MATH 7711, AUTUMN 2019

Ricci-Hessian Equations

[**DG**] stands for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`
[**TC**] for *Tractor Connections in Tractor Bundles* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/tc.pdf`

Given a torsion-free connection ∇ and a smooth vector field v on a manifold M, by contracting the Ricci identity $v^k{}_{,ij} - v^k{}_{,ji} = R_{ijl}{}^k v^l$ in $j = k$ we see that

$$v^k{}_{,ik} - v^k{}_{,ki} \,=\, R_{ik} v^k \tag{1}$$

or, in coordinate-free notation, $\operatorname{div}\nabla v - d(\operatorname{div} v) = r(\,\cdot\,, v)$.

Lemma. *Suppose now that smooth functions α and ψ on a pseudo-Riemannian manifold (M, g) of any dimension m with the Ricci tensor r satisfy the condition*

$$\nabla d\alpha \,+\, q\alpha r \,=\, \psi g \tag{2}$$

where q is a constant and ∇ denotes the Levi-Civita connection. Then

$$(m-1)\,d\psi \,=\, -(q+1)\,r(\nabla\alpha, \cdot\,) \,+\, qs\,d\alpha \,+\, q\alpha\, ds/2,$$

s being the scalar curvature, or, in coordinates,

$$(m-1)\,\psi_{,i} \,=\, -(q+1)R_{ik}\alpha^{,k} \,+\, qs\alpha_{,i} \,+\, q\alpha s_{,i}/2. \tag{3}$$

Proof. Taking the g-trace of (2) we see that $m\psi = \alpha^{,k}{}_{,k} + qs\alpha$. Differentiating this, we obtain

$$m\psi_{,i} \,=\, \alpha^{,k}{}_{ki} \,+\, qs\alpha_{,i} \,+\, q\alpha s_{,i}. \tag{4}$$

The divergence operator applied to the coordinate form $\psi g_{ij} = \alpha_{,ij} + q\alpha R_{ij}$ of (2) yields in turn $\psi^{,k} g_{ik} = \alpha_{,ik}{}^k + qR_{ik}\alpha^{,k} + q\alpha R_{ik}{}^{,k}$. As $\psi^{,k} g_{ik} = \psi_{,i}$, while symmetry of the Hessian $\nabla d\alpha$ and (1) give $\alpha_{,ik}{}^k = \alpha_{,ki}{}^k = \alpha^{,k}{}_{,ik} = \alpha^{,k}{}_{ki} + R_{ik}\alpha^{,k}$, and $2R_{ik}{}^{,k} = s_{,i}$ from the Bianchi identity for the Ricci tensor [**DG**, formula (38.13)], we can rewrite the last equality as $\psi_{,i} = \alpha^{,k}{}_{ki} + (q+1)R_{ik}\alpha^{,k} + q\alpha s_{,i}/2$. Subtracting (4) from this, we get (3).

Corollary. *Under the assumptions of the lemma, $\overline{\nabla}_w(v, \alpha, \psi) = 0$ for $v = \nabla\alpha$ and any vector field w, where $\overline{\nabla}$ is the connection given by*

$$\overline{\nabla}_w(v, \alpha, \psi) = \left(\nabla_w v - \psi w + q\alpha r w,\; d_w\alpha - g(w, v),\; d_w\psi + \frac{2(q+1)r(w, v) - 2qs\,g(w, v) - q\alpha\, d_w s}{2(m-1)}\right)$$

in the vector bundle $E = TM \oplus [M \times \mathrm{I\!R}^2]$ over M obtained as the direct sum of TM and the product plane bundle $M \times \mathrm{I\!R}^2$.

More precisely, α and ψ satisfy (2) *if and only if $(v, \alpha, \psi) = 0$, with $v = \nabla\alpha$, is a $\overline{\nabla}$-parallel section of E.*

When $m \geq 3$ and $q = 1/(m-2)$, we call $E = TM \oplus [M \times \mathrm{I\!R}^2]$ the *tractor bundle* of the m-dimensional pseudo-Riemannian manifold (M, g), and refer to $\overline{\nabla}$ as the *tractor connection* in E. See [**TC**].