MATH 7711, AUTUMN 2019

Inner Products up to a Factor

[**AC**] stands for *Algebraic Curvature Tensors* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/ac.pdf`

In these notes, $\mathcal{T}$ is always a finite-dimensional real vector space, and 'relation' means *a binary relation involving two nonzero vectors in* $\mathcal{T}$. Given a (possibly degenerate) symmetric bilinear form $(\,,\,)$ in $\mathcal{T}$ and vectors $u, v \in \mathcal{T}$, we say that u, v are $(\,,\,)$-orthogonal, or have equal $(\,,\,)$-inner-squares, if $(u, v) = 0$ or, respectively, $(u, u) = (v, v)$. A vector $(\,,\,)$-orthogonal to itself will be called $(\,,\,)$-*null*.

The Angle-Geometry Lemma. *For two Euclidean inner products* $\langle\,,\rangle$ *and* $(\,,\,)$ *in* $\mathcal{T}$*, the following three conditions are equivalent.*

(i) $(\,,\,)$ *is a scalar multiple of* $\langle\,,\rangle$,
(ii) $\langle\,,\rangle$ *and* $(\,,\,)$ *define the same angle function, that is,* $\langle u, v\rangle/[\langle u, u\rangle\langle v, v\rangle]^{1/2} = (u, v)/[(u, u)(v, v)]^{1/2}$ *whenever* $u, v \in \mathcal{T} \smallsetminus \{0\}$,
(iii) $\langle\,,\rangle$ *and* $(\,,\,)$ *lead to the same orthogonality relation between nonzero vectors.*

Proof. Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Now assume (iii). Thus, $\langle\,,\rangle$ and $(\,,\,)$ give rise to the same relation $\sim$ between vectors $u, v \in \mathcal{T} \smallsetminus \{0\}$, where $u \sim v$ if and only if u, v are orthogonal and of equal lengths (since this amounts to orthogonality of both pairs u, v and $u + v, u - v$). Therefore, a fixed $\langle\,,\rangle$-orthonornal basis is $(\,,\,)$-orthogonal with all vectors of the same length r, and so $(\,,\,) = r^2\langle\,,\rangle$ (as both sides of the equality agree on any pair of vectors from the basis in question). Consequently, (iii) $\Rightarrow$ (i).

The Null-Cone Lemma. *Let* $(\,,\,)$ *be a symmetric bilinear form in a vector space* $\mathcal{T}$ *endowed with an indefinite pseudo-Euclidean inner product* $\langle\,,\rangle$*. Then* $(\,,\,)$ *is a nonzero scalar multiple of* $\langle\,,\rangle$ *if and only if* $\langle\,,\rangle$ *and* $(\,,\,)$ *have the same null vectors.*

Proof. It suffices to establish the 'if' part. Let $\langle\,,\rangle$ and $(\,,\,)$ have the same null vectors. Then they define the same relation $\sim$ between vectors $u, v \in \mathcal{T} \smallsetminus \{0\}$, with $u \sim v$ meaning that u, v are orthogonal and have opposite inner squares (as $u \sim v$ then clearly amounts to requiring both $u + v$ and $u - v$ to be null). Denoting by p the negative index of $\langle\,,\rangle$, so that $0 < p < m = \dim \mathcal{T}$, we may fix an $\langle\,,\rangle$-orthonornal basis $e_1, \ldots, e_m$, with $\langle e_i, e_i\rangle = -1$ for $i \le p$ and $\langle e_k, e_k\rangle = 1$ for $k > p$. Thus, $e_k \sim e_i$ as well as $e_k \sim (e_i + e_j)/\sqrt{2}$ and $e_i \sim (e_k + e_l)/\sqrt{2}$ whenever $1 \le i < j \le p < k < l \le m$. (Note: j, or l, is to be ignored and deleted if $p = 1$ or, respectively, $p = m$.) Since $\sim$ also corresponds to $(\,,\,)$, all such e_k (and $(e_k + e_l)/\sqrt{2}$) have the same $(\,,\,)$-inner-square c, opposite to that of all e_i (and $(e_i + e_j)/\sqrt{2}$), which clearly gives $(e_k, e_l) = (e_i, e_j) = 0$. Thus, $(\,,\,) = c\langle\,,\rangle$, as both sides agree on any pair of vectors from our basis, and $c \ne 0$ (or else all vectors in $\mathcal{T}$ would be $(\,,\,)$-null). This completes the proof.