MATH 7711, AUTUMN 2019

Distributions and the Frobenius Theorem

[**DG**] stands for *Differential Geometry* at
`http://www.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`

By a p-dimensional *distribution* on a manifold M we mean a smooth vector subbundle D of fibre dimension p in the tangent bundle TM. Given a p-dimensional distribution $D \subseteq TM$, a submanifold P of M is said to be an *integral manifold* of D if $T_xP = D_x$ for every $x \in P$ (thus, $\dim P = p$), and a distribution D on M is called *integrable* if every point of M lies in an integral manifold of D. Obvious examples of integrable distributions are the vertical distribution of fibrations, with the fibres serving as integral manifolds.

Submanifolds, including integral manifolds of distributions, *are not assumed here to have the subset topology* [**DG**, Section 9]. *Connectedness* of a submanifold always refers to its own topology.

Given a distribution D on a manifold M, let D^{nrm} be the *normal bundle* of D, that is, the quotient vector bundle $(TM)/D$. The *curvature* of D is the smooth vector-bundle morphism $\varOmega : D^{\wedge 2} \to D^{\mathrm{nrm}}$ such that $\varOmega(v, w) = \pi[v, w]$ for any smooth local sections v, w of D, where $[\,,\,]$ is the Lie bracket [**DG**, Section 6] and $\pi : TM \to D^{\mathrm{nrm}}$ denotes the quotient projection morphism. (About well-definedness of $\varOmega$, see Exercise 2 below.)

The Frobenius Theorem. *For a distribution to be integrable, it is necessary and sufficient that its curvature be identically zero.*

Furthemore, a distribution is integrable if and only if it is, locally, the vertical distribution of a fibration with connected fibres.

In other words, integrability of a distribution $D \subset TM$ means precisely that the set of smooth local sections of D is closed under the Lie-bracket operation, while every point of a manifold with a fixed integrable distribution D has a neighborhood which is the total space of a smooth locally trivial bundle, and the (connected) fibres of that bundle are integral manifolds of D.

Necessity in the Frobenius Theorem is obvious: if two vector fields u, v on a manifold are tangent to a given submanifold, so is $[u, v]$, cf. [**DG**, Theorem 6.1]. For a direct proof of necessity, see the lines following formula (14).

It is convenient to generalize the notion of an integral manifold as follows. Given a distribution D on a manifold M, we will say that a submanifold N of M (or, a smooth mapping $\varphi : Q \to M$, where Q is a manifold) is *tangent* to D if $T_xN \subseteq D_x$ (or, respectively, $d\varphi_z(T_zQ) \subseteq D_x$) whenever $x \in N$ (or, $z \in Q$).

To prove sufficiency in the Frobenius theorem, let us fix a p-dimensional distribution D on a manifold M of dimension m, a point $x \in M$, and a local coordinate system $x^1, \dots, x^p, y^{p+1}, \dots, y^m$, for which we also use the concise notation x^j, y^λ, using from now on the convention that the ranges of the indices j, k, λ, μ are

$$1 \le j, k \le p \quad \text{and} \quad p+1 \le \lambda, \mu \le m. \tag{1}$$

Without much loss of generality, we may also require that the image of our coordinate chart is an open rectangle in $\mathbb{R}^m$ (that is, the Cartesian product of m open intervals in $\mathbb{R}$), and that our distribution D is transverse to the span of the last $m-p$ coordinate directions or, equivalently, is a horizontal distribution (connection) for the fibration of the coordinate domain with the fibres defined by equating the first p coordinates x^j to constants, so that, under the coordinate identification,

$$(x^1,\ldots,x^p,y^{p+1},\ldots,y^m)\,\mapsto\,(x^1,\ldots,x^p) \tag{2}$$

is the bundle projection. To this end, we write down a basis of D_x followed by the values at x of the coordinate vector fields, then cancel in the resulting $(m+p)$-tuple every vector which is a linear combination of vectors precedining it, and finally *rearrange our coordinate functions* so that the $m-p$ coordinate vector fields still left after the cancellation will be ∂_λ, $\lambda=p+1,\ldots,m$. (The "rectangle" requirement is achieved by replacing the coordinate domain with a smaller neighborhood of x.) It follows (see Exercise 3) that by restricting dx^j, $j=1,\ldots,p$, to D_x, we obtain a basis of D_x^*, and that the same will be true for all nearby points. Thus, on a neighborhood of x, the restrictions of dx^j to D form a system of local trivializing sections of D^*. Therefore, for some unique smooth functions H_j^λ, defined on a neighborhood of x,

$$dy^\lambda\,=\,H_j^\lambda\,dx^j\qquad\text{on }\;D, \tag{3}$$

meaning that, at every point near x, the restrictions of dy^λ to the fibre of D are the corresponding linear combinations of the basis provided by the restrictions of dx^j. For the vector fields $e_j=\partial_j+H_j^\mu\partial_\mu$, it is obvious from Exercise 4 that

(4) e_j,∂_λ and $dy^j,dy^\lambda-H_k^\lambda\,dx^k$ are local trivializing sections of TM and T^*M,

since the same is true for $\partial_j,\partial_\lambda$ and dy^j,dy^λ. Also (see Exercise 5)

(5) the two local trivializing systems in (4) are each other's duals.

In view of (3), the fibre of D at any point near x is contained in the simultaneous kernel of the linear functionals $dy^\lambda-H_k^\lambda\,dx^k$. As these functionals are linearly independent, their simultaneous kernel is of dimension p, that is, coincides with the fibre of D. In other words, for a vector u tangent to M at a point near x,

(6) u lies in the fibre of D if and only if $(dy^\lambda-H_k^\lambda\,dx^k)(u)=0$, $\lambda=p+1,\ldots,m$.

By (4) – (6), e_j are local sections of D, linearly independent at each point. Thus,

(7) $e_j=\partial_j+H_j^\mu\partial_\mu$ and $\pi\partial_\lambda$ are local trivializing sections of D and D^{nrm}.

We now describe the components of the curvature $\varOmega$ of D relative to (7): for smooth local sections u,v of D, one has $u=u^je_j$ and $v=v^ke_k$, with suitable functions u^j,v^k, and then $\varOmega(u,v)=\varOmega_{jk}^\lambda u^jv^k\pi\partial_\lambda$, where (see Exercise 6)

$$\varOmega_{jk}^\lambda\,=\,\partial_jH_k^\lambda-\partial_kH_j^\lambda+H_j^\mu\partial_\mu H_k^\lambda-H_k^\mu\partial_\mu H_j^\lambda. \tag{8}$$

As usual, ∂_j denote both the coordinate vector fields and the correponding directional derivatives, that is, partial derivatives $\partial/\partial x^j$ relative to the given coordinate system (and similarly for ∂_λ).

Given a smooth mapping φ from a q-dimensional manifold Q into our coordinate domain and local coordinates z^a in Q, $a = 1,\dots,q$, it is clear from (6) that φ is tangent to D if and only if, with x^j and y^λ standing for $x^j \circ \varphi$ and $y^\lambda \circ \varphi$,

$$\partial_a y^\lambda \;=\; (\partial_a x^j)\, H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)\,. \tag{9}$$

In fact, $(dy^\lambda)(u) = \partial_a y^\lambda$ and $(dx^k)(u) = \partial_a x^j$, both evaluated at $\varphi(z)$, for the $d\varphi_z$-image u of the coordinate vector ∂_a at $z \in Q$, cf. [**DG**, formula (5.17)].

Two special cases of (9) are particularly important. In one, Q is an open interval $I \subseteq \mathrm{I\!R}$ and $z^1,\dots,z^q$ is the standard coordinate t. With $(\ )^{\cdot} = d/dt$, (9) becomes

$$\dot y^\lambda \;=\; \dot x^j\, H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)\,, \tag{10}$$

the equation characterizing *curves tangent to the distribution* D. Since (10) is a system of ordinary differential equations imposed on a curve $t \mapsto (x^1,\dots,x^p,y^{p+1},\dots,y^m)$,

(11) if the image $t \mapsto (x^1,\dots,x^p)$ of the curve under the projection (2) is fixed, (10) has a unique solution for any given initial data $y^\lambda(t_0)$

at a fixed initial parameter $t_0 \in I$. Thus, *every one-dimensional distribution is integrable* (and it has $\Omega = 0$ due to skew-symmetry of $\Omega(u,v)$ in u,v). Also, given a connection (horizontal distribution) H in a bundle $\mathrm{pr} : E \to B$, a smooth curve $\gamma : I \to B$, where I is an open interval, as well as fixed data $t \in I$ and $y \in E_{\gamma(t)}$, there exist an open interval $I' \subseteq I$ containing t and a unique curve $\varphi : I' \to E$ with $\pi \circ \varphi = \gamma$ and $\varphi(t) = y$ which is *horizontal* (that is, tangent to H). One then calls φ a *horizontal lift* of the curve $\gamma : I' \to B$. The *H-parallel transport* from $t \in I$ to $s \in I$ along a smooth curve $\gamma : I \to B$ is the mapping from an open subset of $E_{\gamma(t)}$ into $E_{\gamma(s)}$, defined by $y \mapsto \varphi(s)$, with $\varphi : I' \to E$ chosen as above, for those y for which $s \in I'$.

Another important special case of (9) occurs when P is an integral manifold of D contained in our coordinate domain. The projection (2) restricted to P is locally diffeomorphic, since P is transverse to its fibres. The inverse mapping theorem [**DG**, Theorem 74.2] allows us to use the inverse local diffeomorphism, and hence treat suitable open submanifolds of P, under our coordinate identification, as graphs of mappings $(x^1,\dots,x^p) \mapsto (y^{p+1},\dots,y^m)$, tangent to the distribution D, so that (9) reads

$$\partial_j y^\lambda \;=\; H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)\,. \tag{12}$$

We may view (12) as a system of first-order partial differential equations, imposed on the unknown functions y^λ (with $p+1 \le \lambda \le m$) of the independent variables x^j (with $1 \le j \le p$). Clearly, D restricted to the coordinate domain is integrable if and only if the system (12) is *completely integrable* in the sense that, for every $z \in \mathrm{I\!R}^p$ and every $w \in \mathrm{I\!R}^{m-p}$ for which (z,w) lies in the coordinate domain, there exists a solution $(x^1,\dots,x^p)\mapsto(y^{p+1},\dots,y^m)$ with the value w at z, defined on a neighborhood of z in $\mathrm{I\!R}^p$. As mentioned before,

(13) integral manifolds of D are, locally, the same as graphs of solutions to (12).

Complete integrability of (12) implies in turn that

$$\partial_k H_j^\lambda + H_k^\mu \partial_\mu H_j^\lambda \,=\, \partial_j H_k^\lambda + H_j^\mu \partial_\mu H_k^\lambda \tag{14}$$

as one sees rewriting the equalities $\partial_k\partial_j y^\lambda = \partial_j\partial_k y^\lambda$ with the aid of (12). Since, (14) amounts, by (8), to $\varOmega = 0$, we thus obtain another proof of necessity in the Frobenius Theorem.

We now prove sufficiency in the Frobenius theorem by assuming (14), fixing

$$(z,w)\,=\,(x_0^1,\dots,x_0^p,y_0^{p+1},\dots,y_0^m) \quad \text{in the coordinate domain,} \tag{15}$$

and showing, via induction on $q=1,\dots,p$, that there exists a unique solution to (12) for $j=1,\dots,q$, defined on an open rectangle in $\mathrm{I\!R}^q\times\{(x_0^{q+1},\dots,x_0^p)\}$ containing z, and having the value w at z. If $q=1$, this is clear from (11) for the image curve $t\mapsto(t,x_0^1,\dots,x_0^p)$. Suppose now that our claim holds with q replaced by some $q-1\in\{1,\dots,p-1\}$, on some open rectangle in $\mathrm{I\!R}^{q-1}\times\{(x_0^q,\dots,x_0^p)\}$. Applying (11) to the image curve $t\mapsto(x^1,\dots,x^{q-1},t)$, where $x^1,\dots,x^{q-1}$ are fixed, with the initial data $y^\lambda(x_0^q)$ at $t_0=x_0^q$ provided by the values at $(x^1,\dots,x^{q-1},x_0^q,\dots,x_0^p)$ of y^λ for the solution assumed to exist, and writing H_j^λ rather than $H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)$, we find unique functions y^λ of the variable x^q, defined on an open interval (depending on $x^1,\dots,x^{q-1}$ and containing x_0^q), such that

$$\partial_j y^\lambda = H_j^\lambda \;\text{ for } j<q \;\text{ at } x^q=x_0^q, \;\text{ and } \partial_q y^\lambda = H_q^\lambda \;\text{ for all } x^q, \tag{16}$$

where the first equality refers to the fact that the functions y^λ involve $x^1,\dots,x^{q-1}$ as parameters. In view of the regularity theorem for systems of ordinary differential equations with parameters [**DG**, Theorem 80.3], our y^λ treated as functions of $x^1,\dots,x^q$ are smooth and defined on an *open* set in $\mathrm{I\!R}^q\times\{(x_0^{q+1},\dots,x_0^p)\}$ containing z and, by making this set smaller, we may replace it with an open rectangle. We complete the induction step by showing that, identically in the rectangle,

$$\partial_q(\partial_j y^\lambda - H_j^\lambda) = (\partial_j y^\mu - H_j^\mu)\partial_\mu H_q^\lambda \;\text{ for } j<q, \tag{17}$$

with H_j^λ standing for $H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)$. Namely, (16) will then give $\partial_j y^\lambda = H_j^\lambda$ everywhere for all $j\le q$. (With $\partial_j y^\lambda - H_j^\lambda$ viewed as functions of x^q, (17) is a system of linear homogeneous ordinary differential equations, while $\partial_j y^\lambda - H_j^\lambda$ all vanish at $x^q=x_0^q$, cf. (16).) To obtain (17), note that, by (16), $\partial_q\partial_j y^\lambda = \partial_j\partial_q y^\lambda = \partial_j[H_q^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)]$. The left-hand side in (17) thus equals $\partial_j[H_q^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)] - \partial_q[H_j^\lambda(x^1,\dots,x^p,y^{p+1},\dots,y^m)]$, that is, in view of the chain rule and (14) – (16), $(\partial_j y^\mu - H_j^\mu)\partial_\mu H_q^\lambda$.

The ‘if’ part of the final clause in the Frobenius Theorem is obvious since vertical distributions of fibrations are integrable. It ‘only if’ part is in turn immediate from the regularity theorem for systems of ordinary differential equations with parameters [**DG**, Theorem 80.3]. Specifically, with fixed (z,w) in (15) now denoted by (z_0,w_0), allowing $w\in\mathrm{I\!R}^{m-p}$ to vary near w_0, let us consider the assignment

$$(x^1,\dots,x^p,w)\,\mapsto\,(x^1,\dots,x^p,y^{p+1},\dots,y^m),\tag{18}$$

where $(y^{p+1},\dots,y^m)$ stands for the value at $(x^1,\dots,x^p)$ of the solution having the value w at $z_0=(x_0^1,\dots,x_0^p)$. We now proceed to show that

(19)
restricted to a smaller neighborhood U of (z_0,w_0),
(18) is a diffeomorphism onto an open set in $\mathrm{I\!R}^m$.

The 'only if' part of the final clause in the Frobenius Theorem easily follows from (19): making U even smaller, we may replace it with an open rectangle, so that the diffeomorphism in (19) sends the fibres of a fibration (which are p-dimensional open rectangles) onto integral manifolds of D (which are graphs of solutions to (12)). To prove (19), we use the inverse mapping theorem. Namely, the differental (18) at (z_0,w_0) is the following matrix, evaluated at $(x^1,\dots,x^p,w)=(z_0,w_0)$:

$$\begin{bmatrix}[\partial x^j/\partial x^k] & [\partial x^j/\partial w^\mu]\\ [\partial y^\lambda/\partial x^k] & [\partial y^\lambda/\partial w^\mu]\end{bmatrix}.\tag{20}$$

Here $[\partial x^j/\partial w^\mu]$ is the zero $p\times(m-p)$ matrix, and $[\partial x^j/\partial x^k]$ the identity $p\times p$ matrix (which is obvious); however, $[\partial y^\lambda/\partial w^\mu]$ is also the identity matrix, of size $(m-p)\times(m-p)$, since at $(x^1,\dots,x^p,w)=(z_0,w_0)$ we have $y^\lambda=w^\lambda$. Thus, the determinant of (20) at (z_0,w_0) equals 1.

Given a distribution D (not necessarily integrable) on a manifold M,

(21)
if P and P' are two integral manifolds of D, then every point
$x\in P\cap P'$ lies in a common open submanifold of P and P'.

This is immediate from (13) along with the fact that a solution to (12) with a fixed initial condition (15) is unique on some neighborhood of z. The uniqueness property is in turn obvious if one applies Exercise 7 to curves $t\mapsto(x^1,\dots,x^p)$ which are radial segments emanating from z. In other words, (21) states that

(22)
the intersection of two integral manifolds of a distribution is
an open subset of both, relative to their manifold structures.

Next, with the word 'countable' always meaning *finite or countably infinite*,

(23)
the union of any nonempty countable family of integral manifolds of a distribution has a unique structure of an integral manifold containing each element of the family as an open subset.

In fact, as in [**DG**, formula (3.1) on p. 6] we turn the union into a disjoint-union manifold, so that the open-subset condition follows, and the submanifold property, being local, holds for the union as well.

One says that an integral manifold P of a (possibly nonintegrable) distribution D on a manifold M is *maximal* if P is connected and not contained in any other connected integral manifold of D. One then also calls P a *leaf* of D.

The Leaf Theorem. *The leaves of a distribution are pairwise disjoint. Every connected integral manifold P is contained in a unique leaf P', and P is open in P'.*

To prove the Leaf Theorem, we fix a p-dimensional distribution D on a manifold M of dimension m, and define the *integrability set* $\mathcal{J}$ to be the union of all integral manifolds of D. An equivalence relation $\sim$ in $\mathcal{J}$ arises if one declares that $x \sim y$ whenever there exists a curve $\gamma : [a, b] \to M$ with $\gamma([a, b]) \subseteq \mathcal{J}$ and $(\gamma(a), \gamma(b)) = (x, y)$, which is piecewise smooth in the sense of [**DG**, Section 3] and tangent to D (meaning: $\dot\gamma(t) \in D_{\gamma(t)}$ whenever $\dot\gamma(t)$ exists). Every connected integral manifold P of D is obviously contained in a single equivalence class of $\sim$, as piecewise smooth curves in P are obviously tangent to D.

The proof of the Leaf Theorem will thus be complete if we show that any given equivalence class P of $\sim$ is an integral manifold of D (and, consequently, the equivalence classes are the same as the leaves). To this end, we first note that, due to the final sentence of the last paragraph, P is the union of all connected integral manifolds intersecting it. The union of the maximal atlases of all these manifolds is – due to (21) – a C^∞ atlas on P. For the Hausdorff and countability axioms we use (13): coordinate charts mentioned in the lines surrounding (2) may be chosen from a countable subatlas for M (and then, subjected to permutations of the coordinates, will still form a countable family), and any two distinct points clearly have disjoint coordinate neighborhoods, of this type, in M.

The existence and disjointness of leaves is an interesting fact even in the case of integrable distributions, leading to a partition of the underlying manifold known as the *leaf decomposition*.

We need the following well-known result. For a proof, see [**DG**, Lemma 9.3]:

The continuous-versus-smooth lemma. *For a smooth mapping $\varphi : Q \to M$ between manifolds and a submanifold P of M such that $\varphi(Q) \subseteq P$, the resulting mapping $\varphi : Q \to P$ is continuous if and only if it is smooth.*

Note that the hypotheses of the lemma do not imply that $\varphi : Q \to P$ is smooth (or continuous). An example is a figure-eight curve in $\mathrm{I\!R}^2$ with two different submanifold structures [**DG**, Section 9]: Q stands for one structure, P for the other, and $\varphi : Q \to P$ is the identity.

The Leaf-Mapping Theorem. *If $\varphi : Q \to M$ is a smooth mapping between manifolds and $\varphi(Q) \subseteq P$, where P is an integral manifold of an integrable distribution D on M, then φ is smooth as a mapping $Q \to P$.*

To prove the theorem, we may assume, in view of the Leaf Theorem, that P is a leaf of D while, due to the continuous-versus-smooth lemma, it suffices to show that $\varphi : Q \to P$ is then continuous. Given $z \in Q$, the 'only if' part of the final clause in the Frobenius Theorem allows us to choose a neighborhood N of $\varphi(z)$ with a fibration $\mathrm{pr} : N \to B$ such that the fibres of pr are connected integral manifolds of D. According to the Leaf Theorem, $N \cap P$ is the union of a family of fibres of pr, which are open submanifolds of P, and so, due to their mutual disjointness, the family is at most countable (Exercise 9). For a connected neighborhood U of z in Q with $\varphi(U) \subseteq N$, the image $\mathrm{pr}(\varphi(U))$ thus is a nonempty pathwise connected countable subset of B, which implies that it is the one-point set $\{y\}$, where $y = \mathrm{pr}(\varphi(z))$. Consequently, the continuous mapping $\varphi : U \to N$ takes values in the fibre $F = \mathrm{pr}^{-1}(y)$, and so it is continuous as a mapping $U \to F$, since the fibres of a fibration have the subset topology [**DG**, Theorem 9.6]. This completes the proof.

The uniqueness corollary. *An integral manifold P of an integrable distribution D on a manifold M has only one manifold structure that makes it a submanifold of M.*

In fact, denoting by P', P'' two such manifold structures, we can apply the Leaf-Mapping Theorem to the pairs $(P, Q) = (P', P'')$ and $(P, Q) = (P'', P')$, concluding the the identity mapping is a diffeomorphism $P' \to P''$.

Given a diffeomorphism $\varphi : M \to M'$ between manifolds and a distribution D on M, one defines the φ-image of D to be the distribution $(d\varphi)D$ on M with $[(d\varphi)D]_{\varphi(x)} = d\varphi_x(D_x)$ for every $x \in M$. The φ-images of integral manifolds (or, leaves) of D then are integral manifolds (or, respectively, leaves) of $(d\varphi)D$. In the case where, in addition, $M' = M$ and $(d\varphi)D = D$, we say that D is *φ-invariant*, or *invariant under φ*.

A *left-invariant distribution* on a Lie group G is one invariant under all left translations. Left-invariant distributions D on G stand in a canonical bijective correspondence with vector subspaces $\mathfrak{p}$ of its Lie algebra $\mathfrak{g}$, which is always identified [**DG**, Section 8] with the space of left-invariant vector fields on G. The correspondence in question assigns to D the space $\mathfrak{p}$ of left-invariant sections of D. Equivalently, $\mathfrak{p}$ determines D via the formula $D_x = \{u_x : u \in \mathfrak{p}\}$. Clearly, for D and $\mathfrak{p}$ related as above, D is integrable if and only if $\mathfrak{p}$ is a Lie subalgebra of $\mathfrak{g}$. Also, whenever $\mathfrak{p}$ is not a Lie subalgebra of $\mathfrak{g}$, the corresponding D has no integral manifolds: if one existed, its images under left translations would all be integral manifolds, implying integrability of D.

The Lie algebra $\mathfrak{p}$ of any Lie subgroup P of G is canonically identified [**DG**, Section 12, Problem 3] with a Lie subalgebra of $\mathfrak{g}$, namely, the one formed by left-invariant vector fields on G that are tangent to P at every point of P. Note that $\mathfrak{p} \subseteq \mathfrak{g}$ will not change if P is replaced by its identity component. According to the next theorem, one obtains in this way a bijective correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.

The Lie-Subgroup Theorem. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G.*

(i) *Every Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is the Lie algebra of a unique connected Lie subgroup P of G. The subgroup P is the leaf, containing the identity element $1 \in G$, of the left-invariant distribution D canonically associated with $\mathfrak{p}$.*

(ii) *Any submanifold P of G which is also a subgroup of G must necessarily be a Lie subgroup of G, as well as an integral manifold of an integrable left-invariant distribution on G, and the manifold structure of P is the only one that makes P a submanifold of M.*

(iii) *For a connected submanifold P of G, the following conditions are equivalent*:

(a) *P is a subgroup of G,*

(b) *P is a Lie subgroup of G,*

(c) *P is the leaf through 1 of an integrable left-invariant distribution on G.*

To prove the theorem, we first note that the leaf P in (i), or in (iii-c), coincides with the translation image $x^{-1}P$, for any $x \in P$, as the latter is also a leaf through 1. The resulting closedness of P under the operation $(x,y) \mapsto x^{-1}y$ shows (see Exercise 10) that P is a subgroup of G, and so (iii-c) implies (iii-a).

Conversely, (iii-a) yields (iii-c). In fact, given P as in (iii-a), the translation images (left cosets) xP, for $x \in G$, form a left-invariant partition of G. The distribution D provided by the tangent spaces of the cosets, being left-invariant, is smooth (Exercise 11), and integrable, with the cosets serving as (connected) integral manifolds. Every coset is thus contained, as an open submanifold, in a unique leaf of D (see the Leaf Theorem), and every leaf of D is a union of cosets. As the leaves are connected, disjointness and the just-mentioned openness of the cosets in a leaf shows that the leaves of D are precisely the cosets of P, one of which is P itself.

The final paragraph of the theorem now follows with the phrase 'connected Lie subgroups' replaced by *connected submanifolds which are also subgroups*. In fact, we just showed that the leaf of D through 1 is an example of the latter. Conversely, if P is one of the latter, the translation images (left cosets) xP, for $x \in G$, form a left-invariant partition of G. The distribution provided by the tangent spaces of the cosets, being left-invariant, is smooth (Exercise 11), and integrable, with the cosets serving as the leaves.

The Image-Group Theorem. *The image $\varphi(G)$ of any Lie-group homomorphism $\varphi : G \to H$ is a Lie subgroup of H.*

This is immediate: we may assume that G is connected, and then easily conclude that $P = \varphi(G)$ satisfies (c), and hence (b), in part (iii) of the Lie-Subgroup Theorem, applied to H instead of G.

Exercise 1. Verify that $[\phi u, \psi v] = \phi\psi[u,v] + \phi(d_u\psi)v - \psi(d_v\phi)u$ for any vector fields u, v and functions ϕ, ψ on a manifold.

Exercise 2. Let e_j be local trivializing sections of a distribution D on a manifold M, with the normal-bundle projection $\pi : TM \to D^{\mathrm{nrm}}$. Expanding smooth local sections u, v of D as $u = u^j e_j$ and $v = v^k e_k$, show that $\pi[u,v] = \varOmega_{jk} u^j v^k$, where $\varOmega_{jk} = \pi[e_j, e_k]$.

Exercise 3. Given a subspace $\mathcal{D}$ of a real vector space $\mathcal{T}$, with $\dim \mathcal{D} = p$ and $\dim \mathcal{T} = m$, along with a basis $t_1, \ldots, t_m$ of $\mathcal{T}$, the corresponding dual basis $u^1, \ldots, u^m$ of $\mathcal{T}^*$, and a basis $d_1, \ldots, d_p$ of $\mathcal{D}$, prove that $d_1, \ldots, d_p, t_{p+1}, \ldots, t_m$ is a basis of $\mathcal{T}$ if and only if the restrictions of $u^1, \ldots, u^p$ to $\mathcal{D}$ form a basis of $\mathcal{D}^*$.

Exercise 4. Show that, if $d_1, \ldots, d_p, t_{p+1}, \ldots, t_m$ is a basis of a real vector space $\mathcal{T}$, then so is $\hat{d}_1, \ldots, \hat{d}_p, t_{p+1}, \ldots, t_m$, as long as each $\hat{d}_j$ equals d_j plus some linear combination (possibly depending on j) of the vectors $t_{p+1}, \ldots, t_m$.

Exercise 5. Verify (5).

Exercise 6. Prove (8).

Exercise 7. Show that, due to the chain rule, the composition of a smooth curve $t \mapsto (x^1, \ldots, x^p)$ with $(x^1, \ldots, x^p) \mapsto (y^{p+1}, \ldots, y^m)$ satisfying (12) is a solution $t \mapsto (x^1, \ldots, x^p, y^{p+1}, \ldots, y^m)$ of the system (10) of *ordinary* differential equations.

Exercise 8. Prove that, for distributions D and D' on manifolds M and M', a distribution $D \times D'$ on $M \times M'$ is defined by the formula $(D \times D')_{(x,x')} = D_x \times D_{x'}$, where $T_{(x,x')}(M \times M') = T_x M \times T_{x'} M'$ according to the standard identification [**DG**, Section 9, Problem 28]. Verify that $P \times P'$ is an integral manifold of $D \times D'$ whenever P and P' are integral manifolds of D and D', respectively.

Exercise 9. Show that any family of disjoint open sets in a manifold is (at most) countable.

Exercise 10. Verify that a subset of a group is a subgroup if and only if it is nonempty and closed under the operation $(x, y) \mapsto x^{-1} y$.

Exercise 11. Show that left-invariant distributions on Lie groups are smooth.

Exercise 12. Given a horizontal distribution (connection) H in a fibration (bundle) $\mathrm{pr} : E \to B$, a local section $U \to E$, defined on an open set $U \subseteq B$ is called *parallel* if its image, as a submanifold of E, constitutes an integral manifold of H. Assuming connectedness of U, prove that a parallel section of E defined on U is uniquely determined by its value at one point.