface metrics with the sign patterns $++$ and $--$, having equal nonzero constant Gaussian curvatures*;

(v) *W is of genus $\boldsymbol{21}^-/\boldsymbol{21}^-$ and g is, locally, a product of two pseudo-Riemannian surface metrics with equal nonzero constant Gaussian curvatures, which both have the sign pattern $-+$*;

(vi) *W is of genus $\boldsymbol{21}^+/\boldsymbol{21}^-$ and g is, locally, the result of complexifying a positive-definite surface metric with a nonzero constant Gaussian curvature, as described in* Example 45.5 *of* §45;

(vii) *W belongs to one of the five Petrov-Segre genera*

$$\boldsymbol{3}/\boldsymbol{2}^+ , \quad \boldsymbol{3}/\boldsymbol{2}^- , \quad \boldsymbol{2}^+/\boldsymbol{2}^+ , \quad \boldsymbol{2}^+/\boldsymbol{2}^- , \quad \boldsymbol{2}^-/\boldsymbol{2}^- , \tag{41.12}$$

and g is locally isometric to one of the five metrics representing these genera, and described in Example 44.1 *of* §44.

For a proof, see end of §46. ■

§42. Some nondiagonalizable Weyl tensors

This section deals with those classes/genera of Weyl tensors which are not diagonalizable, but can be realized by locally symmetric pseudo-Riemannian metrics in dimension four. (See (39.8) and Proposition 40.2.) Specifically, Lemmas 42.1 and 42.3 below characterizes such Weyl tensors by expressing them in terms of tensor products of suitable bivectors. A similar characterization of the (diagonalizable) genus $\boldsymbol{21}^+/\boldsymbol{21}^-$ is provided by Lemma 42.4.

We also observe (Remark 42.6 below) that the particular structure of the curvature tensor of locally symmetric metrics with nondiagonalizable Weyl tensors allows a simple existence proof for local Killing fields.

Lemma 42.1. *Let g be a metric of the Lorentzian sign pattern $-+++$ on a 4-manifold M, and let $W=W(x)$ denote its Weyl tensor at a point $x\in M$. Then, the following two conditions are equivalent*:

(i) *W is of the Petrov-Segre class **2***;

(ii) *There exist nonzero bivectors β, γ at x with*

$$W = \beta\otimes\beta - \gamma\otimes\gamma , \tag{42.1}$$

$$(42.2)\qquad \langle\beta,\beta\rangle \,=\, \langle\beta,\gamma\rangle \,=\, \langle\gamma,\gamma\rangle \,=\, 0\,,$$

and

$$(42.3)\qquad \gamma \,=\, *\beta\,, \quad \text{for a suitable orientation of} \quad T_xM\,.$$

Furthermore, if W *is parallel and satisfies* (i) *at every point, some neighborhood* U *of any given point of* M *admits parallel bivector fields* β *and* γ *satisfying* (42.1) – (42.3) *everywhere in* U*.*

Proof. Let us assume (i). According to (39.7) and Proposition 39.2, there exists a complex basis $\alpha_1, \alpha_2, \alpha_3$ of $[T_xM]^{\wedge 2}$ in which $h = (\,,)_{\mathbf{c}}$ and $\mathsf{W} = W(x)$ have the canonical form (39.5)II) with $\lambda = 0$, $\delta = 1$ and $\varepsilon = 1$. Setting $\beta = \alpha_1$ and $\gamma = i\alpha_1$ we then clearly have (42.3) (as $\gamma = *\alpha_1 = *\beta$). Also, (42.2) is obvious since $(\beta,\beta)_{\mathbf{c}} = 1$ (by (39.5)II)), while $(\,,)_{\mathbf{c}}$ is complex-bilinear and $\langle\,,\rangle = \mathrm{Re}\,(\,,)_{\mathbf{c}}$. Finally, we have (42.1) since both sides yield the same values when applied to the real basis $\alpha_1, i\alpha_1, \alpha_2, i\alpha_2, \alpha_3, i\alpha_3$ of $[T_xM]^{\wedge 2}$, as one sees using (39.5)II), (5.14) and the relation $\langle\,,\rangle = \mathrm{Re}\,(\,,)_{\mathbf{c}}$. Thus, (i) implies (ii). Conversely, assuming (42.1) – (42.3), we can obtain a complex basis $\alpha_1, \alpha_2, \alpha_3$ of $[T_xM]^{\wedge 2}$ leading to the canonical form (39.5)II) (with details as above) by setting $\alpha_1 = \beta$ and choosing any $\alpha_2, \alpha_3 \in [T_xM]^{\wedge 2}$ with $h(\beta,\alpha_2) = h(\alpha_2,\alpha_3) = h(\alpha_3,\alpha_3) = 0$ and $h(\beta,\alpha_3) = h(\alpha_2,\alpha_2) = 1$, where $h = (\,,)_{\mathbf{c}}$. (Such α_2, α_3 exist by Lemma 3.14 with $r = 1$.) Hence (i) follows from (ii). Finally, if W is parallel, β (and consequently γ) can be chosen, locally, to form parallel bivector fields, according to Remark 40.1(b). This completes the proof. ■

Remark 42.2. For a four-dimensional Lorentzian manifold (M, g), the requirement that the Weyl tensor $W = W(x)$ at a given point $x \in M$ be of the Petrov-Segre class ***2*** amounts to the condition

$$(42.4)\qquad \dim_{\mathbf{C}} \mathrm{Ker}\, W \,=\, 2\,, \qquad W \circ W \,=\, 0\,.$$

where $W = W(x)$ is treated as a complex-linear operator $[T_xM]^{\wedge 2} \to [T_xM]^{\wedge 2}$. This is clear from (39.7) and (39.5), along with Proposition 39.2.)

Lemma 42.3. *Suppose that* g *is a pseudo-Riemannian metric of the neutral sign pattern* $- - + +$ *on a* 4*-manifold* M*, while* $W = W(x)$ *denotes the Weyl tensor of* g *at a point* $x \in M$*, and* $\pm$ *is one of the signs* $+$ *and* $-$*. Then*

(i) W *is of the Petrov-Segre genus* ***3/2***$^{\pm}$ *if and only if there exists a nonzero bivector* β *at* x *such that*

$$(42.5)\qquad W \,=\, \pm\,\beta\otimes\beta\,, \qquad \langle\beta,\beta\rangle \,=0\,,$$

and, for a suitably chosen orientation in T_xM*,*

$$(42.6)\qquad *\beta \,=\, \beta\,.$$

(ii) W *belongs to the Petrov-Segre genus* ***2***$^{\pm}$/***2***$^{\pm}$ *if and only if some nonzero bivectors* β*,* γ *at* x *satisfy the conditions*

$$(42.7)\qquad W \,=\, \pm\,(\beta\otimes\beta \,+\, \gamma\otimes\gamma)\,,$$

along with (42.2) *and*

$$(42.8)\qquad *\beta \,=\, \beta\,, \qquad *\gamma \,=\, -\,\gamma\,, \quad \text{for a suitable orientation of} \quad T_xM\,.$$

(iii) W *has the Petrov-Segre genus* ***2***$^{+}$/***2***$^{-}$*, if and only if, for some nonzero bivectors* β*,* γ *at* x*, we have* (42.1) *along with* (42.2) *and* (42.8).

Furthermore, if W *is parallel and has, at every point, the property described in* (i)*, or* (ii)*, or* (iii)*, then some neighborhood* U *of any given point of* M *admits a parallel bivector field* β *or, respectively, parallel bivector fields* β *and* γ *satisfying* (42.5) *and* (42.6)*, or* (42.7)*,* (42.2) *and* (42.8) *or, respectively,* (42.1)*,* (42.2) *and* (42.8) *everywhere in* U*.*

Proof. For any fixed orientation of T_xM, $\mathsf{W} = W^+(x)$ belongs to the subclass $\underline{\boldsymbol{2}}^\pm$ if and only if some basis $\alpha_1, \alpha_2, \alpha_3$ of Λ_x^+M brings $\mathsf{W} = W(x)$ and $h = \langle\,,\rangle$ (the inner product in Λ_x^+M) into the canonical form (39.5)II) with $\lambda = 0$, $\delta = -1$ and $\varepsilon = \pm 1$. (See (39.10) and Proposition 39.2.) Setting $\beta = \alpha_1$ we then have $\langle\beta, \beta\rangle = 0$ (by (39.5)II)). Furthermore, $W^+(x) = \pm\beta\otimes\beta$; in fact, by (39.5)II) and (5.14), both sides yield the same values when applied to α_1, α_2 and α_3.

Conversely, if $W^+(x) = \pm\beta\otimes\beta$ for some $\beta \in \Lambda_x^+M$ with $\beta \neq 0$ and $\langle\beta, \beta\rangle = 0$, then $W^+(x)$ represents the subclass $\underline{\boldsymbol{2}}^\pm$. To see this, note that a basis $\alpha_1, \alpha_2, \alpha_3$ of Λ_x^+M that makes $\mathsf{W} = W^+(x)$ and $h = \langle\,,\rangle$ appear as the canonical form (39.5)II) (with $\lambda = 0$, $\delta = -1$ and $\varepsilon = \pm 1$) may be obtained by applying Lemma 3.14 to $V = \Lambda_x^+M$ with $n = 3$, $r = 1$ and $u_1 = \beta = \alpha_1$, which produces $\alpha_2 = w_1$ and $\alpha_3 = v_1$ (with v_1, w_1 chosen as in Lemma 3.14). Then, by (5.14), we obtain the representation (39.5)II) for $\mathsf{W} = W^+(x)$. (Note that $\langle\alpha_2, \alpha_2\rangle$ equals 1 rather than -1, since $h = \langle\,,\rangle$ has in Λ_x^+M the sign pattern $- - +$ (see the comment following (37.25)).

Assertions (i) – (iii) now follows from the above applied to W^+ and W^- separately.

Finally, if W is parallel, β (or, respectively, β and γ) can be chosen, locally, to form parallel bivector fields, according to Remark 40.1(b) applied to $\mathsf{W} = W^+(x)$ (or, respectively, to both $\mathsf{W} = W^+(x)$ and $\mathsf{W} = W^-(x)$). This completes the proof. ∎

Lemma 42.4. *Let* (M, g) *be an orientable pseudo-Riemannian Einstein* 4*-manifold of the neutral sign pattern* $- - + +$*. Then, the following two conditions are equivalent*:

- (a) (M, g) *is locally symmetric and its Weyl tensor* W *is of the Petrov-Segre genus* $\boldsymbol{21}^+/\boldsymbol{21}^-$;
- (b) *The scalar curvature* s *of* g *is nonzero and its curvature tensor* R *is, locally, given by*

$$R \;=\; \frac{\mathrm{s}}{8}\,(\beta\otimes\beta \,-\, \gamma\otimes\gamma) \tag{42.9}$$

for some parallel bivector fields β*,* γ *satisfying the conditions* (42.8) *and*

$$\langle\beta, \beta\rangle \;=\; 2\,, \qquad \langle\gamma, \gamma\rangle \;=\; -\,2\,. \tag{42.10}$$

Proof. (b) implies (a) in view of Schur's Theorem 5.1 along with (5.33) (for $n = 4$) and (39.10); in view of (5.14), we obtain the canonical forms (39.5)I) for both $W^+(x)$ and $W^-(x)$, at any point x, by using orthonormal bases of Λ_x^-M and Λ_x^-M that include the elements $\beta(x)$ and $\gamma(x)$.

Conversely, assume (a). From (39.10) and Remark 40.1(c) applied to both $\mathsf{W} = W^+(x)$ and $\mathsf{W} = W^-(x)$ we obtain, locally, the existence of parallel bivector fields β, γ with (42.8), (42.10) and $W\beta = \mathrm{s}\beta/6$, $W\gamma = \mathrm{s}\gamma/6$. Now (42.9) follows from

(5.33) since, by (5.14) and Remark 40.1(c), both sides agree as operators acting on bivectors (when tested on β, γ, and bivectors orthogonal to both β and γ). This completes the proof. ■

Remark 42.5. The same argument leads to completely analogous characterizations of the genera ***21/21***, ***21***$^+$/***21***$^+$, ***21***$^-$/***21***$^-$, ***21***$^+$/***21***$^-$. (The only difference is that the plus/minus signs appearing in (42.9) and (42.10) need modification.) The modified formulae then are really nothing else than (10.13) and (10.14) (with the difference between the factors $\mathrm{s}/8$ and $\mathrm{s}/4$ explained by different normalizations of β and γ in (42.10) compared to (10.14)). This in turn may be used to prove the product-of-surfaces cases in Theorem 41.5 and 41.6.

Remark 42.6. Suppose that we are given a 4-dimensional pseudo-Riemannian manifold (M, g) and a point $x \in M$, and $W = W(x)$ is the Weyl tensor of (M, g) at x. We now assume that either

(i) g is Lorentzian, that is, has the sign pattern $- - + +$, and its Weyl tensor W is of the Petrov-Segre class ***2***, or
(ii) g has the neutral sign pattern $- - + +$, and W represents one of the genera (41.12); specifically, these are
  a) Genus ***2***$^+$/***2***$^+$,
  b) Genus ***2***$^-$/***2***$^-$,
  c) Genus ***2***$^+$/***2***$^-$,
  d) Genus ***3/2***$^+$,
  e) Genus ***3/2***$^-$.

If, in addition, (M, g) is locally symmetric (and so one of conditions (i), (ii)a) – e) holds at *every* point $x \in M$), we denote r the rank of the Weyl tensor of (M, g) acting on bivectors at any point of M, i.e., the fibre dimension of the subbundle $W([TM]^{\wedge 2})$ of $[TM]^{\wedge 2}$. Then, by (39.7) and (39.5), $r = 1$ in cases (i) and (ii)a), b, c), and $r = 2$ in cases (ii)d), e).

Furthermore, in all cases, $\dim[\mathfrak{isom}(M, g)] \le 10 - 2r$ and every point of M has a connected neighborhood U such that $\dim[\mathfrak{isom}(U, g)] = 10 - 2r$. (Notation as in Remark 17.6(i).) Finally, (M, g) is infinitesimally homogeneous and, consequently, locally homogeneous.

This is an easy consequence of Proposition 17.26, with the dimension of the centralizer $\mathcal{C}_x$ easily verified to be $6 - 2r$ as a consequence of Lemmas 42.1, 42.3 along with Lemmas 37.7 – 37.9 and formula (2.28).

Remark 42.7. Every locally symmetric pseudo-Riemannian Einstein 4-manifold is infinitesimally homogeneous, and hence (by Lemma 17.20), also locally homogeneous. In fact, in the case of nondiagonalizable Weyl tensors, this was established in Remark 42.7, while in the remaining cases (listed in Theorems 41.4 – 41.6), Killing fields are easily constructed in the corresponding geoemtric models; cf. Example 17.19.

§43. Petrov's example

As already stated at the beginning of §41, the class of locally symmetric pseudo-Riemannian Einstein metrics in dimension four contains, beside the "obvious" examples, also some *exotic* ones. In the case of metrics with the Lorentzian sign pattern $- + + +$, such exotic metrics form just one local-isometry type, described in Example 43.1 below; its uniqueness is established in the proof of Theorem 41.5, given at the end of this section.

The construction of the example and the classification theorem just mentioned are both due to Petrov. (See Petrov, 1969, especially formula (25.23) on p. 154, p. 352, and formula (24.1) on p. 142.)

Cahen and Wallach (1970) proved a much more general result, which amounts to a local classification of locally symmetric Lorentzian metrics in all dimensions. Compared with that, our discussion is not only limited to the four-dimensional case, but also further restricted just to those locally symmetric Lorentz 4-manifolds which are also *Einstein*. (The latter restriction is significant since locally symmetric Lorentz metrics, unlike *Riemannian* ones, need not be locally isometric to products of Einstein metrics.)

Example 43.1. The *locally symmetric Lorentzian Petrov metric* on $\mathbf{R}^4$, here denoted g, is defined as follows. Let x^j and e_j, $j = 1, \ldots, 4$, denote the Cartesian coordinates in $\mathbf{R}^4$, and, respectively, the vectors of the standard basis of $\mathbf{R}^4$. We declare g to be a special case of the metric given by (41.2) – (41.4), namely

$$\begin{aligned} &g_{11} = g_{22} = g_{34} = g_{43} = 1\,, \\ &g_{44} = (x^1)^2 - (x^2)^2\,, \quad \text{and} \quad g_{jk} = 0 \quad \text{otherwise,} \end{aligned} \tag{43.1}$$

that is,

$$[g_{jk}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \qquad f = (x^2)^2 - (x^1)^2\,, \tag{43.2}$$

where $g_{jk} = g(e_j, e_k)$ and the e_j are treated as constant vector fields. In view of Corollary 41.2, g is a locally symmetric Lorentzian Ricci-flat metric. Furthermore, the Weyl tensor W of g represents, at every point, the Petrov-Segre class ***2***. (This is a consequence of Lemma 42.1; for details, see Proposition 43.2 below.)

Proposition 43.2. *The Weyl tensor W of the Petrov metric g described in* Example 43.1 *above is of the Petrov-Segre class **2** at every point.*

Proof. Since g is Ricci-flat (Corollary 41.2(ii)), we have $W = R$ (see (5.10)) and so (41.6) with $f = [(x^2)^2 - (x^1)^2]/2$ gives (42.1) with $\beta = \beta^2$, $\gamma = \beta^1$ (where β^a are defined as in Lemma 41.1(ii)). On the other hand, since $[g_{ac}]$ in (41.3) is, in this case, the identity matrix, formula (41.5) gives $\beta = e_2 \wedge e_3$, $\gamma = e_1 \wedge e_3$. Using Lemma 41.1(i), we now obtain (42.2). On the other hand, in view of (43.2), Proposition 37.1(ii) can be applied to the quadruple $(a, b, c, d) = (e_1, e_3, e_2, e_4)$, proving that β and γ satisfy (42.3) for the orientation that makes e_1, e_2, e_3, e_4 *negative-oriented* at each point. In view of Lemma 42.1, this completes the proof. ■

The following result provides a local classification of locally symmetric Lorentzian Einstein metrics in dimension four with class ***2*** Weyl tensors. Note that our assertion says 'locally isometric' without adding *'up to a factor'*. The latter phrase would, in fact, be redundant here since, according to Remark 41.3, the positive-constant multiples of the Petrov metric are all isometric to it.

Theorem 43.3 (Petrov, 1969). *Suppose that g is a locally symmetric Lorentzian Einstein metric on a 4-manifold M whose Weyl tensor W is of the Petrov-Segre class **2**. Then (M,g) is locally isometric to the Petrov metric on $\mathbf{R}^4$, described in* Example 43.1.

Proof. By Proposition 40.2(i), (M,g) is Ricci-flat, and hence $W=R$ (see (5.10)). Let us now fix a point $x\in M$. According to the final clause in Lemma 42.1, we can find parallel bivector fields $\beta,\ \gamma$ defined on some oriented connected neighborhood U of x and such that conditions (42.1) – (42.3) hold everywhere in U. Writing $\beta^+=\beta,\ \beta^-=\gamma$ and $R=W$, we thus have

$$R\,=\,\beta^+\otimes\beta^+\,-\,\beta^-\otimes\beta^-\,,\qquad \nabla\beta^\pm\,=\,0\,,\qquad \beta^\pm\,\neq\,0\,. \tag{43.3}$$

$$\langle\beta^\pm,\,\beta^\pm\rangle\,=\,\langle\beta^+,\,\beta^-\rangle\,=\,0\,,\qquad \beta^\pm\,=\,0\,,\qquad \beta^-\,=\,*\beta^+\,, \tag{43.4}$$

The superscripts $\pm$ used here should not be confused with a similar notation for Λ^+_xM-components of bivectors, appearing in (6.15).

Throughout this argument, we will repeatedly "make U smaller", that is, replace U with a suitable connected open subset of U, containing x, for which we will still use the same symbol U.

Thus, making U smaller, we may assume that there exist C^∞ vector fields ξ^+, ξ^- and w, all defined on U, such that

$$\beta^+\,=\,\xi^+\wedge\,w\,,\qquad \beta^-\,=\,\xi^-\wedge\,w\,, \tag{43.5}$$

$$\nabla w\,=\,0\,, \tag{43.6}$$

and

$$g(\xi^\pm,\,\xi^\pm)=1\,,\quad g(\xi^+,\,\xi^-)=0\,,\quad g(\xi^\pm,\,w)=0\quad\text{and}\quad g(w,w)=0\,. \tag{43.7}$$

To see this, note that the first relation in (43.4), combined with Lemma 37.7, guarantees the existence of *vectors* $\xi^\pm(x),\ w^\pm(x)\in T_xM$ satisfying the conditions

$$\beta^\pm\,=\,\xi^\pm\wedge w^\pm\,,\quad g(\xi^\pm,\,\xi^\pm)=1\,,\quad g(\xi^\pm,\,w^\pm)=0\quad\text{and}\quad g(w^\pm,\,w^\pm)=0 \tag{43.8}$$

at the point x. By spreading these vectors through parallel transports (Remark 4.6), we obtain vector fields, for which relations (43.8) will remain valid throughout (a smaller version of) U, since $\beta^\pm$ and g are all parallel. Furthermore, since, at each point, $\beta^\pm$ determines $w^\pm$ uniquely up to a sign (cf. the final clause of Lemma 37.7), $w^\pm$ is parallel as well (see Remark 4.6.) For any fixed point $y\in U$, let us set $u_2=\xi^+(y)$ and $u_3=w^+(y)$, and let $u_1\in T_yM$ be a nonzero vector orthogonal to both u_2 and u_3. By (43.8), u_3 is null and hence u_1 cannot be null; otherwise, u_1 and u_3 would span a null subspace in T_yM of dimension 2, greater than the maximum value 1 possible in the Lorentz sign pattern $-+++$ (see (3.27)). Normalizing u_1, we now may assume that $g(u_1,u_1)=\pm 1$. However, if we had $g(u_1,u_1)=-1$, u_1+u_2 and u_3 would span a null plane, which again contradicts (3.27). Thus, $g(u_1,u_1)=1$ and, applying Proposition 37.1(ii) to the triple $(a,b,c)=(u_1,u_3,u_2)$ and using (43.5) and the second relation in (43.4), we now obtain, for a suitable orientation,

$$u_1\wedge w^+ \,=\, u_1\wedge u_3 \,=\, *(u_2\wedge u_3) \,=\, *(\xi^+\wedge\, w^+) \,=\, *\beta^+ \,=\, \beta^- \,=\, \xi^-\wedge\, w^-$$

at the point y. The uniqueness clause of Lemma 37.7 now implies that $w^+(y)$ and u_1 are, up to a simultaneus change of signs, the same as $w^-(y)$ and, respectively, the sum of $\xi^-(y)$ and a multiple of $w^-(y)$. Therefore, as $g(u_2,u_1)=g(u_2,u_3)=0$, we have $g(\xi^+,\,\xi^-)=0$. Finally, changing the signs of both ξ^- and w^- if necessary (which will leave (43.8) unaffected), we obtain $w^+=w^-$. Setting $w=w^+=w^-$, we thus have (43.5) – (43.7).

In view of (43.5), (43.7) and (2.22), $\beta^\pm$ treated as skew-adjoint bundle morphisms $TU\to TU$ satisfy

$$\beta^\pm w=0\,,\quad \beta^\pm\xi^\pm=w\,,\quad \beta^\pm\xi^\mp=0\,. \tag{43.9}$$

Since $\beta^\pm(v,v')=g(\beta^\pm v,v')$ by (2.19), using (43.3), (43.7) and (43.9), it is easy to verify that, at any point of U we have, for any tangent vectors v,v',

$$R(v,v')u \,=\, \pm\,[g(u,v)g(w,v')-g(u,v')g(w,v)]\,w \quad\text{if}\quad u=\xi^\pm \ \text{or}\ u=w\,. \tag{43.10}$$

(Both sides clearly vanish when $u=w$, cf. (4.26).) Let us now denote $\mathcal{P}^\pm$ the real-plane subbundles of TU given by

$$\mathcal{P}^\pm \,=\, \beta^\pm(TU) \,=\, \mathrm{Span}\,\{\xi^\pm,w\}\,. \tag{43.11}$$

(See (37.19).) Furthermore, let $\mathcal{X}^\pm$ be the vector space of all C^∞ functions $\phi:U\to\mathbf{R}$ such that

$$\nabla\phi \qquad\text{is a section of}\quad \mathcal{P}^\pm=\,\mathrm{Span}\,\{\xi^\pm,w\}\,, \tag{43.12}$$

and $\nabla d\phi=\mp\,\phi\,w\otimes w$; the local-coordinate form of the last equation is

$$\phi_{,jk} \,=\, \mp\,\phi\,w_jw_k\,. \tag{43.13}$$

Pairs $(\phi,\nabla\phi)$ with $\phi\in\mathcal{X}^\pm$ thus are nothing else than those sections (ϕ,u) of the direct-sum vector bundle $\mathcal{E}=[U\times\mathbf{R}]\oplus\mathcal{P}^\pm$ which are $\mathrm{D}^\pm$-parallel for the connection $\mathrm{D}^\pm$ in $\mathcal{E}$ given by

$$\mathrm{D}^\pm_v(\phi,u) \,=\, (d_v\phi-g(v,u)\,,\ \nabla_vu\,\pm\,\phi g(v,w)w) \tag{43.14}$$

for vector fields v tangent to U. Note that, since $\beta^\pm$ are parallel (see (43.3)), relation (43.11) shows that $\mathcal{P}^\pm$ are parallel subbundles of TU, as defined in Remark 4.7; hence, for any C^1 section u of $\mathcal{P}^\pm$ or $\mathcal{L}$ and any vector field v on U, ∇_vu is again a section of $\mathcal{P}^\pm$. Computing the curvature tensor $R^{[\pm]}$ of $\mathrm{D}^\pm$ from (4.52) (and using the shortcuts offered by Remark 4.4), we now obtain

$$R^{[\pm]}(v,v')(\phi,u) \,=\, (0\,,\ R(v,v')u\,\mp\,[g(u,v)g(w,v')\,-\,g(u,v')g(w,v)])\,.$$

Since u stands here for a section of (43.11), formula (43.10) now gives $R^{[\pm]}=0$, i.e., $\mathrm{D}^\pm$ is flat. Consequently, making U smaller again, we can find $\mathrm{D}^\pm$-parallel sections of $\mathcal{P}^\pm$ defined on U, that realize any prescribed initial value at any point. (See Lemma 11.2.) In particular, $\dim \mathcal{X}^\pm = 3$.

For every $\phi \in \mathcal{X}^\pm$, the function $g(\nabla\phi,\, \nabla\phi)$ is constant. In fact, $(\phi^{,s}\phi_{,s})_{,j} = 2\phi^{,s}\phi_{,sj} = 0$ by (43.13), (43.12) and (43.7). Let us now fix $\phi^\pm \in \mathcal{X}^\pm$ satisfying the initial condition $[\nabla\phi^\pm](x) = \xi^\pm(x)$. Hence, by (43.7), $g(\nabla\phi^\pm,\, \nabla\phi^\pm) = 1$ identically in U. Now, by (43.12), $\nabla\phi^\pm$ is at every point a combination of $\xi^\pm$ and w; the last identity, along with (43.7), now shows that the coefficient of $\xi^\pm$ in that combination must be equal to 1, i.e., $\nabla\phi^\pm$ equals $\xi^\pm$ plus a function times w. Let us now change our notations, modifying $\xi^\pm$ so that from now on it stands for

$$\xi^\pm \,=\, \nabla\phi^\pm\,, \quad \text{that is,} \quad \xi^\pm_j \,=\, \phi^\pm_{,j}\,. \tag{43.15}$$

Since that amounts to adding to the old $\xi^\pm$ a functional multiple of w, relations (43.5) – (43.7) and (43.9) – (43.12) all remain valid with this new meaning of $\xi^\pm$. Also, by (43.13) and (43.15),

$$\nabla\xi^\pm \,=\, \mp\,\phi^\pm w\otimes w\,, \quad \text{i.e.,} \quad \xi^\pm_{j,k} \,=\, \mp\,\phi^\pm w_j w_k\,. \tag{43.16}$$

Making U smaller, we can now find a C^∞ vector field v on U such that

$$\nabla v \,=\, \phi^+\beta^+ \,-\, \phi^-\beta^-\,, \quad \text{that is,} \quad v_{j,k} \,=\, \phi^+\beta^+_{kj} \,-\, \phi^-\beta^-_{kj}\,, \tag{43.17}$$

and

$$g(v,\,\xi^+) \,=\, g(v,\,\xi^-) \,=\, 0\,, \quad g(v,w) \,=\, 1\,, \quad g(v,v) \,=\, (\phi^+)^2 \,-\, (\phi^-)^2\,. \tag{43.18}$$

In fact, let $\mathcal{H} = [\mathrm{Span}\,\{\xi^+,\xi^-\}]^\perp$ be the real-plane subbundle of TU obtained as the orthogonal complement of the subbundle spanned by ξ^+ and ξ^-. Formula

$$\mathrm{D}_u v \,=\, \nabla_u v \,-\, g(v,w)\left[\phi^+\beta^+ u \,-\, \phi^-\beta^- u\right], \tag{43.19}$$

for vector fields u, v tangent to U, now defines a connection D in TU such that the subbundle $\mathcal{H}$ is D-parallel, as defined in Remark 4.7. In fact, since $g(\beta^\pm u, \xi^\pm) = -\,g(u,w)$ and $g(\beta^\pm u, \xi^\mp) = 0$ for all u (due to (43.9) and skew-adjointness of $\beta^\pm$), combining (43.19) with (43.16) and differentiation by parts we obtain $g(\mathrm{D}_u v, \xi^\pm) = 0$ whenever $g(v,\xi^+) = g(v,\xi^-) = 0$. Consequently, the same formula (43.19) (for vector fields v which are sections of $\mathcal{H}$) now defines a "restricted" connection in $\mathcal{H}$, also denoted D. Computing its curvature via (4.52), with the simplifications suggested by Remark 4.4, and using the relation

$$\beta^\pm u \,=\, g(\xi^\pm, u)w \,-\, g(w,u)\xi^\pm \tag{43.20}$$

for any tangent vector u (immediate from (43.5), (43.7), (2.15), and (2.22)), we see that the connection $\tilde{\mathrm{D}}$ in $\mathcal{H}$ is flat. Using Lemma 11.2, we can find a D-parallel vector field v (on a smaller version of U) such that (43.18) holds just at the point x. That such a choice of $v(x)$ is possible is clear since the vectors $\xi^\pm(x)$ are $(+\,+)$-orthonormal by (43.7); thus, $g(x)$ restricted to the orthogonal complement $P^\perp$ of the plane $P\subset T_xM$ spanned by $\xi^+(x)$ and $\xi^-(x)$ must have the sign pattern $-\,+$, while, by (43.7) and (2.4), $w(x)$ is a nonzero null vector in

$P^\perp$, and choosing $u \in P^\perp$ with $g(u,u) = 0$ and $g(u, w(x)) = 1$, we may now set $v(x) = u + [(\phi^+(x))^2 - (\phi^-(x))^2]w(x)/2$, which yields (43.18) at x.

For v selected as above, we clearly have $d_u[g(v,w)] = 0$ in view of (43.6) and (43.19); note that, by (43.20) and (43.7), $g(\beta^\pm u, w) = 0$ for all u. Thus, $g(v,w)$ is constant, and our choice of $v(x)$ now guarantees that $g(v,w) = 1$ everywhere. This implies (43.17) for our v (as $\mathrm{D}\, v = 0$, with D given by (43.19)).

Now, using (43.17), (43.15), we obtain $d[g(v,v) - (\phi^+)^2 + (\phi^-)^2] = 2\phi^-(\beta^- v + \xi^-) - 2\phi^+(\beta^+ v + \xi^+) = 0$ since, by (43.20) with $g(v,w) = 1$ and $g(v, \xi^\pm) = 0$, we have $\beta^\pm v = -\xi^\pm$. The function $g(v,v) - (\phi^+)^2 + (\phi^-)^2$ now is constant, and vanishes identically, which proves (43.18) everywhere in U.

Let $\xi = \xi^\pm$ for a fixed sign $\pm$. By (43.16) and (43.18), we then have $\nabla_v \xi = \mp\, \phi^\pm w$, while (43.17) and (43.9) give $\nabla_\xi v = \pm\, \phi^\pm w$. Therefore, by (4.4), $[v, \xi^\pm] = 0$. Moreover, in view of (43.16), (43.6) and (43.8), $\nabla_u u' = 0$ whenever u, u' are any two of the three vector fields ξ^+, ξ^- and w. Hence, again by (4.4), the vector fields

$$e_1 = \xi^-, \quad e_2 = \xi^+, \quad e_3 = w\,, \quad e_4 = v \tag{43.21}$$

commute with one another, i.e., $[e_j, e_k] = 0$ for all j, k. Corollary 11.6 now implies the existence of a coordinate system x^j, $j = 1,2,3,4$, on a smaller version of U, for which the coordinate vector fields e_j are given by (43.21). Furthermore, by (43.15), (43.7) and (43.18), the partial derivatives $\partial_j \phi^\pm = \partial\phi^\pm/\partial x^j$ are all zero except $\partial_1\phi^- = \partial_2\phi^+ = 1$. Hence ϕ^- and ϕ^+ differ from x^1 and, respectively, x^2 by constants; replacing x^1 with ϕ^- and x^2 with ϕ^+, we obtain a new coordinate system, which we still denote x^j, and which has the same coordinate vector fields (43.21) as before. It is now obvious from (43.18) and (43.7) along with $x^1 = \phi^-$ and $x^2 = \phi^+$ that the metric g has in these coordinates the component functions $g_{jk} = g(e_j, e_k)$ given by (43.1). This completes the proof. ■

We are now in a position to prove a local classification result for locally symmetric Lorentzian Einstein 4-manifolds, namely, Theorem 41.5 of §41:

Proof of Theorem 41.5. Of the six ***a priori*** possible classes listed in (40.2), three (namely, ***111***, ***11*** and ***1***) are excluded by Proposition 40.2. Moreover, in case ***3*** we have $W = 0$ (see (39.7)), and so assertion (a) of Theorem 41.5 follows from (5.10) and (10.1). This leaves just two more possibilities: ***21*** and ***2***.

Case ***21*** leads to assertion (b) of Theorem 41.5. In fact, according to Remark 40.1(c) we can find, locally, a parallel bivector field α with $(\alpha,\alpha)_{\mathbf{c}} = 1$ and $W\alpha = -2\lambda\alpha$ for some $\lambda \in \mathbf{C}$, where $(\,,)_{\mathbf{c}}$ is the complex-bilinear inner product of bivectors given by (37.3). Using Lemma 37.6(ii), at each point x, we now obtain $\alpha = e_1 \wedge e_2$ for some unit orthogonal vectors $e_1, e_2 \in T_xM$. In view of (37.19), the real-plane subbundle $\mathcal{P}$ of TM spanned by e_1 and e_2 is parallel, as defined in Remark 4.7. Thus, $\mathcal{P}$ and $\mathcal{Q} = \mathcal{P}^\perp$ satisfy condition (ii) of Theorem 14.5 and, consequently, also condition (i) in Theorem 14.5. This yields assertion (b) of Theorem 41.5.

Finally, in case ***2***, assertion (c) of Theorem 41.5 is immediate from Theorem 43.3. This completes the proof. ■

§44. Locally symmetric neutral metrics (sign pattern $--++$)

We now proceed to describe several examples of "exotic" locally symmetric Einstein metrics on 4-manifolds, this time with the neutral sign pattern $--++$.

Metrics of this type were classified by Cahen and Parker (1980), but some of them have been known much longer; for instance, the metrics with $f = f^{\pm}$ in our Example 44.1 appear in Petrov's monograph. (See Petrov, 1969, especially Example 2 on p. 256.)

Example 44.1. Let x^j and e_j, $j = 1, \ldots, 4$, stand for the Cartesian coordinates in $\mathbf{R}^4$ and, respectively, the standard basis of $\mathbf{R}^4$. Furthermore, let f be one of the following five quadratic homogeneous polynomial functions of two real variables x^1, x^2:

$$(44.1) \quad f^{\pm} = \pm\,(x^1)^2\,, \quad f^{\pm\pm} = \pm\left[(x^1)^2 + (x^2)^2\right], \quad f^{+-} = (x^1)^2 - (x^2)^2\,,$$

where $\pm$ is one of the signs $+$ or $-$. Treating the e_j as constant vector fields on $\mathbf{R}^4$, we can now define a pseudo-Riemannian metric g on $\mathbf{R}^4$ by declaring its components functions $g_{jk} = g(e_j, e_k)$ to be

$$(44.2) \quad \begin{aligned} &g_{12} = g_{21} = g_{34} = g_{43} = 1\,, \\ &g_{44} = -f\,, \quad \text{and} \quad g_{jk} = 0 \quad \text{otherwise.} \end{aligned}$$

In other words,

$$(44.3) \quad [g_{jk}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \quad f \quad \text{being one of} \quad f^+,\, f^-,\, f^{++},\, f^{--},\, f^{+-}\,.$$

Thus, g has the neutral sign pattern $--++$. Moreover, in view of Corollary 41.2, g is locally symmetric and Ricci-flat.

Finally, depending on whether f is $f^{\pm}$, $f^{\pm\pm}$, or f^{+-}, the Weyl tensor W of g represents the Petrov-Segre genus $\boldsymbol{3/2^{\pm}}$, $\boldsymbol{2^{\pm}/2^{\pm}}$ or, respectively, $\boldsymbol{2^{+}/2^{-}}$. To see this, let us consider the bivector fields $\beta = \beta^1$ and $\gamma = \beta^2$, with β^a defined as in Lemma 41.1(ii)). In view of (44.2) and (41.5), we have $\beta = e_2 \wedge e_3$ and $\gamma = e_1 \wedge e_3$. Therefore, Lemma 41.1(i) shows that β and γ satisfy (42.2). On the other hand, we obtain (42.8) for β from Proposition 37.1(i) applied to $(a, b, c, d) = (e_2, e_3, e_1, e_4)$, while relation (42.8) for γ follows from Proposition 37.1(i) with $(a, b, c, d) = (e_1, e_3, e_2, e_4)$. (In both cases, the Hodge star $*$ corresponds to g along with the orientation that makes e_1, e_2, e_3, e_4 positive-oriented at each point.) Furthermore, by Corollary 41.2(b) and (5.10), we have $R = W$ and so (41.6) gives (42.5), (42.7) or (42.1), depending on whether $f = f^{\pm}$, $f = f^{\pm\pm}$ or, respectively, $f = f^{+-}$. The fact that W is of the required Petrov-Segre genus now follows from Lemma 42.3.

Proposition 44.2. *Let (M, g) be a locally symmetric Einstein 4-manifold with a metric g of the neutral sign pattern $--++$. If the Weyl tensor W of (M, g) represents one of the five Petrov-Segre genera $\boldsymbol{3/2^{\pm}}$, $\boldsymbol{2^{\pm}/2^{\pm}}$ and $\boldsymbol{2^{+}/2^{-}}$, listed in* (41.12)*, then g is locally isometric to one of the five metrics described in* Example 44.1.

Proof. Let W first be of the genus $\boldsymbol{3/2^{\pm}}$, for some fixed sign $\pm$. By Proposition 40.2(i), (M, g) is Ricci-flat, so that $W = R$ (cf. (5.10)). Let us fix a point $x \in M$. In view of the final clause in Lemma 42.3, we can find a nonzero parallel bivector field β defined on some oriented connected neighborhood U of x and satisfying conditions (42.5), (42.6) everywhere in U. We thus have, with $W = R$,

$$R = \pm\beta\otimes\beta\,, \qquad \nabla\beta = 0\,, \qquad \beta \neq 0\,. \tag{44.4}$$

As in the proof of Theorem 43.3, we will "make U smaller" whenever convenient. As the first example, by making U smaller we may assume that there exist C^∞ vector fields ξ, w defined on U and such that

$$\beta = \xi\wedge w\,, \qquad g(\xi,\xi) = g(w,w) = g(\xi,w) = 0\,, \tag{44.5}$$

and, with ∇ standing as usual for the Levi-Civita connection of g,

$$\nabla\xi = \nabla w = 0\,. \tag{44.6}$$

In fact, the existence of *vectors* $\xi(x)$, $w(x) \in T_xM$ satisfying (44.5) at x is immediate from (42.6) and Lemma 37.8. Spreading these vectors away from x through radial parallel transports (Remark 4.6), we obtain vector fields ξ, w on (a possibly smaller version of) U, for which relations (44.5) will remain valid, since β and g are parallel. Let us now denote $\mathcal{P}$ the real-plane subbundle of TU given by

$$\mathcal{P} = \beta(TU) = \operatorname{Span}\{\xi,\, w\}\,. \tag{44.7}$$

(See (37.19).) Since β is parallel, $\mathcal{P}$ clearly is a ∇-parallel subbundle of TU, as defined in Remark 4.7. Thus, ∇ has an obvious restriction to a connection in $\mathcal{P}$ and, according to Example 4.3, that "restricted connection" in $\mathcal{P}$ is flat. Thus, by Lemma 11.2, if we make U smaller again, we can find two parallel sections of $\mathcal{P}$ on U whose values at x are $\xi(x)$ and $w(x)$. Due to the their parallel-transport origins, ξ and w now must coincide with those parallel sections everywhere in U, which proves (44.6).

From (44.5) and (2.27) we now obtain

$$\beta^2 = 0\,. \tag{44.8}$$

Also, by (44.5), (2.15), and (2.22), for any tangent vectors v, v', u we have

$$\beta v = g(\xi,v)w - g(w,v)\xi\,, \tag{44.9}$$

$$g(\beta v, w) = g(\beta v, \xi) = 0\,, \tag{44.10}$$

$$R(v,v')u = \pm\,[g(\xi,v)g(w,v') - g(\xi,v')g(w,v)]\,\beta u\,. \tag{44.11}$$

Here $\pm$ is, again, the fixed sign appearing in (44.4) and in the genus ***3/2***$^\pm$.

Combining (44.6) with (4.22) and Poincaré's Lemma (Corollary 11.3), we can find C^∞ functions θ, χ on a smaller version of U with

$$w = \nabla\theta\,, \qquad \xi = \nabla\chi\,. \tag{44.12}$$

Also, making U smaller, we can find C^∞ vector fields u, v on U satisfying the inner-product relations

$$(44.13)\qquad \begin{aligned} g(u,\xi) \,=\, g(v,w) \,=\, 1\,, \quad g(u,w) \,=\, g(v,\xi) \,=\, 0\,,\\ g(u,u) \,=\, g(u,v) \,=\, 0\,, \quad g(v,v) \,=\, \mp\,\chi^2\,, \end{aligned}$$

and the differential equations $\nabla u = \mp\,\chi w \otimes w$, $\nabla v = \mp\,\chi\beta$, the local-coordinate forms of which are

$$(44.14)\qquad u_{j,k} \,=\, \mp\,\chi w_j w_k\,, \qquad v_{j,k} \,=\, \pm\,\chi\beta_{jk}\,,$$

with the same fixed sign $\pm$ as in the genus ***3/2***$^\pm$. To see this, let us consider the connections D and $\tilde{\mathrm{D}}$ in TU given by

$$(44.15)\ \mathrm{D}_v u \,=\, \nabla_v u \,\pm\, \chi g(v,w)\beta u\,, \quad \tilde{\mathrm{D}}_v u \,=\, \nabla_v u \,\pm [\chi g(u,w)\beta v \,-\, \theta g(u,\xi)]\,\beta v\,,$$

for vector fields v tangent to U. We may now compute the curvature tensors of both connections from (4.52), with the shortcuts provided by Remark 4.4. Using (44.12), (44.6), (44.4), (44.8) and (44.11), we easily see that D is flat. A similar but slightly longer computation involving, in addition, (44.9) and (44.10), shows that $\tilde{\mathrm{D}}$ is flat as well. Applying Lemma 11.2, and making U smaller again, we can find a D-parallel vector field u and a $\tilde{\mathrm{D}}$-parallel vector field v, defined on U, and having any prescribed values at x.

Let us now select, and fix, such D-parallel u and $\tilde{\mathrm{D}}$-parallel v with the property that their values $u(x)$, $v(x)$ satisfy (44.13) just at the point x. This can be done by initially selecting $u = u(x)$ and $v = v(x)$ so as to have the first four relations in (44.13). (Note that $\xi \wedge w \neq 0$ at x by (44.5) and (44.4), i.e., ξ and w are linearly independent, and so we may find a vector orthogonal to one of them, but not to the other.) If we now replace u by $u - g(u,u)\xi/2$ and v by $v - g(u,v)\xi - [g(v,v) \pm \chi]w$, at the point x, we will clearly have (44.13) at x (in view of (44.5)).

For the vector fields u, v selected above, the functions $g(u,\xi)$, $g(v,w)$, $g(u,w)$, $g(v,\xi)$, $g(u,u)$, are all constant; in fact, they are automaically constant whenever u is D-parallel and v is $\tilde{\mathrm{D}}$-parallel. (To see this, use the Leibniz rule (4.5) for ∇, along with (44.6), (44.10) and the fact that $g(\beta u, u) = 0$ due to skew-adjointness of β.) This yields the first five relations in (44.13) which, by (44.9), implies

$$(44.16)\qquad \beta u \,=\, w\,, \quad \beta v \,=\, -\,\xi\,.$$

Consequently, we have (44.14) in view of (44.14).

Computing $\nabla[g(u,v)]$ via (44.14), we now obtain $(u^s v_s)_{,j} \,=\, \pm\,\chi[\beta_{sj}u^s - w_j]$ (as $w^s v_s = 1$), so that $\nabla[g(u,v)] = \chi(\beta u - w) = 0$ from (44.16). Similarly, using (44.14) we get $\nabla[g(v,v) \pm \chi^2] = \pm\,2\chi(\beta v + \xi) = 0$ in view of (44.16). Thus, the functions $g(u,v)$ and $g(v,v) \pm \chi^2$ are both constant and, as they are zero at x, they must vanish on U, which proves the last two relations in (44.13). Consequently, all seven relations (44.13) now hold everywhere in U.

By (44.5), (44.6), (44.13), (44.16) and (44.14), we now have $\nabla\xi = \nabla w = 0$, $\nabla_\xi u = \nabla_\xi v = \nabla_w u = \nabla_w v = 0$ and $\nabla_u v = \nabla_v u = \mp\,\chi w$. Hence, by (4.4), the vector fields

$$(44.17)\qquad e_1 \,=\, u\,, \quad e_2 \,=\, \xi\,, \quad e_3 \,=\, w\,, \quad e_4 \,=\, v$$

commute with one another, i.e., $[e_j, e_k] = 0$ for all j, k. In view of Corollary 11.6, there must now exist a coordinate system x^j, $j = 1, 2, 3, 4$, whose domain is a smaller version of U and for which the coordinate vector fields e_j are given by (44.17). Furthermore, by (44.12), (44.5) and (44.13), the partial derivatives $\partial_j\chi = \partial\chi/\partial x^j$ are all zero except $\partial_1\chi = d_u\chi = 1$. Hence χ and x^1 differ by a constant and so, replacing x^1 with χ we obtain a new coordinate system, which we still denote x^j, and which clearly has the same coordinate vector fields (44.17). Since we now have $\chi = x^1$, it is immediate from (44.5) and (44.13) that the components $g_{jk} = g(e_j, e_k)$ of g in these coordinates are given by (44.2) with $f = \pm\,(x^1)^2$. This proves our assertion for the genera $\boldsymbol{3/2^{\pm}}$.

Let us now consider the remaining case, where the genus of W is one of $\boldsymbol{2^+/2^+}$, $\boldsymbol{2^-/2^-}$ or $\boldsymbol{2^+/2^-}$. Since this part of our argument is virtually identical to the proof of Theorem 43.3 given in §43, our presentation will be brief. In particular, we will work with a fixed point $x \in M$, and U will stand for an oriented connected neighborhood of x which will be made "smaller and smaller" as needed, without further comments.

By Proposition 40.2(i), (M, g) is Ricci-flat. The final clause of Lemma 42.1 allows us to choose nonzero parallel bivector fields $\beta^+ = \beta$, $\beta^- = \gamma$ defined on U and satisfying (42.2) and (42.8) plus a third condition which, depending on the genus, is (42.7) or (42.1), in both cases with $W = R$. To discuss all these cases simultaneously, we will write

$$R \;=\; \varepsilon^+\beta^+\otimes\,\beta^+ \;+\; \varepsilon^-\beta^-\otimes\,\beta^-\,, \tag{44.18}$$

$$\langle\beta^\pm,\,\beta^\pm\rangle \;=\; \langle\beta^+,\,\beta^-\rangle \;=\; 0\,, \qquad \nabla\beta^\pm \;=\; 0\,, \qquad *\beta^\pm \;=\; \pm\,\beta^\pm \;\neq\; 0\,. \tag{44.19}$$

There must now exist C^∞ vector fields ξ^+, ξ^- and w on U such that

$$\beta^+ = \xi^+\wedge\, w\,, \qquad \beta^- = \xi^-\wedge\, w\,, \tag{44.20}$$

$$\nabla w \;=\; 0\,, \tag{44.21}$$

and

$$g(\xi^\pm,\,u^\pm) \;=\; g(\xi^\pm,\,w) \;=\; g(w,w) \;=\; 0\,, \quad g(\xi^+,\,\xi^-) \;=\; 1\,. \tag{44.22}$$

In fact, Lemma 37.9 combined with (44.19) guarantees the existence of *vectors* at x satisfying (44.20) and (44.22). Spreading these vectors through radial parallel transports (Remark 4.6), we obtain vector fields, still satisfying the same relations and, since β^+ and β^- determine w uniquely up to a sign and are themselves parallel, (44.21) follows.

In view of (44.20), (44.22) and (2.22), $\beta^\pm$ treated as skew-adjoint bundle morphisms $TU \to TU$ satisfy

$$\beta^\pm w \;=\; 0\,, \quad \beta^\pm\xi^\pm \;=\; 0\,, \quad \beta^\pm\xi^\mp \;=\; w\,. \tag{44.23}$$

Since $\beta^\pm(v, v') = g(\beta^\pm v, v')$ by (2.19), using (44.18), (44.22) and (44.23), it is easy to verify that, at any point of U we have, for any tangent vectors v, v',

$$R(v,v')u^\pm \;=\; \varepsilon^\mp\left[g(u^\mp,v)g(w,v') - g(u^\mp,v')g(w,v)\right]w \;\;\text{if } u^\pm = \xi^\pm \text{ or } u^\pm = w\,. \tag{44.24}$$

(Both sides are zero when $u^\pm = w$; cf. (4.26).) Let us now denote $\mathcal{P}^\pm$ the real-plane subbundles of TU given by

$$\mathcal{P}^\pm \,=\, \beta^\pm(TU) \,=\, \mathrm{Span}\,\{\xi^\pm, w\}\,, \tag{44.25}$$

cf. (37.19). Furthermore, let $\mathcal{X}$ be the vector space of all pairs $(\phi^+,\,\phi^-)$ of C^∞ functions $\phi^\pm : U \to \mathbf{R}$ such that

$$\nabla\phi^\pm \qquad \text{is a section of} \quad \mathcal{P}^\pm \,=\, \mathrm{Span}\,\{\xi^\pm, w\}\,, \tag{44.26}$$

and $\nabla d\phi^\pm = -\,\varepsilon^\mp\phi^\mp w\otimes w$; in local coordinates, the last equation reads

$$\phi^\pm_{,jk} \,=\, -\,\varepsilon^\mp\phi^\mp\, w_j w_k\,. \tag{44.27}$$

Quadruples $(\phi^+,\,\phi^-,\,\nabla\phi^+,\,\nabla\phi^-)$ with $(\phi^+,\,\phi^-)\in\mathcal{X}$ are nothing else than those sections (ϕ^+,ϕ^-,u^+,u^-) of the direct-sum vector bundle $\mathcal{E} = [U\times\mathbf{R}^2]\oplus\mathcal{P}^+\oplus\mathcal{P}^-$ which are D-parallel for the connection D in $\mathcal{E}$ given by $\mathrm{D}_v(\phi^+,\phi^-,u^+,u^-) = (Y^+,\,Y^-,\,Z^+,\,Z^-)$ with $Y^\pm = d_v\phi^\pm - g(v,u^\pm)$ and $Z^\pm = \nabla_v u^\pm + \varepsilon^\mp\phi^\mp g(v,w)w$, for vector fields v tangent to U. Computing the curvature tensor R^{D} of D from (4.52), with the simplifications described in Remark 4.4, we now obtain

$$R^{\mathrm{D}}(v,v')(\phi^+,\phi^-,u^+,u^-) \,=\, \big(0\,,\;0\,,\;A^+,\;A^-\big)$$

with $A^\pm = R(v,v')u^\pm - \varepsilon^\mp\,[g(u^\mp,v)g(w,v') - g(u^\mp,v')g(w,v)]\,w$. Since $u^\pm$ stands here for a section of (44.25), formula (44.24) now gives $R^{\mathrm{D}} = 0$, i.e., D is flat. Consequently, making U smaller again, we can find D-parallel sections of $\mathcal{P}^\pm$, defined on U, that realize any prescribed initial value at any point. (See Lemma 11.2.) In particular, $\dim\mathcal{X} = 6$.

For every $(\phi^+,\phi^-)\in\mathcal{X}$, the functions $g(\nabla\phi^\pm,\,\nabla\phi^\pm)$ and $g(\nabla\phi^+,\,\nabla\phi^-)$ are constant in view of (44.27), (44.26) and (44.22). Let us now fix $(\phi^+,\phi^-)\in\mathcal{X}$ satisfying the initial conditions $[\nabla\phi^\pm](x) = \xi^\pm(x)$. By (44.22), $g(\nabla\phi^\pm,\,\nabla\phi^\pm) = 0$ and $g(\nabla\phi^+,\,\nabla\phi^-) = 1$ identically in U. By (44.26), $\nabla\phi^\pm$ is at every point a combination of $\xi^\pm$ and w; the inner-product identities just established, along with (44.22), now show that the coefficients of $\xi^\pm$ in those combinations must be equal to 1, i.e., $\nabla\phi^\pm$ equals $\xi^\pm$ plus a function times w. We can now change our notations, replacing the old meaning of $\xi^\pm$ with a new one, given by

$$\xi^\pm \,=\, \nabla\phi^\pm\,, \quad \text{that is,} \quad \xi^\pm_j \,=\, \phi^\pm_{,j}\,. \tag{44.28}$$

Since that amounts to adding to the old $\xi^\pm$ a functional multiple of w, relations (44.20) – (44.22) and (44.23) – (44.26) all remain valid with this new meaning of $\xi^\pm$. Also, by (44.27) and (44.28),

$$\nabla\xi^\pm \,=\, -\,\varepsilon^\pm\,\phi^\mp w\otimes w\,, \quad \text{that is,} \quad \xi^\pm_{j,k} \,=\, -\,\varepsilon^\pm\,\phi^\mp w_j w_k\,. \tag{44.29}$$

Making U smaller, we can now find a C^∞ vector field v on U such that

$$\nabla v \,=\, -\,\varepsilon^+\phi^+\beta^+ \,-\, \varepsilon^-\phi^-\beta^-\,, \quad \text{i.e.,} \quad v_{j,k} \,=\, -\,\varepsilon^+\phi^+\beta^+_{kj} \,-\, \varepsilon^-\phi^-\beta^-_{kj}\,, \tag{44.30}$$

and

$$g(v,\,\xi^+)\,=\,g(v,\,\xi^-)\,=\,0\,,\quad g(v,w)\,=\,1\,,\qquad g(v,v)\,=\,-\,\varepsilon^+(\phi^+)^2\,-\,\varepsilon^-(\phi^-)^2\,. \tag{44.31}$$

In fact, let $\mathcal{H} = [\mathrm{Span}\,\{\xi^+,\xi^-\}]^\perp$ be the real-plane subbundle of TU obtained as the orthogonal complement of the subbundle spanned by ξ^+ and ξ^-. Formula

$$\tilde{\mathrm{D}}_u v\,=\,\nabla_u v\,-\,g(v,w)\left[\varepsilon^+\phi^+\beta^+u\,+\,\varepsilon^-\phi^-\beta^-u\right], \tag{44.32}$$

for vector fields u,v tangent to U, now defines a connection $\tilde{\mathrm{D}}$ in TU such that the subbundle $\mathcal{H}$ is $\tilde{\mathrm{D}}$-parallel, as defined in Remark 4.7. In fact, since $g(\beta^\pm u,\xi^\pm)=0$ and $g(\beta^\pm u,\xi^\mp)=-\,g(u,w)$ for all u (due to (44.23) and skew-adjointness of $\beta^\pm$), combining (44.32) with (44.29) and differentiation by parts we obtain $g(\tilde{\mathrm{D}}_u v,\xi^\pm)=0$ whenever $g(v,\xi^+)=g(v,\xi^-)=0$. Consequently, the same formula (44.32) (for vector fields v which are sections of $\mathcal{H}$) now defines a "restricted" connection in $\mathcal{H}$, also denoted $\tilde{\mathrm{D}}$. Computing its curvature via (4.52), with the simplifications suggested by Remark 4.4, and using the relation

$$\beta^\pm u\,=\,g(\xi^\pm,u)w\,-\,g(w,u)\xi^\pm \tag{44.33}$$

for any tangent vector u (immediate from (44.20), (44.22), (2.15), and (2.22)), we see that the connection $\tilde{\tilde{\mathrm{D}}}$ in $\mathcal{H}$ is flat. Using Lemma 11.2, we can find a $\tilde{\mathrm{D}}$-parallel vector field v which is a section of $\mathcal{H}$ (on a smaller version of U) and satisfies (44.31) just at the point x. To see that such a choice of $v(x)$ is possible, note that we can find $v=v(x)$ satisfying the first three relations in (44.31) at x (since $w(x)$ is not a combination of $\xi^+(x)$ and $\xi^-(x)$). To obtain the fourth relation in (44.31), it then suffices to replace v with $v-[g(v,v)+\varepsilon^+(\phi^+)^2+\varepsilon^-(\phi^-)^2]w/2$.

For v selected as above, we clearly have $d_u[g(v,w)]=0$ in view of (43.6) and (44.32); note that, by (44.33) and (44.22), $g(\beta^\pm u,w)=0$ for all u. Thus, $g(v,w)$ is constant, and our choice of $v(x)$ now guarantees that $g(v,w)=1$ everywhere. This implies (44.30) for our v (as $\tilde{\mathrm{D}}\,v=0$, with $\tilde{\mathrm{D}}$ given by (44.32)).

Now, using (44.30), (44.28), we obtain

$$d[g(v,v)+\,\varepsilon^+(\phi^+)^2+\,\varepsilon^-(\phi^-)^2]=2\varepsilon^+\phi^+(\beta^+v+\xi^+)+2\varepsilon^-\phi^-(\beta^-v+\xi^-)=0$$

since, by (44.33) with $g(v,w)=1$ and $g(v,\xi^\pm)=0$, we have $\beta^\pm v=-\,\xi^\pm$. The function $g(v,v)-\,\varepsilon^+(\phi^+)^2-\,\varepsilon^-(\phi^-)^2$ is therefore constant, and hence vanishes identically, which proves (44.31) everywhere in U.

Let $\xi=\xi^\pm$ for a fixed sign $\pm$. By (44.29) and (44.31), we then have $\nabla_v\xi=-\varepsilon^\pm\phi^\mp w$, while (44.30) and (44.23) give $\nabla_\xi v=-\varepsilon^\pm\phi^\mp w$. Therefore, by (4.4), $[v,\,\xi^\pm]=0$. Moreover, in view of (44.29), (43.6) and (43.8), $\nabla_u u'=0$ whenever u,u' are any two of the three vector fields ξ^+, ξ^- and w. Hence, again by (4.4), the vector fields

$$e_1\,=\,\xi^-,\quad e_2\,=\,\xi^+,\quad e_3\,=\,w\,,\quad e_4\,=\,v \tag{44.34}$$

commute with one another, i.e., $[e_j,e_k]=0$ for all j,k. Corollary 11.6 now implies the existence of a coordinate system x^j, $j=1,2,3,4$, on a smaller version of U, for which the coordinate vector fields e_j are given by (44.34). Furthermore, by (44.28), (44.22) and (44.31), the partial derivatives $\partial_j\phi^\pm = \partial\phi^\pm/\partial x^j$ are all zero except $\partial_1\phi^+ = \partial_2\phi^- = 1$. Hence (as in the proof of Theorem 43.3) we can replace x^1 with ϕ^+ and x^2 with ϕ^-, obtaining a new coordinate system, which we still denote x^j, and which has the same coordinate vector fields (44.34) as before. It is now obvious from (44.31) and (44.22) that the component functions $g_{jk} = g(e_j, e_k)$ of the metric g in these coordinates are given by (44.2) with $f = \varepsilon^+(x^1)^2 + \varepsilon^-(x^2)^2$. This completes the proof. ■

§45. Complex-analytic metrics and complexifications

The results presented here go back to Cahen and Parker (1980).

In full analogy with real-analytic pseudo-Riemannian metrics on real manifolds, one can speak of *complex-analytic metrics on complex manifolds*. Our interest in such metrics arises from their usefulness in creating further examples of locally symmetric Einstein metrics g in dimension four (with the neutral sign pattern $- - + +$). Namely, if we start with a (real) surface metric h having a nonzero constant Gaussian curvature, and form its "local complexification", or *complex-analytic extension*, which is a complex-analytic metric $h^{\mathbf{c}}$ on a complex surface, then its real part $g = \mathrm{Re}\, h^{\mathbf{c}}$ is a real 4-dimensional locally symmetric Einstein metric and, in addition, its Weyl tensor W is of the Petrov-Segre genus $\boldsymbol{21}^+/\boldsymbol{21}^-$ at every point.

The aim of this section is to verify the claim just made about g (see Example 45.5 below). We also establish its converse (Proposition 45.7), which is a classification result stating that, up to local isometries, the only metrics g with the properties just listed are those obtained from the above construction.

Let f be a real-valued, real-analytic function of m real variables $x^1, \dots, x^m$, defined on a (connected) domain $U \subset \mathbf{R}^m$. There exists a *complex-analytic extension* of f, that is, a complex-valued, complex-analytic function $f^{\mathbf{c}}$ of m complex variables $z^1, \dots, z^m$, which is defined on a connected open set Ω in $\mathbf{C}^m$ with $U = \Omega \cap \mathbf{R}^m$, and coincides with f on U. Such an extension $f^{\mathbf{c}}$ is unique once Ω is fixed, and has the same power-series expansion at any point of U as f. Dealing with $f^{\mathbf{c}}$, we will often denote it f and call it simply f *treated as a complex-analytic function*.

Let M be a complex manifold of some complex dimension m (cf. §23). By a *complex-analytic metric* g on M we mean an assignment to each point $x \in M$ of a nondegenerate complex-bilinear symmetric form $g(x) : T_xM \times T_xM \to \mathbf{C}$ whose dependence on x is complex-analytic, as described in the next paragraph.

Specifically, since our discussion is local, we may as well fix a complex-analytic local coordinate system z^j in M, $j = 1, \dots, m$, thus identifying the coordinate domain with a region Ω in $\mathbf{C}^m$. A complex-analytic metric g on Ω now is described by its component functions g_{jk} with $g_{jk} = g(e_j, e_k)$, where e_j, $j = 1, \dots, m$, are the vectors of the standard basis of $\mathbf{C}^m$, treated as constant vector fields. The requirements of complex-analyticity, nondegeneracy, and symmetry in the above definition now mean, respectively, that the g_{jk} are all complex-analytic, while $\det[g_{jk}] \neq 0$ and $g_{jk} = g_{kj}$ at every point of Ω.

Any complex-analytic metric g has a well-defined Levi-Civita connection ∇, gradient operator (also denoted ∇), curvature tensor R, Ricci tensor Ric and scalar curvature function s, all defined by the same local-coordinate formulae (4.1), (4.25), (4.35), (4.40) as in the real case, with $[g^{jk}] = [g_{jk}]^{-1}$ and

$$[g^{jk}] \,=\, g(\nabla z^j,\, \nabla z^k)\,, \tag{45.1}$$

as in (2.8) and (2.11). The only difference lies in the required regularity: Since the operators $\partial_j = \partial/\partial z^j$ now are the complex (Cauchy-Riemann) partial derivatives, all functions they are applied to, or resulting from their application, must be complex-analytic.

As for the coordinate-free meaning of these objects, it is completely analogous to that for real metrics. For instance, at any point x, $R(x)$ sends vectors v, w, u tangent at x, complex-trilinearly, to a vector $R(v,w)u$; $\mathrm{Ric}(x)$ is a complex-bilinear form sending vectors v, w to a complex scalar $\mathrm{Ric}(v,w)$; and ∇ associates with holomorphic (i.e., complex-analytic) vector fields v, w another such field $\nabla_v w$. Again, ∇ is characterized by being the unique torsionfree complex-analytic connection compatible with g (cf. Remark 4.1).

In particular, we may speak of complex-analytic metrics which are *locally symmetric* or *Einstein*, that is, satisfy $\nabla R = 0$ (i.e., $R_{jkl}{}^m{}_{,p} = 0$) or, respectively, (0.1).

Complex-analytic metrics give rise to very easy constructions of (real) pseudo-Riemannian Einstein metrics. (See the beginning of this section.) Before discussing such constructions, we need the following simple fact from linear algebra.

Lemma 45.1. *Let V be a finite-dimensional complex vector space with a fixed nondegenerate complex-bilinear symmetric form $h : V \times V \to \mathbf{C}$, and let $g : V \times V \to \mathbf{R}$ be the real-bilinear form $g = \mathrm{Re}\, h$. Furthermore, let $F : V \to V$ be the complex-linear operator corresponding via h to a given complex-bilinear form B on V, so that $h(Fv,w) = B(v,w)$ for all $v, w \in V$. Then the same F is the unique real-linear operator $V \to V$ corresponding via $g = \mathrm{Re}\, h$ to the real-bilinear form $\mathrm{Re}\, B$.*

This is immediate if we take the real part of the equality $h(Fv,w) = B(v,w)$ and use uniqueness of F. ■

Example 45.2. Any (real) pseudo-Riemannian metric h in any real dimension m which is real-analytic can be *locally complexified*, which produces a complex-analytic metric $h^{\mathbf{c}}$ in the complex dimension m. A local complexification of h is obtained by fixing a local coordinate system x^j, $j = 1, \ldots, m$, in which h has real-analytic component functions h_{jk}, and then declaring $h^{\mathbf{c}}$ to be the metric whose component functions, in a suitable domain of the m complex coordinates $z^1, \ldots, z^m$, are the same h_{jk}, now treated as complex-analytic functions of the variables $z^1, \ldots, z^m$ (see the beginning paragraph of this section). In other words, the components of $h^{\mathbf{c}}$ are the complex-analytic extensions of the h_{jk}. For notational convenience, we will sometimes use the same symbol h for both h and $h^{\mathbf{c}}$. Due to uniqueness of the analytic continuation, all relations valid for the original metric h that appear in local coordinates as polynomial equalities involving the h_{jk} and their partial derivatives up to any given order, will also hold for $h^{\mathbf{c}}$. Thus, for instance, $h^{\mathbf{c}}$ is locally symmetric, or Einstein, if so is h. As another example, every real surface metric h satisfies the relation $\mathrm{Ric} = \kappa h$, where κ is its Gaussian curvature. (See Remark 10.1.) If h is real-analytic, the same relation must holds for $h^{\mathbf{c}}$:

$$\mathrm{Ric}^{\mathbf{c}} \,=\, \kappa\, h^{\mathbf{c}}\,. \tag{45.2}$$

Note that Schur's Theorem 5.1 remains valid, with the same proof, for complex-analytic metric. Note that this does not contradict (45.2): Since the conclusion of the complex version of Schur's Theorem is true only in *complex* dimensions other than 2, it does not force the Gaussian curvature κ in (45.2) to be constant.

Example 45.3. The *real part* $g = \mathrm{Re}\, h$ of any complex-analytic metric h in the complex dimension m is a (real) pseudo-Riemannian metric in the real dimension $n = 2m$; in fact, g is nondegenerate by Remark 3.11. Moreover, g must have the neutral sign pattern (m, m), i.e., $- \ldots - + \ldots +$ with m minuses and m pluses. (In fact, we have $g(iv, iw) = -g(v, w)$ for tangent vectors v, w, i.e., multiplication by i establishes, at any point, an algebraic equivalence between g and $-g$.) Let the symbols ∇, R, Ric and s stand, as usual, for the Levi-Civita connection of g, its curvature tensor R, Ricci tensor Ric and scalar curvature s, all interpreted in the usual fashion; thus, at any point x, $R(x)$ sends three vectors v, w, u to a vector $R(v, w)u$. Also, let Rc denote the Ricci tensor of g regarded, at each point x, not as a bilinear form on, but as a linear operator in the tangent space; the components of Rc thus are $R_j^k = R_{jl} g^{lk}$. If we mark the analogous objects for the complex-analytic metric h with the superscript $^{\mathbf{c}}$, then

$$\nabla = \nabla^{\mathbf{c}}, \quad R = R^{\mathbf{c}}, \quad \mathrm{Rc} = \mathrm{Rc}^{\mathbf{c}}, \tag{45.3}$$

$$\mathrm{Ric} = 2\,\mathrm{Re}\,(\mathrm{Ric}^{\mathbf{c}}), \tag{45.4}$$

$$\mathrm{s} = 2\,\mathrm{Re}\,(\mathrm{s}^{\mathbf{c}}). \tag{45.5}$$

In fact, since $\nabla^{\mathbf{c}}$ is torsionfree and compatible with g, it must coincide with ∇ (Remark 4.1). Therefore, $R = R^{\mathbf{c}}$ in view of formula (4.52). However, with the Ricci tensor the situation is different: 'Trace' in (4.34) stands for the real trace for g and the complex trace for h, and these two traces, rather than being equal, are related by (3.4). This gives (45.4). Relation $\mathrm{Rc} = \mathrm{Rc}^{\mathbf{c}}$ now is immediate from (45.4) and Lemma 45.1 for $B = \mathrm{Ric}(x)$, at any point x. This proves (45.3). Finally, since $\mathrm{s} = \mathrm{Trace}\,\mathrm{Rc}$, equality (45.5) is immediate from the last relation in (45.3) and (3.4).

Lemma 45.4. *Let V be a two-dimensional complex vector space endowed with a nondegenerate, complex-valued, bilinear symmetric form h. Then*

- (i) *There exists a complex-linear operator $\gamma : V \to V$ such that*
  - a) *γ is an involution, that is, $\gamma^2 = \mathrm{Id}$, and*
  - b) *γ is skew-adjoint relative to h in the sense that, for $v, w \in V$, $h(\gamma v, w) + h(v, \gamma w) = 0$.*
- (ii) *An operator γ with* a) *and* b) *is unique up to a sign, and satisfies the relation*

$$h(v, u)w - h(w, u)v = -h(\gamma v, w)\gamma u \tag{45.6}$$

for all $u, v, w \in V$. In terms of the real-bilinear form $g = \mathrm{Re}\, h$ and the operator $\beta = i\gamma$, (45.6) *can also be rewriten as*

$$h(v, u)w - h(w, u)v = g(\beta v, w)\beta u - g(\gamma v, w)\gamma u. \tag{45.7}$$

Proof. Nonzero complex-linear operators $\gamma : V \to V$ are in a bijective orrespondence with nonzero complex-bilinear forms B on V, with B given by $B(v,w) = h(\gamma v, w)$, $v, w \in V$. Skew-adjointness of γ means that B is skew-symmetric, which (as $\dim V = 2$) makes B, and γ, unique up to a nonzero factor (see Remark 3.8). Choosing a basis v, w of V with $h(v,v) = h(w,w) = 1$ and $h(v,w) = 0$, and setting $\lambda = h(\gamma v, w)$, we clearly have, for a fixed skew-adjoint $\gamma \neq 0$, $\gamma v = \lambda w$ and $\gamma w = -\lambda w$. Hence $\gamma^2 = -\lambda^2 \cdot \mathrm{Id}$ and, using an appropriate complex factor, we see that γ as in (i) exists and is unique up to a change of sign.

According to Remark 3.2, any skew-adjoint involution γ gives rise to a direct-sum decomposition $V = V_+ \oplus V_-$ of V into the (± 1)-eigenspaces $V_\pm$ of γ. Note that, as γ is assumed skew-adjoint, it cannot be a multiple of Id, and so $V_\pm$ must both be 1-dimensional complex subspaces of V; otherwise, one of them would coincide with V, giving $\gamma = \pm\,\mathrm{Id}$. Also, since γ is skew-adjoint, its eigenspaces $V_\pm$ are both h-null subspaces of V. We may thus choose a basis u^+, u^- of V with $u^\pm \in V_\pm$ and $h(u^\pm, u^\pm) = 0$, $h(u^\pm, u^\mp) = 1$. Since both sides of (45.6) are skew-symmetric in $v, w \in V$ and $\dim V = 2$, it suffices to prove (45.6) for $v = u^+$, $w = u^-$, and any fixed $u \in V$. Relation (45.6) is consequently reduced to $h(u^+, u)u^- - h(u^-, u)u^+ = -\gamma u$, which holds whenever $u = u^\pm$ since both sides then become $\mp u^\pm$. We have thus established (45.6). Now (45.6) follows from (45.6); in fact, the right-hand sides of both relations coincide in view of the obvious equality $g(\beta v, w)\beta = i\,[\mathrm{Re}\, h(i\gamma v, w)]\gamma = -\,i\,[\mathrm{Im}\, h(\gamma v, w)]\gamma$. This completes the proof. ■

Example 45.5. Let g be the 4-dimensional (real) pseudo-Riemannian metric obtained as the real part $g = \mathrm{Re}\, h^{\mathbf{c}}$ of a complex-analytic metric $h^{\mathbf{c}}$ in the complex dimension 2 which itself is the result of complexifying a (real) pseudo-Riemannian surface metric h. We will then say, briefly, that g is (the real 4-dimensional metric) *obtained by complexifying the real surface metric* h. If, in addition, h has a nonzero constant Gaussian curvature κ, then g is a *locally symmetric Einstein metric of the neutral sign pattern* $--++$, and its Weyl tensor W has, at each point, the Petrov-Segre genus ***21***$^+$/***21***$^-$. In fact, $h^{\mathbf{c}}$ satisfies (45.2). Since κ is real, taking the real parts of both sides of (45.2) and using (45.4) we obtain the equality $\mathrm{Ric} = 2\kappa g$ satisfied by the Ricci tensor Ric of g and the original Gaussian curvature κ of h. Thus, according to (5.3), g is Einstein, with the scalar curvature

$$\mathrm{s} \,=\, 8\kappa\,. \tag{45.8}$$

Moreover, since h is locally symmetric, so must be both $h^{\mathbf{c}}$ (Example 45.2), and g (Example 45.3). Finally, according to Remark 10.1 we have (10.2) with $K = \kappa$ and $g = h$, which can also be rewritten as $R_{jkl}{}^m = \kappa\left(h_{jl}\delta_k^m - h_{kl}\delta_j^m\right)$. In other words, relation

$$R(v,w)u \,=\, \kappa\,[h(v,u)w \,-\, h(w,u)v]\,, \tag{45.9}$$

for all tangent vectors u, v, w, is satisfied by h and its curvature R; hence (see Example 45.2) it will remain valid if we let h and R stand for $h^{\mathbf{c}}$ and its curvature $R^{\mathbf{c}}$). Using (45.7), we can further rewrite (45.9), for $h = h^{\mathbf{c}}$, as

$$R(v,w)u \,=\, \kappa\,[g(\beta v, w)\beta u \,-\, g(\gamma v, w)\gamma u]\,, \tag{45.10}$$

where R now is the curvature tensor of the metric $g = \mathrm{Re}\, h^{\mathbf{c}}$. In view of (45.8) and (2.20), this is precisely (42.9) (with $\mathrm{s} \neq 0$, since $\kappa \neq 0$), so that W has the genus $\boldsymbol{21}^+/\boldsymbol{21}^-$ as a consequence of Lemma 42.4.

Remark 45.6. Our next objective is show, in Proposition 45.7 below, that the metrics g described in Example 45.5 are, essentially, the only possible 4-dimensional locally symmetric Einstein metrics M having a Weyl tensor W of genus $\boldsymbol{21}^+/\boldsymbol{21}^-$. To this end, we need to come up with a local description, suited to this purpose, of real pseudo-Riemannian metrics h on surfaces Σ, with a nonzero constant Gaussian curvature κ. The coordinate system we select consists of coordinate functions φ each of which satisfies the equation $\nabla d\varphi = -\kappa\varphi h$, that is,

$$\varphi_{,jk} \,=\, -\kappa\varphi h_{jk}\,. \tag{45.11}$$

(We are here retracing our steps in §12 and §13 that led to the proof of Theorem 14.2(i) in §14; the functions φ are restrictions to Σ of linear homogeneous functions on the pseudo-Euclidean vector space into which Σ is locally embedded.) Let us fix a point $x \in \Sigma$ and use a connected neighborhood U of x which will be made smaller whenever necessary. Pairs $(\varphi, \nabla\varphi)$ with $\varphi : U \to \mathbf{R}$ satisfying (45.11) are nothing else than those sections $\psi = (\varphi, u)$ of the direct-sum vector bundle

$$\mathcal{E} \,=\, [U \times \mathbf{C}] \oplus TU \tag{45.12}$$

which are D-parallel for the connection D in $\mathcal{E}$ given by

$$\mathrm{D}_v(\varphi, u) \,=\, (d_v\varphi - h(v,u)\,,\ \nabla_v u \,+\, \kappa\varphi v)\,, \tag{45.13}$$

where v is any vector (field) tangent to U (and $\kappa \neq 0$ is fixed). Computing the curvature tensor R^{D} of D via (4.52) (with the simplifications provided by Remark 4.4), we obtain

$$R^{\mathrm{D}}(v,w)(\phi, u) \,=\, (0\,,\ R(v,w)u \,-\, \kappa[h(v,u)w \,-\, h(w,u)v])\,. \tag{45.14}$$

and so D is flat by (45.9). Making U smaller, we may now choose D-parallel sections $\psi = (\xi, u)$, $\chi = (\eta, v)$ defined on U and realizing any prescribed values at x. (See Lemma 11.2.) Furthermore, formula

$$((\varphi, u), (\varphi', u')) \,=\, h(u, u') \,+\, \kappa\varphi\varphi' \tag{45.15}$$

defines a pseudo-Riemannian fibre metric (,) in $\mathcal{E}$, which is easily verified to be compatible with D. Thus, the (,)-inner product of any two parallel sections is constant on U. Therefore, we may choose our parallel sections ψ, χ in such a way that $\xi(x) = \eta(x) = 0$ and $(\psi, \psi) = \varepsilon_1\kappa$, $(\chi, \chi) = \varepsilon_2\kappa$, and $(\psi, \chi) = 0$. Here $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ are chosen so as to have the following property:

(45.16) The signs $(-\,\mathrm{sgn}\,(\varepsilon_1\kappa),\ -\,\mathrm{sgn}\,(\varepsilon_2\kappa))$ form the sign pattern of h.

(Note that $\kappa \neq 0$.) Since $d\xi$, $d\eta$ now form, at x, an (orthogonal) basis of T_x^*M, the inverse mapping theorem shows that the functions $x^1 = \xi$ and $x^2 = \eta$, restricted to a smaller version of U, form a coordinate system. The matrix $[h_{jk}]$ of the component functions of h in these coordinates is now characterized by its inverse matrix $[h^{jk}]$ with (2.11); namely, in the (real) local coordinates ξ, η varying near $\xi = \eta = 0$,

$$(45.17) \qquad [h^{jk}] = [h_{jk}]^{-1} = -\kappa \begin{bmatrix} \varepsilon_1 + \xi^2 & \xi\eta \\ \xi\eta & \varepsilon_2 + \eta^2 \end{bmatrix}.$$

We *do not have to verify* that, given $\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ and $\kappa \neq 0$, formula (45.17) really defines a surface metric h having the sign pattern as in (45.16) and the (constant) Gaussian curvature κ; that assertion follows since we proved the existence of coordinates in which such a metric (which we know exists) has the form (45.17).

Proposition 45.7. *Any locally symmetric Einstein metric g in dimension four which has the neutral sign pattern $--++$ and whose Weyl tensor W is of the Petrov-Segre genus* ***21***$^+$/***21***$^-$ *is locally isometric to a metric obtained by complexifying a surface metric h with the nonzero constant Gaussian curvature $\kappa = \mathrm{s}/8$, as described in* Example 45.5*, s being the scalar curvature of g.*

Proof. Let us define κ by (45.8). As in the proof of Theorem 43.3 (§43), we fix a point $x \in M$ and use an oriented connected neighborhood U of x which will be made smaller (but still denoted U) any time a need arises.

By Lemma 42.4, we have (42.8) – (42.10) at every point of a neighborhood U of x, with some parallel bivector fields β and γ on U (cf. the last clause of Lemma 42.4). In view of (45.8) and (2.20), equality (42.9) amounts to condition (45.10) satisfied by the curvature tensor R of g and all vectors u, v, w tangent to U. Here β and γ are treated, with the aid of g, as skew-adjoint bundle morphisms $TU \to TU$); viewed as such morphisms, they commute (by (42.8) and (37.30)) and satisfy $\beta^2 = -\mathrm{Id}$, and $\gamma^2 = \mathrm{Id}$ (by (42.10) and (37.32)). For their composite morphism $J = \beta\gamma = \gamma\beta$ we thus have $J^2 = -\mathrm{Id}$, so that J forms an almost complex structure in U (§9). We will from now on treat TU as a *complex vector bundle*, for which J is the operator of multiplication by i. Since β and γ are skew-adjoint and commute, J is self-adjoint relative to g, and so (cf. Remark 3.18), g is the real part of a unique complex-bilinear symmetric fibre metric h in the complex bundle TU, given by (3.35), that is,

$$(45.18) \qquad h(v, w) = g(v, w) - ig(Jv, w).$$

Thus, J is parallel, since so are β and γ. Consequently, the Levi-Civita connection ∇ of g also constitutes a connection in the *complex* bundle TU, while the complex fibre metric h is ∇-parallel (compatible with ∇), that is, we have a Leibniz rule for ∇ and h.

Let us now consider the complex vector bundle $\mathcal{E}$ over M obtained as the direct sum $\mathcal{E} = [U \times \mathbf{C}] \oplus TU$. Sections ψ of $\mathcal{E}$ thus are nothing else than pairs (φ, u) consisting of a function $\varphi : U \to \mathbf{C}$ and a vector field u on U. We now define a connection D in $\mathcal{E}$ by (45.13). As before, the curvature tensor R^{D} of D is given by (45.14) and so D is flat in view of (45.9). Also, formula (45.15) defines, again, a fibre metric $(\,,\,)$ in $\mathcal{E}$, which is this time complex-valued, complex-bilinear and symmetric, but as before is compatible with D. Making U smaller, we may now choose D-parallel sections $\psi = (\xi, u)$, $\chi = (\eta, v)$, defined on U, with $\xi(x) = \eta(x) = 0$ and $(\psi, \psi) = \varepsilon_1\kappa$, $(\chi, \chi) = \varepsilon_2\kappa$, $(\psi, \chi) = 0$, where

$\varepsilon_1, \varepsilon_2 \in \{1, -1\}$ are arbitrary. Since $\kappa \neq 0$, the gradients $\nabla\xi = u$, $\nabla\eta = v$ form, at x, an h-orthogonal complex basis of T_xM, and the inverse mapping theorem implies that the functions $z^1 = \xi$ and $z^2 = \eta$, restricted to a smaller version of U, form a $\mathbf{C}^2$-valued coordinate system (i.e., $\mathrm{Re}\,\xi$, $\mathrm{Im}\,\xi$, $\mathrm{Re}\,\eta$, $\mathrm{Im}\,\eta$ form a real coordinate system). The matrix $[h_{jk}]$ of the component functions of h in these coordinates is now characterized by its inverse matrix $[h^{jk}]$ with (45.1). Thus, in the complex local coordinates ξ, η varying near $\xi = \eta = 0$, h is characterized by (45.17), and so it is a local complexification of a real surface metric (with the sign pattern (45.16)) and with the constant Gaussian curvature κ. This completes the proof. ∎

Remark 45.8. The sign pattern (45.16) of a real surface metric h leading to g as in the above proof is completely arbitrary. For instance, we can obtain any such g by complexifying a *positive-definite* surface metric h. Thus, the local-isometry types of locally symmetric Einstein metrics g in dimension four whose Weyl tensors represent the Petrov-Segre genus $\boldsymbol{21}^+/\boldsymbol{21}^-$ are completely determined by just one invariant, the scalar curvature s with $\mathrm{s} \neq 0$.

§46. Pseudo-complex projective spaces

This sections deals with yet another family of examples of locally symmetric pseudo-Riemannian Einstein metrics in dimension 4, with the neutral sign pattern $--++$, which are *exotic* in the sense of being different from "obvious" examples mentioned at the beginning of §41. Although we choose to introduce these examples using the (seemingly most convenient) route of a local-coordinate formula, the manifolds in question can also be obtained through a natural geometric construction, paralleling that of complex projective or hyperbolic spaces (see Example 10.6). This is why manifolds with such metrics will be referred to as *pseudo-complex projective spaces.*

Besides a construction of such metrics, we also present here a classification result that characterizes them uniquely up to local isometries (see Proposition 46.10 below). Namely, the pseudo-complex projective spaces are, essentially, the only possible locally symmetric Einstein 4-manifolds whose Weyl tensor W is of the Petrov-Segre genus $\boldsymbol{3}/\boldsymbol{21}^-$ at each point.

The end of this section is devoted to a proof of Theorem 41.6, due to Cahen and Parker (1980).

Both here and in §49 below we deal with pseudo-Riemannian metrics g in even dimensions $n = 2p$, for integers $p \geq 1$, whose component functions g_{jk} form the block matrix

$$(46.1)\qquad [g_{jk}] = \begin{bmatrix} \mathbf{0} & \mathfrak{G} \\ \mathfrak{G}^* & \mathbf{0} \end{bmatrix},$$

where $\mathfrak{G}$ is a $p \times p$ matrix of real-valued C^∞ functions, and $\mathfrak{G}^*$ is the transpose of $\mathfrak{G}$. In other words, using the ranges of indices given by

$$(46.2)\qquad \begin{aligned} j, k, l, m &\in \{1, 2, \ldots, 2p\}\,, \\ a, b, c, d &\in \{1, 2, \ldots, p\}\,, \\ \lambda, \mu, \nu, \rho &\in \{p+1, \ldots, 2p\}\,, \end{aligned}$$

let us consider a $2p$-dimensional manifold U covered by a global coordinate system x^j, $j = 1, \ldots, 2p$, and any metric g on U with

$$g_{ab} = g_{\lambda\mu} = 0 \quad \text{for} \quad a, b \in \{1, \ldots, p\} \quad \text{and} \quad \lambda, \mu \in \{p+1, \ldots, 2p\}\,. \tag{46.3}$$

As usual, $g_{jk} = g(e_j, e_k)$, where e_j stand for the coordinate vector fields (see (2.1)). Thus, in (46.1) we have $\mathfrak{G} = [g_{a\lambda}]$ with C^∞ functions $g_{a\lambda} : U \to \mathbf{R}$. Note that, necessarily, $\det \mathfrak{G} \neq 0$ everywhere in U, and, by (46.3), the reciprocal metric components g^{jk} with $[g^{jk}] = [g_{jk}]^{-1}$ (see (2.8)) are given by

$$g^{ab} = g^{\lambda\mu} = 0\,, \qquad [g^{a\lambda}] = [g_{a\lambda}]^{-1} \tag{46.4}$$

(indices as in (46.2)). Furthermore, the vector subbundles $\mathcal{P}^\pm$ of TU given by

$$\mathcal{P}^+ = \operatorname{Span}\{e_1, \ldots, e_p\}\,, \qquad \mathcal{P}^- = \operatorname{Span}\{e_{p+1}, \ldots, e_{2p}\}\,, \tag{46.5}$$

both have the fibre dimension p and, by (46.3), are both null (that is, their fibres $\mathcal{P}^\pm_x$ at every point x are null subspaces of T_xU, cf. (3.26)). It follows now from (3.27) that g has the neutral sign pattern (p, p), i.e., $- \ldots - + \ldots +$ with p minuses and p pluses. We also have the direct-sum decomposition

$$TU = \mathcal{P}^+ \oplus \mathcal{P}^-\,. \tag{46.6}$$

Finally, the bundle morphism $\alpha : TU \to TU$ given by

$$\alpha = \mp \operatorname{Id} \quad \text{on} \quad \mathcal{P}^\pm\,, \tag{46.7}$$

with $\mathcal{P}^\pm$ given by (46.5), is skew-adjoint at each point. To see this, note that, treated (with the aid of g) as a twice-contravariant tensor field, α is a *bivector field* on U, i.e., its component functions $\alpha^{jk} = g^{jl}\alpha_l{}^k$ satisfy $\alpha^{jk} = -\alpha^{kj}$. More precisely, since, by (46.7), $\alpha_c{}^b = -\delta_c^b$, $\alpha_\lambda{}^\mu = \delta_\lambda^\mu$ and $\alpha_\lambda{}^c = \alpha_b{}^\mu = 0$, (46.4) gives

$$\alpha^{bc} = \alpha^{\lambda\mu} = 0\,, \quad \alpha^{b\mu} = -\alpha^{\mu b} = g^{b\mu}\,, \quad \alpha = g^{b\mu}\, e_b \wedge e_\mu\,, \tag{46.8}$$

e_j being again the coordinate vector fields. (In $g^{a\lambda}\, e_a \wedge e_\lambda$ we have, of course, summation over $a = 1, 2$ and $\lambda = 3, 4$.)

Lemma 46.1. *For a metric g of the form* (46.1) *on a coordinate domain U of any even dimension $n = 2p$, the following four conditions are equivalent*:

(a) $\mathcal{P}^\pm$ *with* (46.5) *are parallel subbundles of TU, as defined in* Remark 4.7;
(b) *With the ranges of indices as in* (46.2)*, all Christoffel symbols other than Γ_{ab}^c and $\Gamma_{\lambda\mu}^\nu$ are identically zero, that is,*

$$\Gamma_{ab}^\lambda = \Gamma_{\lambda\mu}^a = \Gamma_{a\lambda}^b = \Gamma_{\lambda a}^b = \Gamma_{a\lambda}^\mu = \Gamma_{\lambda a}^\mu = 0\,. \tag{46.9}$$

(c) *The differential 2-form $\omega = g_{a\lambda}\, dx^a \wedge dx^\lambda$ on U is closed, i.e., $d\omega = 0$;*
(d) *There exists, in a neighborhood of any point of U, a* potential *for g, by which we mean a C^∞ function ϕ such that*

$$g_{a\lambda} = \partial_a \partial_\lambda \phi \qquad \text{for} \quad a = 1, \ldots, p \quad \text{and} \quad \lambda = p+1, \ldots, 2p\,, \tag{46.10}$$

with $\partial_j = \partial/\partial x^j$, $j = 1, \ldots, 2p$.

Proof. Using the covariant-derivative formula (4.13) to describe parallel transports along curves (in terms of solving the local-coordinate form of the equation $\nabla_{\dot{x}} w = 0$), we easily see that (a) is equivalent to (b).

Defining ω as in (c), we have $d\omega = dg_{a\lambda} \wedge dx^a \wedge dx^\lambda$. Since $dg_{a\lambda} = \partial_b g_{a\lambda}\, dx^b + \partial_\mu g_{a\lambda}\, dx^\mu$, it is now easy to see that $2\, d\omega = (\partial_a g_{b\lambda} - \partial_b g_{a\lambda})\, dx^a \wedge dx^b \wedge dx^\lambda + (\partial_\mu g_{a\lambda} - \partial_\lambda g_{a\mu})\, dx^a \wedge dx^\lambda \wedge dx^\mu$. Skew-symmetry of the parenthesized expressions in a and b or, respectively, λ and μ, implies that $d\omega = 0$ if and only if $\partial_a g_{b\lambda} = \partial_b g_{a\lambda}$ and $\partial_\lambda g_{a\mu} = \partial_\mu g_{a\lambda}$. Using (4.9) and (46.3) one easily sees that this is in turn equivalent to requiring the functions Γ_{jkl} defined by (4.6) to satisfy the conditions $\Gamma_{\lambda ab} = \Gamma_{a\lambda\mu} = 0$ (for indices as in (46.2)). However, in view of (46.3), those conditions are nothing else than (46.9). This proves that (b) is equivalent to (c).

Finally, according to Poincaré's Lemma for 2-forms (Remark 11.5), ω in (c) is closed if and only if, locally in U, $\omega = d\vartheta$ with a 1-form ϑ of class C^∞. Writing $\vartheta = h_j\, dx^j$, we have $2\, d\vartheta = (\partial_j h_k - \partial_k h_j)\, dx^j \wedge dx^k$, summed over $j, k = 1, \ldots, 2p$. Splitting this last sum into terms involving $dx^a \wedge dx^b$, $dx^a \wedge dx^\lambda$ and $dx^\lambda \wedge dx^\mu$ (indices as in (46.2)), we see that condition $\omega = d\vartheta$ amounts to $\partial_a h_b - \partial_b h_a = \partial_\lambda h_\mu - \partial_\mu h_\lambda = 0$ and $g_{a\lambda} = \partial_a h_\lambda - \partial_\lambda h_a$; the first of these relations states that $d\vartheta^+ = d\vartheta^- = 0$ for $\vartheta^+ = h_a\, dx^a$, $\vartheta^- = h_\lambda\, dx^\lambda$. Thus, in view of Poincaré's Lemma *for* 1-*forms* (Corollary 11.3), closedness of ω means nothing else than the existence, locally in U, of C^∞ functions ψ, χ with $h_a = \partial_a \psi$, $h_\lambda = \partial_\lambda \chi$ (i.e., $\vartheta^+ = d\psi$, $\vartheta^- = d\chi$) and $g_{a\lambda} = \partial_a \partial_\lambda \chi - \partial_\lambda \partial_a \psi = \partial_a \partial_\lambda \phi$, where $\phi = \psi - \chi$. Thus, (c) implies (d); while, choosing ϕ as in (d) we obtain $\omega = d\xi$ with $\xi = \partial_\lambda \phi\, dx^\lambda$, and so $d\omega = 0$. Hence (c) also follows from (d), which completes the proof. ■

Lemma 46.2. *Let a metric g on a coordinate domain U of some even dimension $n = 2p$ satisfy* (46.3) *and* (46.10) *with a C^∞ function $\phi : U \to \mathbf{R}$. In other words, g is assumed to be of the form* (46.1) *and have a potential ϕ. With indices as in* (46.2)*, we then have, for Γ_{jkl} defined by* (4.6)*,*

$$\begin{aligned} &\Gamma_{ab\lambda} = \partial_a \partial_b \partial_\lambda \phi\,, \quad \Gamma_{\lambda\mu a} = \partial_\lambda \partial_\mu \partial_a \phi\,, \\ &\Gamma_{jkl} = 0 \qquad \text{otherwise,} \end{aligned} \tag{46.11}$$

while the the components of the curvature and Ricci tensors of g are

$$\begin{aligned} &R_{a\lambda b\mu} = \partial_a \partial_\lambda \partial_b \partial_\mu \phi - \Gamma_{ab}^c \Gamma_{\lambda\mu c}\,, \\ &R_{\lambda a b\mu} = R_{a\lambda\mu b} = -R_{a\lambda b\mu}\,, \qquad R_{\lambda a\mu b} = R_{a\lambda b\mu}\,, \\ &R_{jklm} = 0 \qquad \text{otherwise,} \end{aligned} \tag{46.12}$$

that is,

$$\begin{aligned} &R_{a\lambda b}{}^c = \partial_\lambda \Gamma_{ab}^c\,, \qquad R_{\lambda a\mu}{}^\nu = \partial_a \Gamma_{\lambda\mu}^\nu\,, \\ &R_{\lambda ab}{}^c = -R_{a\lambda b}{}^c\,, \qquad R_{a\lambda\mu}{}^\nu = -R_{\lambda a\mu}{}^\nu = \\ &R_{jkl}{}^m = 0 \qquad \text{otherwise,} \end{aligned} \tag{46.13}$$

and

$$R_{ab} = R_{\lambda\mu} = 0\,, \qquad R_{a\lambda} = R_{\lambda a} = -R_{a\lambda b\mu}\, g^{b\mu}\,. \tag{46.14}$$

Furthermore,

(i) *The bivector field α on U characterized by* (46.7)*, or* (46.8)*, is parallel.*

(ii) *At every point* $x \in U$, *and for any fixed indices* a, b, λ, μ *with* (46.2),

$$R\beta = R\gamma = 0 \quad \text{if} \quad \beta = e_a \wedge e_b\,, \quad \gamma = e_\lambda \wedge e_\mu\,, \tag{46.15}$$

where the curvature tensor R *of* g *acts on bivectors at* x *according to* (5.13). *Finally, if* g *happens to be an Einstein metric, we have*

$$R\alpha = \frac{\mathrm{s}}{n}\,\alpha\,, \tag{46.16}$$

with α *as in* (i), s *being the scalar curvature of* g.

Proof. Relations (46.11) are immediate from (4.9), (46.3) and (46.10). The first equality in (46.12) now follows easily from (4.31) along with (46.11) and (46.4). (Note that, by (46.4) and (4.6), $\Gamma_{ab}^c = g^{c\mu}\Gamma_{ab\mu}$.) The remainder of (46.12) now is a direct consequence of (4.32), (4.31), (46.11) and (46.4). As for (46.13), the first two formulae are immediate from (4.25) and assertion (b) in Lemma 46.1 (which holds, since we are assuming assertion (d)), while the last three may be obtained either in the same way or, equivalently, as consequences of the first two along with (46.12) and (46.4). (Note that $R_{jklm} = R_{jkl}{}^p g_{pm}$, cf. (4.30).)

Since $R_{jk} = g^{lm}R_{jlkm}$ by (4.37), we now easily obtain (46.14) using (46.12), (46.4) and the fact that, by (46.12), $R_{a\lambda b\mu}$ is symmetric both in a, b and in λ, μ.

Finally, the bivector field α is parallel since formula (46.7) provides a natural definition of the corresponding bundle morphism $\alpha : TU \to TU$ in terms of the subbundles $\mathcal{P}^\pm$ of TU given by (46.5), which are parallel according to Lemma 46.1(a), (d). On the other hand, by (46.12), $R_{jkab} = R_{jk\lambda\mu} = 0$ for all j, k. Since the components of β and γ are all zero, except possibly for β^{ab}, β^{ba} and $\gamma^{\lambda\mu}$, $\gamma^{\mu\lambda}$, we now obtain $2(R\beta)_{jk} = R_{jklm}\beta^{lm} = R_{jkab}\beta^{ab} = 0$, $2(R\gamma)_{jk} = R_{jklm}\gamma^{lm} = R_{jk\lambda\mu}\gamma^{\lambda\mu} = 0$, which proves (46.15). Finally, since α is parallel (see (i)), (46.16) is immediate from (5.20) and (5.10). This completes the proof. ■

Remark 46.3. Let g be a metric of the form (46.1), that is, (46.3), on a coordinate domain U of an even dimension $n = 2p$, and let us suppose that U is *rectangular* in the sense that the subset of $\mathbf{R}^n$ corresponding to U under the coordinate identification is an open rectangle, i.e., a Cartesian product of n open intervals in $\mathbf{R}$. The subset N of U obtained by arbitrarily fixing the values of the last (or, respectively, first) p coordinates, if nonempty, is a p-dimensional submanifold of U covered by a global rectangular coordinate system consisting of the functions x^a (or, respectively, x^λ) restricted to N. Clearly, N then is an integral manifold of the subbundle $\mathcal{P}^+$ (or, respectively, $\mathcal{P}^-$) of TU given by (46.5), as defined in Lemma 4.8. Any such integral manifold N of $\mathcal{P}^\pm$ carries a torsionfree connection $\nabla^\pm$ defined by declaring its component functions to be the Christoffel symbols Γ_{ab}^c (or, respectively, $\Gamma_{\lambda\mu}^\nu$) restricted to N (cf. (4.2), (4.3)); we will refer to $\nabla^\pm$ as the *submanifold connection* of N. It can be easily shown that the submanifold connection of any such N depends just on the subbundles $\mathcal{P}^\pm$ and the Levi-Civita connection of g, but not on the coordinate system x^j used here to describe it; however, that fact is not relevant for our purposes and can be safely ignored. The coordinates x^a (or, x^λ) on any N as above identify N with an open rectangle U^+ (or, U^-) in $\mathbf{R}^p$, which allows us to treat $\nabla^\pm$ as a torsionfree connection in $U^\pm$. However, what we obtain in this way is usually not a single connection in $U^\pm$, but rather a p-parameter family of connections, with a separate connection for each individual integral manifold of $\mathcal{P}^{\pm}$. In fact, in general, $\partial_\lambda \varGamma_{ab}^c$ and $\partial_a \varGamma_{\lambda\mu}^\nu$ may both be nonzero, even under additional assumptions such as (46.10). Condition (46.10) does, however, imply that the *submanifold connection* $\nabla^{\pm}$ *of any integral manifold of* $\mathcal{P}^+$ *or* $\mathcal{P}^+$ *is flat.* This is clear since assertion (b) in Lemma 46.1, which holds as a consequence of (46.10), along with (4.25), shows that the curvature tensor of ∇^+ (or, ∇^-) has, in the coordinates x^a (or, x^λ), the component functions equal to the components $R_{abc}{}^d$ (or, $R_{\lambda\mu\nu}{}^\rho$) of the curvature tensor of g. On the other hand, $R_{abc}{}^d = R_{\lambda\mu\nu}{}^\rho = 0$ by (46.13).

Lemma 46.4. *Let* g *be a metric of the form* (46.1) *on a* 4*-dimensional coordinate domain* U*, and let an orientation of* U *be chosen in such a way that the basis of the tangent space* T_xU *formed by the coordinate vector fields* e_j*,* $j = 1, \ldots, 4$*, is positive-oriented or negative-oriented at every point* $x \in U$ *depending on whether* $\det \mathfrak{G} = \det[g_{a\lambda}]$*, with indices as in* (46.2)*, is negative or, respectively, positive. Then, at every point* $x \in U$*, the bivector spaces* $\Lambda_x^{\pm}M$ *defined as in* (6.4) *for this orientation, can be characterized as follows.*

(a) *The space* Λ_x^+M *consists of all combinations*

$$\zeta = \zeta^{a\lambda} e_a \wedge e_\lambda \qquad (\text{summed over} \quad a = 1, 2 \quad \text{and} \quad \lambda = 3, 4)\,, \tag{46.17}$$

where $[\zeta^{a\lambda}]$ *is any* 2×2 *matrix with* $g_{a\lambda}\zeta^{a\lambda} = 0$*. In other words, elements of* Λ_x^+M *are precisely those bivectors* ζ *at* x *whose components* ζ^{jk} *satisfy the conditions*

$$\zeta^{ab} = \zeta^{\lambda\mu} = 0\,, \quad \zeta^{a\lambda} = -\,\zeta^{\lambda a}\,, \quad g_{a\lambda}\zeta^{a\lambda} = 0\,. \tag{46.18}$$

(b) *The space* Λ_x^-M *is spanned by the bivectors* $\alpha = \alpha(x)$*, characterized by* (46.7) *or* (46.8)*, and* $\beta = e_1 \wedge e_2$*,* $\gamma = e_3 \wedge e_4$*.*

(c) *If, moreover,* g *happens to be an Einstein metric, and* s *is the constant scalar curvature of* g*, then the anti-self-dual Weyl tensor* W^- *represents, at each point, either*

i) *The Petrov-Segre class* ***3****, when* $\mathrm{s} = 0$*, or*

ii) *The subclass* ***21***$^-$*, when* $\mathrm{s} \neq 0$*.*

Proof. Let $e^j = \nabla x^j$, $j = 1, 2, 3, 4$, be the differentials of the coordinate functions, treated as vector fields with the aid of g. By (2.10), (2.11) and (46.3), we thus have

$$e^\lambda = g^{a\lambda} e_a\,, \tag{46.19}$$

$$g(e^\lambda, e_\mu) = \delta_\mu^\lambda\,, \quad g(e^a, e^b) = g(e^\lambda, e^\mu) = 0\,, \tag{46.20}$$

with indices as in (46.2). Therefore, by (46.4),

$$e^3 \wedge e^4 = [g^{13}g^{24} - g^{23}g^{14}]\, e_1 \wedge e_2 = [\det \mathfrak{G}]^{-1}\, e_1 \wedge e_2\,. \tag{46.21}$$

Consequently, $e^3 \wedge e_3 \wedge e^4 \wedge e_4 = -\,[\det \mathfrak{G}]^{-1}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4$, i.e., the basis of the tangent space formed, at each point, by e^3, e_3, e^4, e_4, is positive-oriented. We now have

$$\begin{aligned} &* (e^3 \wedge e_4) = e^3 \wedge e_4\,, \;\; *(e^4 \wedge e_3) = e^4 \wedge e_3\,, \;\; *(e^3 \wedge e^4) = -\,e^3 \wedge e^4\,, \\ &* (e_3 \wedge e_4) = -\,e_3 \wedge e_4\,, \;\; *(e_1 \wedge e_2) = -\,e_1 \wedge e_2\,. \end{aligned} \tag{46.22}$$

In fact, each of the first four equalities is easily verified by applying Proposition 37.1(i) to the appropriate quadruple (a,b,c,d) which is $(e^3,\, e_4,\, e_3,\, e^4)$ for the first equality and, respectively, $(e^4,\, e_3,\, e_4,\, e^3)$, $(e^3,\, e^4,\, e_3,\, e_4)$ and $(e_3,\, e_4,\, e^3,\, e^4)$ for the other three; note that (a,b,c,d) then have the required inner-product properties in view of (46.20). The fifth relation follows from the third and (46.21). Also,

$$*(e^3\wedge e_3) \,=\, -\,e^4\wedge e_4\,,\;\; *(e^4\wedge e_4) \,=\, -\,e^3\wedge e_3\,. \tag{46.23}$$

To see this, let us set $v_1 = (e^3 - e_3)/\sqrt{2}$, $v_2 = (e^3 + e_3)/\sqrt{2}$, $v_3 = (e^4 - e_4)/\sqrt{2}$, $v_4 = (e^4 + e_4)/\sqrt{2}$, thus defining a $(-\,+\,-\,+)$-orthonormal basis v_1, v_2, v_3, v_4 of the tangent space which, as one sees computing $v_1\wedge v_2\wedge v_3\wedge v_4$, is also positive-oriented. By (37.13), we have $*(v_1\wedge v_2) = -\,v_3\wedge v_4$, $*(v_3\wedge v_4) = -\,v_1\wedge v_2$, while $v_1\wedge v_2 = \sqrt{2}\,e^3\wedge e_3$ and $v_3\wedge v_4 = \sqrt{2}\,e^4\wedge e_4$, which proves (46.23).

Since $\dim[\Lambda^\pm_x M] = 3$, assertions (a) and (b) will follow if we show that every ζ of the form (46.17) with $g_{a\lambda}\zeta^{a\lambda} = 0$ is in $\Lambda^+_x M$ and, in (b), $\alpha,\beta,\gamma\in\Lambda^-_x M$. The latter statement follows, as $*\beta = -\,\beta$ and $*\gamma = -\,\gamma$ by (46.22), while $*\alpha = -\,\alpha$ in view of (46.23) (In fact, $\alpha \,=\, g^{a\lambda}\,e_a\wedge e_\lambda \,=\, e^\lambda\wedge e_\lambda \,=\, e^3\wedge e_3 + e^4\wedge e_4$ by (46.8) and (46.19).) As for (a), setting $\zeta^\lambda_\mu = g_{a\mu}\zeta^{a\lambda}$ in (46.17), we obtain $\zeta = \zeta^\lambda_\mu\, e^\mu\wedge e_\lambda$, with $\zeta^\lambda_\lambda = 0$ (that is, $\zeta^4_4 = -\,\zeta^3_3 = 0$). Therefore, $\zeta = \zeta^3_4\, e^4\wedge e_3 \,+\, \zeta^4_3\, e^3\wedge e_4 \,+\, \zeta^3_3(e^3\wedge e_3 \,-\, e^4\wedge e_4)$, and so $*\zeta = \zeta$ by (46.22) and (46.23). This yields (a), and completes the proof. ■

Lemma 46.5. *Let g be a metric of the form* (46.1) *defined on a coordinate domain U of dimension $n = 2p \ge 4$ and satisfying condition* (d) *of* Lemma 46.1*.*

(i) *If the component functions of the curvature tensor of g satisfy*

$$R_{a\lambda b\mu} \,=\, -\,K\,[g_{a\lambda}g_{b\mu} \,+\, g_{b\lambda}g_{a\mu}] \tag{46.24}$$

for $a,b = 1,\dots,p$ and $\lambda,\mu = p+1,\dots,2p$, with a nowhere-zero real-valued function K, then g is a locally symmetric Einstein metric, while K in (46.24) *is constant and*

$$K \,=\, \frac{2\,\mathrm{s}}{n(n+2)}\,, \tag{46.25}$$

where s is the scalar curvature of g.

(ii) *If $n = 4$, then the following two conditions are equivalent*:

a) *Equality* (46.24) *holds for some nowhere-zero function K*;

b) *g is a locally symmetric Einstein metric and its Weyl tensor W represents, at each point, the Petrov-Segre genus* ***3/21***$^-$.

Proof. Suppose that (46.24) holds. Contracting (46.24) with $g^{b\mu}$, we obtain $\mathrm{Ric} = (p+1)Kg$, so that g is Einstein and K with (46.25) must be constant by Schur's Theorem 5.1. Furthermore, the decomposition (46.6) of TM into the parallel subbundles $\mathcal{P}^\pm$ (Lemma 46.1(a)) gives rise to a similar decomposition of the bivector bundle $[TU]^{\wedge 2}$ into four summands: $[\mathcal{P}^+]^{\wedge 2}$, $[\mathcal{P}^-]^{\wedge 2}$, $\mathbf{R}\alpha$, and a subbundle we denote $\mathcal{E}$, spanned, respectively, by all sections of the form $e_a\wedge e_b$; $e_\lambda\wedge e_\mu$; $\alpha = g^{a\lambda}\,e_a\wedge e_\lambda$ (see (46.7), (46.8)); and, for $\mathcal{E}$, all combinations $\zeta = \zeta^{a\lambda}\,e_a\wedge e_\lambda$ with 2×2 matrices such that $g_{a\lambda}\zeta^{a\lambda} = 0$. The first three summands are obviously parallel since so are $\mathcal{P}^\pm$ (Lemma 46.1(a)) and α (Lemma 46.2(i)). On the other hand, $\mathcal{E}$ is orthogonal to them, by (37.3), and so, for dimensional reasons, $\mathcal{E}$ must coincide with the orthogonal complement of their span. Hence $\mathcal{E}$ is parallel as well. In view of (46.15), and (46.16), $R\beta = R\gamma = 0$ for sections β of $[\mathcal{P}^+]^{\wedge 2}$ and γ of $[\mathcal{P}^-]^{\wedge 2}$, and $R\alpha = \mathrm{s}\alpha/n$. Also, for sections ζ of $\mathcal{E}$, $R\zeta = 2K\zeta$ (and so, by (46.25), $R\zeta = 4\mathrm{s}\,\zeta/[n(n+2)]$). In fact, $2(R\zeta)_{jk} = R_{jklm}\zeta^{lm} = 2R_{jkb\mu}\zeta^{b\mu}$, which equals $2K\zeta_{a\lambda}$ when $j = a$, $k = \lambda$ (in view of (46.24) with $g_{b\mu}\zeta^{b\mu} = 0$), and 0 for $j = a$ and $k = b$ or $j = \lambda$ and $k = \mu$ (see (46.12)). Therefore, the curvature tensor of R acting on bivectors as a bundle morphism $R : [TU]^{\wedge 2} \to [TU]^{\wedge 2}$ is, restricted to each summand, a constant multiple of the identity. Consequently, R is parallel, since so are the four summand subbundles of $[TU]^{\wedge 2}$. This proves (i).

If, in addition, $n = 4$, the eigenvalues of the curvature operator R acting on bivectors become 0, $\mathrm{s}/4$ and $\mathrm{s}/12$, and the corresponding eigenspace subbundles of $[TU]^{\wedge 2}$ are $[\mathcal{P}^+]^{\wedge 2} \oplus [\mathcal{P}^-]^{\wedge 2}$, $\mathbf{R}\alpha$ and $\mathcal{E}$. For a suitable orientation of U, The first two of these subbundles span $\Lambda^- M$, while the third one is nothing else than $\Lambda^+ M$. (See Lemma 46.4.) Since $W = R - \mathrm{s}/12$ by (5.10) with $n = 4$, this shows that $W^+ = 0$ identically, while W^- is diagonalizable at each point with the eigenvalues $-\mathrm{s}/12$, $-\mathrm{s}/12$ and $\mathrm{s}/6$. The the Petrov-Segre classes of W^+ and W^- thus are ***3*** and, respectively, ***21***. (See (39.7).) Since the eigenvector α corresponding to the simple eigenvalue of W^- satisfies $\langle \alpha, \alpha \rangle = -2 < 0$ (by (2.17) with $\alpha^2 = \mathrm{Id}$, which in turn is clear from (46.7)), W^+ belongs to the subclass ***21***$^-$ (cf. the paragraph following (39.10)). Thus, W has the Petrov-Segre genus ***3/21***$^-$.

Conversely, let us suppose that $n = 4$ and g is a locally symmetric Einstein metric with a Weyl tensor of genus ***3/21***$^-$.

Since W^- then automatically represents the Petrov-Segre subclass ***21***$^-$ (Lemma 46.4(c)ii)), the class ***3*** forming the remaining component of the genus must correspond to W^+, so that W^+ is identically zero. (See (39.7).) As $W = R - \mathrm{s}/12$ by (5.10) with $n = 4$, this means that $R\zeta = \mathrm{s}\,\zeta/6 = 2K\zeta$ (cf. (46.25)), for every ζ as in (46.17), that is, for any bivector ζ whose components satisfy (46.18). Hence $4K\zeta_{a\lambda} = 2(R\zeta)_{a\lambda} = R_{a\lambda lm}\zeta^{lm} = 2R_{a\lambda b\mu}\zeta^{b\mu}$ whenever $g_{b\mu}\,\zeta^{b\mu} = 0$, and hence the expression $C_{a\lambda b\mu} = R_{a\lambda b\mu} + Kg_{b\lambda}g_{a\mu}$ satisfies $C_{a\lambda b\mu}\zeta^{b\mu} = 0$ whenever $g_{b\mu}\,\zeta^{b\mu} = 0$. Hence $C_{a\lambda b\mu} = S_{a\lambda}g_{b\mu}$ for some $S_{a\lambda}$ and, summing the last equality against $g^{b\mu}$, we find that $2S_{a\lambda} = C_{a\lambda b\mu}g^{b\mu} = [R_{a\lambda b\mu} + Kg_{b\lambda}g_{a\mu}]g^{b\mu} = -R_{a\lambda} + Kg_{a\lambda} = -2Kg_{a\lambda}$. (This follows from (46.25), (46.14) and the relation $2g_{b\mu}g^{b\mu} = g_{jk}g^{jk} = 4$, cf. (46.4).) Hence $S_{a\lambda} = -Kg_{a\lambda}$, which gives (46.24). This completes the proof. ∎

Remark 46.6. Lemma 46.5 has reduced the question of classifying metrics with the properties listed in condition (ii)b) of Lemma 46.5 to solving (46.24), which is a system of nonlinear fourth-order partial differential equations imposed on a potential function ϕ with (46.10). (Thus is clear from (46.4), (46.12) and (46.11) with $\Gamma_{ab}^c = g^{c\mu}\Gamma_{ab\mu}$, which in turn follows from (46.4) and (4.6).) The system in question is, however, invariant under an infinite-dimensional pseudogroup of transformations; in fact, relations (46.3) and (46.10) remain valid, for a given metric g, if we replace the x^j and ϕ with new coordinates $\tilde{x}^j$ and a new potential function $\tilde{\phi}$, as long as the new $\tilde{x}^a$ (or, $\tilde{x}^\lambda$) depend only on the old $\tilde{x}^b$ (or, respectively, $\tilde{x}^\mu$), with indices as in (46.2), while

$$\tilde{\phi} = \phi + A(x^1, \ldots, x^p) + B(x^{p+1}, \ldots, x^{2p}) \qquad (46.26)$$

with arbitrary C^∞ functions A, B of p real variables. It is therefore not surprising that, rather than directly solving the system in question for ϕ as a function of the original x^j, the following lemma just states what a modified version (46.26) of a solution ϕ must look like in some new coordinates $\tilde{x}^j$ of the type just described.

Lemma 46.7. *Let a metric g on an n-dimensional coordinate domain U, $n = 2p$ have the form* (46.1) *and satisfy* (46.10) *for a C^∞ function $\phi : U \to \mathbf{R}$.*

(i) *If ϕ is given by*

$$e^{K\phi} = 1 + Q_{a\lambda} x^a x^\lambda \tag{46.27}$$

with summation over $a = 1, \ldots, p$ and $\lambda = p+1, \ldots, 2p$, for some nonzero real constant K, and a constant real 2×2 matrix $[Q_{a\lambda}]$ such that $\det [Q_{a\lambda}] \neq 0$, then the curvature tensor of g satisfies (46.24) *with the same K as in* (46.27)*.*

(ii) *Suppose that* (46.24) *holds for some nowhere-zero function K and $n \geq 4$. Then K is constant, and a suitable neighborhood of any point of U admits a new coordinate system in which g is still given by* (46.3) *and* (46.10)*, even if we change the notation so that the x^j stand for the new coordinates and ϕ denotes a new potential function for g, defined by* (46.27) *with some $Q_{a\lambda}$ such that $\det [Q_{a\lambda}] \neq 0$, and with the same K as in* (46.24)*.*

Proof. Let us now fix a point $y \in M$. Without loss of generality, we may assume, in either (i) or (ii), that the coordinate domain U is rectangular (see Remark 46.3), while, with indices as in (46.2),

$$x^j = 0 \quad \text{at} \quad y \quad \text{for} \quad j = 1, \ldots, 2p\,, \tag{46.28}$$

and

$$\phi = 0 \quad \text{whenever} \quad x^1 = \ldots = x^p = 0 \quad \text{or} \quad x^{p+1} = \ldots = x^{2p} = 0\,. \tag{46.29}$$

In fact, under the assumptions of (i), the rectangle formed by all $(x^1, \ldots, x^{2p})$ is $\mathbf{R}^n$, while (46.29) is obvious, and (46.28) holds if we choose y to be the point with $x^j = 0$, $j = 1, \ldots, 2p$. On the other hand, in (ii), this can be achieved by first replacing U with a smaller, rectangular neighborhood of x (from now on also denoted U), and then choosing a modification of our coordinates x^j and of the potential function ϕ of the type mentioned in Remark 46.6. Specifically, we replace the original coordinates x^j with $x^j - y^j$, where y^j are the components of y. Using the notation x^j, now and in the sequel, for the new coordinates, we thus have (46.28). In these new coordinates x^j, clearly, $\phi = \Phi(x^1, \ldots, x^{2p})$ for some function Φ of $2p$ real variables. Let us now replace $\phi = \Phi(x^1, \ldots, x^{2p})$ by the function

$$\begin{aligned} \tilde{\phi} = \Phi(x^1, \ldots, x^{2p}) - \Phi(x^1, \ldots, x^p, 0, \ldots, 0) \\ - \Phi(0, \ldots, 0, x^{p+1}, \ldots, x^{2p}) + \Phi(0, \ldots, 0)\,. \end{aligned} \tag{46.30}$$

From now on, the symbol ϕ will stand for the new potential function (46.30). (All our assumptions about ϕ are still satisfied, since $\tilde{\phi}$ in (46.30) is of the form (46.26)). Moreover, for this new ϕ we now have $\phi = 0$ at points where all x^a or all x^λ vanish, which clearly implies (46.29).

For the rest of this argument, we adopt the same ranges of indices as in (46.2)). Note that (46.29) easily yields

$$\begin{aligned} &\partial_a\phi \,=\, \partial_a\partial_b\phi \,=\, 0 \quad \text{whenever} \quad x^{p+1} = \ldots = x^{2p} = 0\,, \\ &\partial_\lambda\phi \,=\, \partial_\lambda\partial_\mu\phi \,=\, 0 \quad \text{whenever} \quad x^1 = \ldots = x^p = 0\,. \end{aligned} \tag{46.31}$$

To prove (ii), let us now assume that $n = 2p \geq 4$ and (46.24) holds for some nowhere-zero function K. In view of Lemma 46.5(i) and Schur's Theorem 5.1, K must be constant. Using (46.13), (46.4) and the fact that $R_{jklm} = R_{jkl}{}^p g_{pm}$ (see (4.30)), we can rewrite (46.24) as

$$R_{a\lambda b}{}^c \,=\, -\,K\,[\delta_a^c g_{b\lambda} \,+\, \delta_b^c g_{a\lambda}] \tag{46.32}$$

or, equivalently, since $R_{\lambda a\mu b} = R_{a\lambda b\mu}$ (cf. (46.12)), also as

$$R_{\lambda a\mu}{}^\nu \,=\, -\,K\,[\delta_\lambda^\nu g_{a\mu} \,+\, \delta_\mu^\nu g_{a\lambda}]\,. \tag{46.33}$$

Thus, in view of (46.10) and (46.13), we have

$$\partial_\lambda \tilde{\varGamma}_{ab}^c \,=\, \partial_a \tilde{\varGamma}_{\lambda\mu}^\nu \,=\, 0\,, \tag{46.34}$$

where

$$\tilde{\varGamma}_{ab}^c \,=\, \varGamma_{ab}^c \,+\, K\,[\delta_a^c\,\partial_b\phi + \delta_b^c\,\partial_a\phi]\,, \qquad \tilde{\varGamma}_{\lambda\mu}^\nu \,=\, \varGamma_{\lambda\mu}^\nu \,+\, K\,[\delta_\lambda^\nu\,\partial_\mu\phi \,+\, \delta_\mu^\nu\,\partial_\lambda\phi]\,. \tag{46.35}$$

Relations (46.34) state that the functions $\tilde{\varGamma}_{ab}^c$ (or, $\tilde{\varGamma}_{\lambda\mu}^\nu$) depend only on the variables x^a (or, respectively, x^λ). Therefore, (46.34) implies that the torsionfree connections ∇^+ and ∇^-, with the component functions $\tilde{\varGamma}_{ab}^c$ and, respectively, $\tilde{\varGamma}_{\lambda\mu}^\nu$, each of which is defined on a p-dimensional manifold with the coordinates x^a (or, respectively, x^λ), are defined uniquely, i.e., independent of the remaining p coordinates. Furthermore, by (46.31), $\tilde{\varGamma}_{ab}^c = \varGamma_{ab}^c$ when all x^λ are zero, and $\tilde{\varGamma}_{\lambda\mu}^\nu = \varGamma_{\lambda\mu}^\nu$ when so are all x^a. Hence, according to Remark 46.3, ∇^+ and ∇^- are both flat. In view of Corollary 11.7, we can change the coordinates, replacing each x^a by a function of all the x^b vanishing when $x^1 = \ldots = x^p = 0$, and each x^λ by a function of all the x^μ vanishing when $x^{p+1} = \ldots = x^{2p} = 0$, in such a way that, if these new coordinates are still denoted x^a and x^λ, we have $\tilde{\varGamma}_{ab}^c = \tilde{\varGamma}_{\lambda\mu}^\nu = 0$ identically on a possibly smaller, rectangular coordinate domain, still denoted U. Since $\varGamma_{ab}^c = \varGamma_{ab\lambda} g^{c\lambda}$ and $\varGamma_{\lambda\mu}^\nu = \varGamma_{\lambda\mu a} g^{a\nu}$ (by (4.6) and (46.4)), conditions $\tilde{\varGamma}_{ab}^c = \tilde{\varGamma}_{\lambda\mu}^\nu = 0$ can be rewritten as

$$\varGamma_{ab\lambda} \,+\, K\,[g_{a\lambda}\,\partial_b\phi + g_{b\lambda}\,\partial_a\phi] \,=\, \varGamma_{\lambda\mu a} \,+\, K\,[g_{a\lambda}\,\partial_\mu\phi \,+\, g_{a\mu}\,\partial_\lambda\phi] \,=\, 0\,. \tag{46.36}$$

However, for any function ϕ and constant K we have

$$\partial_j\partial_l\varPsi \,=\, \varPsi\,F_{jl} \quad \text{whenever} \quad \varPsi \,=\, e^{K\phi} \quad \text{and} \quad F_{jl} \,=\, \partial_j\partial_l\phi \,+\, K\,(\partial_j\phi)\partial_l\phi\,. \tag{46.37}$$

In view of (46.10) and (46.11), condition (46.36) is nothing else than $\partial_\lambda F_{ab} = \partial_a F_{\lambda\mu} = 0$, for F_{jl} as in (46.37). Thus, F_{ab} and $F_{\lambda\mu}$ do not depend on the x^λ (or, the x^a) while, by (46.36) and (46.31), they vanish when the x^λ (or, the x^a) are all zero. Therefore, $F_{ab} = F_{\lambda\mu} = 0$. By (46.37), this amounts to

$$\partial_a\partial_b\Psi \,=\, \partial_\lambda\partial_\mu\Psi \,=\, 0 \qquad \text{identically in} \quad U\,. \tag{46.38}$$

Also, by (46.31) and (46.29), $\Psi = 1$ at the point y, and $\partial_a\Psi = \partial_\lambda\Psi = 0$ at y. Setting $Q_{a\lambda} = [\partial_a\partial_\lambda\Psi](y)$, we thus obtain

$$\Psi \,=\, 1 \,+\, Q_{a\lambda}x^a x^\lambda\,. \tag{46.39}$$

In fact, by (46.38), all third-order partial derivatives of Ψ vanish identically, so that Ψ is a quadratic polynomial. The quadratic polynomial $\Psi - 1 - Q_{a\lambda}x^a x^\lambda$ now must be identically zero, since it vanishes, along with its partial derivatives up to order two, at the point y with (46.28). Assertion (ii) now follows from (46.37) and (46.39).

Conversely, to prove (i), note that (46.27) with a constant $K \neq 0$ implies (46.38) for Ψ given by (46.39). From (46.37) we now obtain $\partial_\lambda F_{ab} = \partial_a F_{\lambda\mu} = 0$ (for F_{jl} defined in (46.37)) which, in view of (46.10) and (46.11), amounts to (46.36). Since $\Gamma_{ab}^c = \Gamma_{ab\lambda}g^{c\lambda}$ and $\Gamma_{\lambda\mu}^\nu = \Gamma_{\lambda\mu a}g^{a\nu}$ (see (4.6), (46.4)), this in turn means that $\tilde\Gamma_{ab}^c$ and $\tilde\Gamma_{\lambda\mu}^\nu$ given by (46.35) are identically zero. Hence we also have (46.34) which, by (46.10) and (46.13), is nothing else than (46.32) and (46.33). Now (46.24) follows from (46.13) and (46.4). This completes the proof. ■

Example 46.8. By a *pseudo-complex projective space* we mean a pseudo-Riemannian manifold (M, g) of any even dimension $n = 2p$ that is covered by coordinate systems in which g is given by

$$\begin{aligned} &g_{ab} \,=\, g_{\lambda\mu} \,=\, 0\,, \qquad g_{a\lambda} \,=\, \partial_a\partial_\lambda\phi \\ &\text{for} \quad a, b = 1, \dots, p \quad \text{and} \quad \lambda, \mu = p+1, \dots, 2p\,, \quad \text{where} \\ &\phi \,=\, \frac{1}{K}\,\log\,[1 \,+\, Q_{a\lambda}x^a x^\lambda]\,, \end{aligned} \tag{46.40}$$

with some real constant $K \neq 0$, and some 2×2 matrix $[Q_{a\lambda}]$ of real constants with $\det[Q_{a\lambda}] \neq 0$. According to Lemmas 46.7(i) and 46.5, g then is a locally symmetric Einstein metric of the neutral sign pattern (p, p), (p minuses and p pluses) and, in the case of dimension $n = 4$, its Weyl tensor has the Petrov-Segre genus ***3/21***$^-$. Although g seems to depend on the parameters K and $Q_{a\lambda}$, only K really matters for the local-isometry type of g and, in fact, K is proportional to the scalar curvature of g (see (46.25)). On the other hand, a given metric g with (46.40) can always be rewritten in the form (46.40) with $[Q_{a\lambda}]$ equal to the 2×2 identity matrix. To achieve this, let us replace the coordinates x^λ with $y_a = Q_{a\lambda}x^\lambda$, leaving the x^a unchanged, $a = 1, \dots, p$. In the new coordinates $x^1, \dots, x^p, y_1, \dots, y_p$, g will still have the form analogous to (46.40) (see Remark 46.6), with $Q_{a\lambda}x^a x^\lambda = x^a y_a$, as required.

We now proceed to describe a geometric construction of pseudo-complex projective spaces, mentioned in the beginning paragraph of this section.

Let V be a finite-dimensional real vector space with a fixed pseudo-Euclidean inner product $\langle\,,\rangle$. By a *pseudo-complex structure* in V we then mean any linear operator $\alpha : V \to V$ with

$$\alpha^2 \,=\, \mathrm{Id}\,, \tag{46.41}$$

which is also skew-adjoint relative to $\langle\,,\rangle$. Then, obviously,

$$\langle\alpha v,\,\alpha w\rangle \;=\; -\,\langle v,w\rangle \tag{46.42}$$

for all $v,w\in V$. Thus, α establishes an algebraic equivalence between $\langle\,,\rangle$ and $-\,\langle\,,\rangle$, so that $\langle\,,\rangle$ must have the neutral sign pattern (p,p) (p minuses, p pluses), where $p=n/2$ and $n=\dim V$ is necessarily even. Condition (46.41) also implies that

$$V \;=\; V^+\oplus V^-\,, \tag{46.43}$$

$V^\pm$ being the eigenspace of α for the eigenvalue ± 1. (See Remark 3.2.) Skew-adjointness of α now shows that $\langle v,v\rangle=\pm\,\langle\alpha v,v\rangle=0$ for all $v\in V^\pm$, that is, both $V^\pm$ are null subspaces of V. (Cf. also Remark 3.12.)

A pseudo-complex structure α in a pseudo-Euclidean vector space V always exists, as long as $\dim V=2p$ is even and the inner product of V has the sign pattern (p,p). In fact, any fixed $(-\,\ldots\,-\,+\,\ldots\,+)$-orthonormal basis $e_1,\ldots,e_{2p}$ then gives rise to such α with $\alpha e_c=e_{c+p}$ and $\alpha e_{c+p}=e_c$ for $c=1,\ldots,p$.

A fixed pseudo-complex structure α in a pseudo-Euclidean vector space V gives rise to the action on V of the *pseudo-circle group*, consisting of all operators $F_r:V\to V$, for real numbers $r>0$, characterized by

$$F_r \;=\; r^{\pm 1}\quad\text{times}\quad \mathrm{Id}\quad\text{on}\quad V^\pm\,. \tag{46.44}$$

Each F_r then is easily seen to be an isometry, that is, preserve the inner product $\langle\,,\rangle$ of V. Note that $F_1=\alpha$.

The geometric idea behind the following example is based on viewing a pseudo-complex structure in V as an analogue of an ordinary complex structure in V, compatible with its inner product; the complex-case counterparts of the operators F_r are the complex rotations F_z, that is, multiplications by unit complex numbers z, while $\alpha=F_1$ then is the multiplication by i. The main difference between the two cases is that, for a complex structure, we have $-\,\mathrm{Id}$ rather than Id in (46.41). In other words, instead of being a complex vector space, V is here a module over the algebra of *duplex numbers*.

Example 46.9. Given a pseudo-complex structure α in a pseudo-Euclidean vector space V, we define (M,g) to be the pseudo-Riemannian quotient manifold of the pseudosphere

$$S_1 \;=\; \{u\in V:\langle u,u\rangle=1\}$$

relative to the isometric action of the pseudo-circle group $\{F_r:r\in(0,\infty)\}$ described above. (For a description of a quotient metric, see the paragraph preceding Example 10.6 in §10; the quotient metric is well-defined since $dF_r/dr=F_r\alpha/r$ while α satisfies (46.42), and so the orbits of the action are all nondegenerate.) This (M,g) *is a pseudo-complex projective space as defined in* Example 46.8. In fact, let us fix an element $u=v+w$ of S_1 with $v\in V^+$, $w\in V^-$, so that $\langle v,v\rangle=\langle w,w\rangle=0$, $\langle v,w\rangle=1/2$, and let $\mathrm{pr}:S_1\to M$ be the quotient projection. Also, let us introduce the vector subspaces $\mathcal{T}^\pm$ of V with $\mathcal{T}^+=V^+\cap w^\perp$, $\mathcal{T}^-=V^-\cap v^\perp$. For $x\in\mathcal{T}^+$ and $y\in\mathcal{T}^-$ which are sufficiently close to 0, we have

$$r(x,y) \;=\; \frac{1}{1+2\langle x,y\rangle} \;>\; 0\,, \tag{46.45}$$

and so we can define the assignment

$$U' \ni (x,y) \mapsto \mathrm{pr}\,(v + x + r(x,y)\,[w + y]) \in S_1\,, \tag{46.46}$$

where U' is a suitable neigborhood of $(0,0)$ in $\mathcal{T}^+ \times \mathcal{T}^-$. It is easy to verify that (46.46) is an immersion, and its local inverses, treated as local coordinate systems for S_1 (with the aid of fixed bases in the spaces $\mathcal{T}^\pm$), cast the quotient metric g in the form (46.40) with $K = 2$, which obviously proves our assertion.

Proposition 46.10. *Any locally symmetric Einstein metric g in dimension four with whose Weyl tensor W is of the Petrov-Segre genus $\boldsymbol{3/21^-}$, is locally isometric to a pseudo-complex projective space, defined as in* Example 46.8.

Proof. Let us fix a point $x \in M$. Our assumption means that, for a suitably chose orientation of some connected neighborhood U of x we have $W^+ = 0$ everywhere in U (see (39.7), while W^- is parallel and U admits a parallel bivector field α such that $\langle\alpha,\alpha\rangle = -2$ and, at each point, W^- is diagonalizable with a unique simple eigenvalue realized by the eigenvector $\alpha(y) \in \Lambda_x^- M$. This is clear from Remark 40.1(c) for $\mathsf{W} = W^-$. From (37.32) it now follows that $\alpha^2 = \mathrm{Id}$ at every point, which leads to a decomposition (46.6) into the eigenspace subbundles $\mathcal{P}^\pm$ satisfying (46.7). Hence the $\mathcal{P}^\pm$ are parallel as subbundles of TU (cf. Remark 4.7), since so is α. In view of (3.27) the fibre dimensions of both $\mathcal{P}^\pm$ must equal 2. Using Lemma 4.9, we may now find C^∞ functions x^j, $j = 1,\dots,4$, such that x^1, x^2 are constant in the direction of $\mathcal{P}^-$ and x^3, x^4 are constant in the direction of $\mathcal{P}^+$, while at x the differentials dx^j are linearly independent. According to the inverse mapping theorem, in a smaller version of U the x^j form a coordinate system in which the $\mathcal{P}^\pm$ are given by (46.5), where e_j are the coordinate vector fields. Since α is skew-adjoint, its eigenspace subbundles $\mathcal{P}^\pm$ are both null, and so g has in our coordinates x^j the form (46.3). By Lemma 46.1(a), (d) we now have (46.10), near x, for some potential function ϕ. Using Lemma 46.5(ii) we now obtain (46.24) for some nonzero constant K, while Lemma 46.7(ii) then allows us to modify both ϕ and the x^j, near x, so as to obtain (46.27). According to Example 46.8, this completes the proof. ∎

Proof of Theorem 41.6. In view of Proposition 40.2, of the forty-five Petrov-Segre genera listed in (40.2), only the following eleven may be realized by locally symmetric metrics: $\boldsymbol{3/3}$, $\boldsymbol{3/21^+}$, $\boldsymbol{3/21^-}$, $\boldsymbol{21^+/21^+}$, $\boldsymbol{21^+/21^-}$, $\boldsymbol{21^-/21^-}$, $\boldsymbol{3/2^+}$, $\boldsymbol{3/2^-}$, $\boldsymbol{2^+/2^+}$, $\boldsymbol{2^+/2^-}$, $\boldsymbol{2^-/2^-}$. According to Proposition 44.2, the last five of these eleven cases lead to assertion (vii) of Theorem 41.6. Similarly, in the third or sixth case (genera $\boldsymbol{3/21^-}$, $\boldsymbol{21^+/21^-}$), assertion (iii) or, respectively, (vi) of Theorem 41.6 is immediate from Proposition 46.10 or Proposition 45.7. Moreover, in case $\boldsymbol{3/3}$ we have $W = 0$ (see (39.7)) which, (5.10) and (10.1), implies assertion (i) of Theorem 41.6. This leaves us with just three genera: $\boldsymbol{3/21^+}$, $\boldsymbol{21^+/21^+}$, and $\boldsymbol{21^-/21^-}$. Let us fix a point $x \in M$.

If the genus is $\boldsymbol{3/21^+}$, we may choose an orientation of some connected neighborhood U of x in such a way that $W^- = 0$ everywhere in U, while U admits a parallel bivector field α such that $\langle\alpha,\alpha\rangle = 2$ and, at each point, W^+ is diagonalizable with a unique simple eigenvalue realized by the eigenvector $\alpha(y) \in \Lambda_x^+ M$. This is clear from Remark 40.1(c) for $\mathsf{W} = W^+$. From (37.32) it now follows that $\alpha^2 = -\mathrm{Id}$ at every point, and so (M, g, α) is a Kähler manifold. The Weyl tensor

W acting on bivectors hus has the spectrum (10.20), with $\mathrm{s} \neq 0$ (as $W^+ \neq 0$) and so, by (5.33), the spectrum of R is given by (10.21), with the parallel bivector field $\alpha = \alpha_j$ corresponding to the eigenvalue $\mathrm{s}/4$. Since the curvature operator acting on bivectors via (5.13) uniquely determines the curvature tensor, the latter must equal (10.5) with λ and μ given by (10.10). Consequenly, the Kähler manifold (U, g, α) is a space of constant holomorphic sectional curvature, and assertion (ii) of Theorem 41.6 now is immediate from Theorem 14.4.

Finally, let the genus be $\boldsymbol{21}^+/\boldsymbol{21}^+$ or $\boldsymbol{21}^-/\boldsymbol{21}^-$. For a simultaneous discussion of both cases, let us introduce the parameter δ with $\delta = 1$ for the former genus and $\delta = -1$ for the latter. Applying Remark 40.1(c) to $\mathsf{W} = W^+$ as well as $\mathsf{W} = W^-$, we can find a neighborhood U of x with parallel sections $\alpha^\pm$ of $\Lambda^\pm U$, for both signs $\pm$, such that $\langle \alpha^\pm, \alpha^\pm \rangle = 2\delta$. By (37.32), we now have $[\alpha^\pm]^2 = -\delta$, while, by (37.30), α^+ and α^- commute. The composite $F = \alpha^+\alpha^-$ is therefore self-adjoint, parallel, and satisfies $F^2 = \mathrm{Id}$. According to Remark 3.2, we now have a direct-sum decomposition

$$TU = \mathcal{P} \oplus \mathcal{Q}$$

of TU into the ± 1-eigenspace bundles of F, which are parallel (as defined in Remark 4.7), since so is F, and mutually orthogonal (since F is self-adjoint). Furthermore, the fibres of $\mathcal{P}$ and $\mathcal{Q}$ are 2-dimensional at each point. In fact, choosing $\beta \in \Lambda^+_y U$, at any $y \in U$, so that $\langle \alpha^+, \beta \rangle = 0$ and $\langle \beta, \beta \rangle = \pm 1$, we see from (37.31) and (37.32) (for β rather than α) that $\beta : T_y M \to T_y M$ is an isomorphism which anticommutes with $F(y)$, and so it interchanges its eigenspaces.

Now $\mathcal{P}$ and $\mathcal{Q}$ satisfy condition (ii) of Theorem 14.5 and, hence, also condition (i) in Theorem 14.5. This yields assertion (iv) or assertion (v) of Theorem 41.6, and completes the proof. ■

§47. More on Petrov's curvature types

To conclude our classification of the Weyl tensors $W(x)$ at points x of arbitrary pseudo-Riemannian 4-manifolds (M, g), let us again replace $T_x M$ and the metric g_x by a 4-space $\mathcal{T}$ with an inner product $\langle\,,\rangle$, as in §38, representing one of the three sign patterns (37.1). The classification of the Weyl tensors provided by Proposition 37.2 (with Petrov-Segre classes and genera introduced in §39 and §40) treats them as operators in the bivector space $\mathcal{T}^{\wedge 2}$, and gives their canonical matrix forms of type (39.5), (39.6) in a basis of $\mathcal{T}^{\wedge 2}$ that represents the inner product of bivectors in some standard way. The question now is, how this relates to the Weyl tensors viewed as quadrilinear forms (38.1) on the space $\mathcal{T}$.

The answer is that "standard" bases of the bivector space $\mathcal{T}^{\wedge 2}$ always arise in some canonical manner from "standard" bases of $\mathcal{T}$. In other words, the classification mentioned above remains valid in the quadrilinear approach; or, equivalently, there are no further subtleties.

As an example, diagonalizable Weyl tensors are brought to their canonical form (39.5)I), cf. Proposition 39.2, in a basis of $\mathcal{T}^{\wedge 2}$ which is either $(\,,)_{\mathbf{c}}$-orthonormal (for the Lorentzian sign pattern $-+++$), or consists of orthonormal bases of the three-dimensional summand spaces $\mathcal{B}^+$ and $\mathcal{B}^-$ of (37.23) (for the other two sign patterns in (37.1)). In the Riemannian case, such two bases (if compatible with some natural orientations in $\mathcal{B}^\pm$), are obtained via (37.24) from an orthonormal basis a, b, c, d of $\mathcal{T}$. For a proof, see Remark 6.19. Almost the same argument works also in the case where $\langle\,,\rangle$ has the neutral sign pattern $--++$, the corresponding formula being (37.25). The only extra twist is that, for a pair of orthonormal bases in $\mathcal{B}^{\pm}$ to be of the form (37.25), they must both represent the correct time orientation and space orientation. (See Remark 47.1 below.) Similarly, in the Lorentzian case, every $(\,,\,)_{\mathbf{c}}$-orthonormal basis of $\mathcal{T}^{\wedge 2}$ can be obtained via (37.28). (The proof is an easy variation on Lemma 37.6.)

In the remaining (nondiagonalizable) cases, it is easy to develop similar arguments, based on Lemmas 37.7 – 37.9.

Remark 47.1. A vector subspace V' of a pseudo-Euclidean inner-product space V V is called *timelike* or *spacelike* if the inner product restricted to V' is negative definite (or, respectively, positive definite). Denoting (q^-, q^+) the sign pattern of the inner product $\langle\,,\rangle$ of V, we have, according to Remark 3.13,

(47.1) $\quad q^-$ is the maximum dimension of a timelike subspace of V,

and

(47.2) $\quad q^+$ is the maximum dimension of a spacelike subspace of V.

If $(\,,\,)$ is indefinite, that is, q^- and q^+ are both positive, we can naturally divide the set of all *oriented* timelike subspaces of V having the maximum dimension q^- into two disjoint subsets, such that for two such subspaces V_1, V_2 that do (or, do not) lie in the same subset, the orthogonal projection $V \to V_2$ restricted to V_1 is orientation-preserving (or, respectively, orientation-reversing; to see this, consider the natural projections of V_1 and V_2 onto V/V_+, where V_+ is a maximal spacelike subspace). We will call these two subsets the *time orientations* of V. Similarly, using spacelike subspaces instead of timelike ones, we define the two *space orientations* of V (which can also be described as the time orientations of V endowed with $-\langle\,,\rangle$ instead of $\langle\,,\rangle$.) The set of all $(-\ldots-+\ldots+)$-orthonormal bases of V thus has four connected components, corresponding to the two independent choices of the time and space orientations represented by the first q^- and the last q^+ vectors of the basis.

§48. Lorentzian Einstein metrics in general relativity

Spacetimes of general relativity are pseudo-Riemannian four-manifolds (M, g) of the Lorentz sign pattern $-+++$. The energy-momentum tensor T of matter, accounting for its distribution and motion, then is determined by g via *Einstein's equations*

$$\lambda T = \mathrm{Ric} - \frac{1}{2}\mathrm{s}g\,, \qquad \text{i.e.,} \qquad \lambda T_{jk} = R_{jk} - \frac{1}{2}\mathrm{s}g_{jk}\,, \tag{48.1}$$

where $\lambda \neq 0$ is a universal constant. The main reason for this choice of T is that it guarantees the *energy-momentum conservation law* $\operatorname{div} T = 0$ in view of the Bianchi identity (5.2). Simplified models of the universe are often called *solutions to the Einstein equations*, as they are obtained by prescribing T (which represents a specific physical situation) and then solving (48.1) for the metric g. See, e.g., Besse (1987).

In this context, Lorentzian *Einstein* metrics in dimension 4 are of obvious physical interest, as they correspond to particularly regular or "symmetric" configurations of matter, with T proportional to g. An especially prominent special case is that of *vacuum solutions*, with $T = 0$ everywhere (which describes regions of the spacetime that are devoid of matter); geometrically, this is nothing else than a Ricci-flat metric g (Ric $= 0$, cf. §10), as one sees contracting (48.1).

Let us consider a special case of a *Schwarzschild metric* g given by (18.26), obtained by requiring that the parameters λ and ε satisfy $\lambda > 0$ and $\varepsilon = -1$ and the surface metric (18.24) with the (positive) constant Gaussian curvature λ be positive definite. The metric g will have the Lorentzian sign pattern $- + + +$ provided that the coordinates t, r, x^1, x^2 are subject to the condition $r > 1/3\lambda$.

For such a Schwarzschild metric g, the coordinate function t satisfies the condition $g(\nabla t, \nabla t) < 0$, which allows us to interpret it as *some physical observer's time*. Furthermore, since the components of the metric (18.26) do not depend on the variable t, the coordinate vector field in the direction of t is a Killing field (cf. Example 17.1) or, in other words, shifts along the t-axis are isometries. Thus, our Schwarzschild metric represents a *steady-state cosmological model*. (See also §19, especially Lemma 19.2.) Moreover, g is easily seen to be invariant under space rotations, for which r serves as a radial variable, and x^1, x^2 are coordinates on a sphere of radius $1/\sqrt{\lambda}$.

The Schwarzschild metrics, discovered by Schwarzschild (1916) shortly after Einstein's publication of equations (48.1), provided the simplest models of empty space in which there still exists gravity, such as the vacuum region surrounding a star. (Again, geometrically this amounts to the Schwarzschild metrics' being Ricci-flat, but not flat.) As models of physical reality, they were extremely successful, predicting the existence and correct numerical value of the perihelion precession of planet Mercury, a phenomenon that resisted a Newtonian explanation.

§49. Curvature-homogeneity for neutral Einstein metrics

According to Corollary 7.2 and Remark 6.24, for *Riemannian* Einstein four-manifolds, curvature-homogeneity implies local symmetry.

In this section we describe examples showing that an analogous assertion fails to hold in the general pseudo-Riemannian case. More precisely (see Corollary 49.2 below), a Ricci-flat indefinite metric of the neutral sign pattern $- - + +$ may be curvature-homogeneous without being locally symmetric, or even locally homogeneous. Specific examples of this kind are obtained as special cases of the construction of Ricci-flat metrics given in Corollary 41.2(b).

Lemma 49.1. *Let g be the pseudo-Riemannian metric on an open connected subset M of $\mathbf{R}^4$ with the component functions in the Cartesian coordinates x^j, $j = 1, \ldots, 4$, given by*

$$(49.1)\qquad \begin{aligned} &g_{12} = g_{21} = g_{34} = g_{43} = 1\,,\\ &g_{44} = -f(x^1)\,, \quad \text{and} \quad g_{jk} = 0 \quad \text{otherwise,}\end{aligned}$$

for some fixed C^∞ function $f = f(x^1)$ depending only on the variable x^1. In other words,

$$(49.2)\qquad [g_{jk}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \qquad \text{where} \quad f = f(x^1)\,.$$

Using the prime symbol $'$ *for the partial derivative* $\partial/\partial x^1$, *let us define the function* $\Phi : M \to \mathbf{R}$ *by* $2\Phi = f''$. *Then*

(i) g *is a Ricci-flat metric of the neutral sign pattern* $- - + +$.
(ii) g *is locally symmetric if and only if* Φ *is constant.*
(iii) g *is curvature-homogeneous if and only if either* $\Phi = 0$ *identically, or* $\Phi \neq 0$ *everywhere in* M.
(iv) *The function*

$$\Psi = \frac{\Phi\Phi''}{(\Phi')^2} \tag{49.3}$$

is a local invariant *of the metric* g *restricted to the open set* $U \subset M$ *on which* $\Phi\Phi' \neq 0$, *that is,* Ψ *is preserved by all* g*-isometries between open connected subsets of* U.
(v) *If, in addition,* g *is locally homogeneous, then*

$$\Phi\Phi'' = r(\Phi')^2 \tag{49.4}$$

everywhere in M, *for some constant* $r \in \mathbf{R}$.

Proof. (i) is immediate from Corollary 41.2(b). On the other hand, denoting e_j, $j = 1, \ldots, 4$, the coordinate vector fields (that is, vectors of the standard basis of $\mathbf{R}^4$, treated as constant vector fields), and setting $\beta = e_2 \wedge e_3$, we have $\beta = \beta^1$ (notation of Lemma 41.1(ii)), in view of (41.5), with g_{ab} given by (49.1). Therefore, from Lemma 41.1(ii) and (41.6), we have

$$R = \Phi\,\beta \otimes \beta\,, \qquad \nabla\beta = 0\,, \qquad \beta \neq 0\,, \tag{49.5}$$

since $2\Phi = f'' = \partial_1\partial_1 f$. This obviously implies (ii) (see also Corollary 41.2(a)). To establish (iii) it now clearly suffices to prove curvature-homogeneity of (M, g) under the assumption that $\Phi \neq 0$ everywhere. To this end, let us fix any $x, y \in M$ and define the basis $\bar{e}_j$ of T_yM by $\bar{e}_1 = \rho^{-1}e_1(y)$, $\bar{e}_2 = \rho e_2(y)$, $\bar{e}_3 = e_3(y)$, and $\bar{e}_4 = e_4(y) + \sigma e_3(y)$, with $\rho, \sigma \in \mathbf{R}$ chosen so that $2\sigma = f(y) - f(x)$ and $\rho^2 = \Phi(y)/\Phi(x)$ (note that $\Phi(y)/\Phi(x) > 0$ as $\Phi \neq 0$ everywhere). Setting $\bar{\beta} = \bar{e}_2 \wedge \bar{e}_3$, we now have $R(y) = \Phi(x)\,\bar{\beta} \otimes \bar{\beta}$, and so $g(y)$ and $R(y)$ look in the basis $\bar{e}_j$ of T_yM exactly like $g(x)$ and $R(x)$ in the basis $e_j(x)$ of T_xM. This yields (b).

In view of (41.8) and (4.7) with $\partial_2 f = 0$, the coordinate vector field $\xi = e_2$ is parallel. Also, (2.10) and (49.1) show that the gradient of the coordinate function x^1 is given by $\nabla x^1 = e_2 = \xi$. Taking covariant derivatives of (49.5) (or, of its local-coordinate form $R_{jklm} = \Phi\beta_{jk}\beta_{lm}$, cf. (5.12)), we now obtain $R_{jklm,p} = \Phi'\xi_p\beta_{jk}\beta_{lm}$ and $R_{jklm,pq} = \Phi''\xi_p\xi_q\beta_{jk}\beta_{lm}$ (or, in coordinate-free form, $\nabla R = \Phi'\xi \otimes \beta \otimes \beta$ and $\nabla^2 R = \Phi''\xi \otimes \xi \otimes \beta \otimes \beta$). Let the vector field u and the twice-contravariant tensor field B on the open set U defined in (iv) be given by $\Phi u = \Phi'\xi$ and $\Phi B = \Phi''\xi \otimes \xi$. By (49.5), we now have $\nabla R = u \otimes R$ and $\nabla^2 R = B \otimes R$. Thus, u and B are, at each point of U, uniquely determined by R, ∇R and $\nabla^2 R$, and so they both are local invariants of g restricted to U. This is, consequently, also the case for Ψ given by (49.3), as $B = \Psi u \otimes u$, which proves (iv).

Finally, if g is locally homogeneous, (iv) implies that Ψ is constant on U, and so (49.4) holds on U, with some constant r. (If U happens to be empty, any constant r will do.) Both sides of (49.4) thus coincide on the set $\tilde{U}$ which is the union of U and the set of all $x \in M$ such that Φ is constant on some neighborhood of x. Assertion (v) will now follow if we show that $\tilde{U}$ is dense in M. To this end, let $U' \subset M$ be any nonempty open set. If $\Phi' = 0$ identically on U', we clearly have $U' \subset \tilde{U}$. Otherwise, $\Phi'(x) \neq 0$ for some $x \in U'$ and so there exist points $y \in U'$ arbitrarily close to x with $\Phi\Phi' \neq 0$ at y, i.e., $y \in U \subset \tilde{U}$. Thus, any such U' intersects $\tilde{U}$, which completes the proof. ■

Corollary 49.2. *There exist Ricci-flat pseudo-Riemannian metrics in dimension four which have the neutral sign pattern* $- - + +$ *and are curvature-homogeneous, but not locally homogeneous or locally symmetric.*

In fact, examples are provided by metrics g obtained as in Lemma 49.1 with $f = f(x^1)$ such that $\Phi = f''/2$ is nonzero everywhere but does not satisfy the differential equation (49.4) for any real constant r. (We do not need the fact that local symmetry implies local homogeneity, cf. Remark 42.7.) ■

Remark 49.3. The metrics constructed in Lemma 49.1 also illustrate the fact that the local-isometry types of Ricci-flat pseudo-Riemannian metrics in dimension four with the neutral sign pattern $- - + +$ form an infinite-dimensional "moduli space". More precisely, we can associate a metric of this kind with an arbitrary C^∞ function $Q = Q(\Psi)$ of a real variable Ψ in such a way that the open set U defined in (iv) is nonempty and the local invariant Ψ given by (49.3) satisfies the equation $\Psi_{,jk} = Q(\Psi)\Psi_{,j}\Psi_{,k}$, i.e., $\nabla d\Psi = Q(\Psi)\, d\Psi \otimes d\Psi$. (This obviously means that *different functions* Q *give rise to different local-isometry types of metrics.*) Specifically, since $\xi = e_2$ is parallel (see proof of Lemma 49.1), we have $d\Psi = \Psi'\xi$ and $\nabla d\Psi = \Psi''\xi \otimes \xi$. Equation $\nabla d\Psi = Q(\Psi)\, d\Psi \otimes d\Psi$ now will hold if we choose $f = f(x^1)$ in (49.1) to be any function with the property that Ψ defined by (49.3) for $\Phi = f''/2$ satisfies the ordinary differential equation $\Psi'' = (\Psi')^2 Q(\Psi)$.

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