MATH 7711, AUTUMN 2019

Consequences of the Second Bianchi Identity

[**DG**] stands for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`
[**AC**] for *Algebraic Curvature Tensors* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/ac.pdf`

By a (p,q) tensor, or tensor field, on a manifold we mean one with p upper and q lower indices. Thus, scalars are $(0,0)$ tensors; vectors, $(1,0)$ tensors; 1-forms, $(0,1)$ tensors; the difference of two connections is a $(1,2)$ tensor field; a pseudo-Riemannian metric a $(1,2)$ tensor field; and the curvature tensor of a connection, a $(1,3)$ tensor field.

Given a torsion-free connection ∇ on a manifold M and a smooth $(1,q)$ tensor field A, where $q \geq 0$, we define its *divergence* $a = \mathrm{div}A$ to be the $(0,q)$ tensor field with $a_{i_1\ldots i_q} = A_{i_1\ldots i_q}{}^k{}_{,k}$. The *exterior derivative* of a smooth $(0,2)$ tensor field b is the $(0,3)$ tensor field Z with $Z_{ijk} = b_{jk,i} - b_{ik,j}$. Thus, for the curvature and Ricci tensors R, r of ∇,

$$(1)\qquad \mathrm{div}R = -dr, \quad \text{that is,} \quad R_{ijk}{}^l{}_{,l} = R_{ik,j} - R_{jk,i},$$

since the second Bianchi identity $0 = R_{ijk}{}^l{}_{,q} + R_{jqk}{}^l{}_{,i} + R_{qik}{}^l{}_{,j}$, contracted in $q = l$, yields $0 = R_{ijk}{}^l{}_{,l} + R_{jk,i} - R_{ik,j}$.

From now on we fix a pseudo-Riemannian metric g on M, and denote by ∇ its Levi-Civita connection. This allows us to form the divergence of any smooth $(0,q)$ tensor field L, with $q \geq 1$, by first using g to raise the last index and then taking the divergence of the resulting $(1,q-1)$ tensor field, so that $a = \mathrm{div}L$ is given by $a_{i_2\ldots i_q} = g^{jk}L_{i_2\ldots i_q j,k}$. It is now a trivial exercise to verify that, for a smooth symmetric $(0,2)$ tensor field b and a smooth function f,

$$(2)\qquad 2\,\mathrm{div}(g \wedge b) = -db - (\mathrm{div}b) \wedge g, \qquad \mathrm{div}(fg) = (df) \wedge g,$$

where, for any 1-form, $\xi \wedge b$ is defined by $(\xi \wedge b)_{ijk} = \xi_i b_{jk} - \xi_j b_{ik}$, and $g \wedge b$ is, as in [**AC**], the algebraic curvature tensor field with $(g \wedge b)_{ijkl} = g_{ik}b_{jl} + g_{jl}b_{ik} - g_{jk}b_{il} - g_{il}b_{jk}$.

$$(1)\qquad (m-2)\,\mathrm{div}W = -(m-3)\,dh, \quad \text{that is,} \quad R_{ijk}{}^l{}_{,l} = R_{ik,j} - R_{jk,i},$$